Introduction to Advanced Mathematics (Exam I)

Problem 1. [10 Points] Suppose P, Q and R are statements. Use truth tables to decide whether $P \implies (Q \vee R)$ and $\neg Q \implies (\neg P \vee R)$ are equivalent statements.

Problem 2. [10 Points] Let A and B be sets. Define their symmetric difference by $A\Delta B = (A \setminus B) \cup (B \setminus A)$. We suppose that the sets are contained in some set U, and a superscript c indicates that we are taking the complement in U. Consider the assertion

$$A\Delta B = (A \cup B) \cap (A \cap B)^c.$$

1. Verify the assertion if $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $U = \{1, 2, 3, 4, 5\}$.
2. Prove or disprove the assertion. A Venn diagram does not suffice.

Problem 3. [10 Points] A prime triple is a triple $(n, n + 2, n + 4)$ of natural numbers, so that all three numbers are prime. Show that $(3, 5, 7)$ is the only prime triple.

Problem 4. [10 Points] Verify for all natural numbers n:

$$1 + 5 + \cdots + (4n - 3) = 2n^2 - n.$$

Problem 5. [10 Points] Following Darboux, a bounded function f on an interval $[a, b]$ is integrable if for all $\epsilon > 0$ there are upper and lower sums S_u and S_l, so that $|S_u - S_l| < \epsilon$. Upper and lower sums are similar to Riemann sums, and the precise definition is not relevant to the problem. Spell out what it means that f is not integrable.

Problem 6. [10 Points] Define a relation on the real numbers $\mathbb{R}$ by:

$$R = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a - b \in \mathbb{Q}\}.$$

Two real numbers a and b are related if their difference is rational. You may write $a \sim b$ instead of $(a, b) \in R$. Verify that R is an equivalence relation and find [3], the class of 3.