Intro. to Advanced Mathematics (Midterm)

Problem 1. [12 Points] Prove by induction that for all $n \geq 2$:
\[
\left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{1}{2} \cdot \frac{n+1}{n}.
\]

Problem 2. [12 Points] Define the notions of a function $f : A \rightarrow B$ being 1–1, onto, and bijective.

Let $g : B \rightarrow C$ another function. Suppose that the composition $g \circ f$ is bijective. Show that $g$ is onto and that $f$ is 1–1.

Problem 3. [8 Points] Let $f : A \rightarrow B$ and let $B_1$ and $B_2$ be subsets of $B$. What is $f^{-1}(B_1)$, the inverse image of $B_1$? Prove or disprove
\[
f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2).
\]

Problem 4. [10 Points] Define the notion of an equivalence relation on a set $X$. Consider a relation $\sim$ on the natural numbers $\mathbb{N}$ by setting $a \sim b$ if $\gcd(a, b) \neq 1$. Which of the properties of an equivalence relation are satisfied?

Problem 5. [15 Points] Consider a function defined by
\[
f(x) = \frac{x-1}{x-3}.
\]
Use $A = \mathbb{R} \setminus \{3\} = (-\infty, 3) \cup (3, \infty)$ as domain.

1. Prove or disprove that $f$ is 1–1.

2. What is the image $B = \text{im}(f) \subset \mathbb{R}$ of $f$.

3. Find $g : B \rightarrow A$ so that it is the inverse of $f : A \rightarrow B$.

Problem 6. [10 Points] Prove or disprove
\[
(A \cup B) \Delta C = (A \Delta C) \cup (B \Delta C).
\]

Problem 7. [18 Points] Consider the set $\mathbb{Z} \times \mathbb{Z}$ (pairs of integers) with the operations
\[
(a, b) + (a', b') = (a + a', b + b')
\]
\[
(a, b) \times (a', b') = (aa' + bb', ab + a'b)
\]
Verify the distributive law and find the units with respect to addition and multiplication. State the Cancellation Axiom (for any commutative ring), and prove or disprove it for $\mathbb{Z} \times \mathbb{Z}$ with the given operations.