ALGEBRAIC REALIZATION FOR CYCLIC GROUP ACTIONS

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Abstract. Suppose G is a finite cyclic group and M a closed smooth G–manifold. In this paper we will show that there is a nonsingular real algebraic G–variety X which is equivariantly diffeomorphic to M and all G–vector bundles over X are strongly algebraic.

1. Introduction

Let G be a compact Lie group. A *real algebraic G–variety* is a G–invariant subset

$$\{x \in \Omega \mid p_1(x) = \cdots = p_n(x) = 0\}$$

in a real G–module Ω that is the set of common zeros of a finite collection $\{p_1, \ldots, p_n\}$ of polynomials [9]. If M is a closed smooth G–manifold and X is a nonsingular real algebraic G–variety that is equivariantly diffeomorphic to M, then we say that M is *algebraically realized* and that X is an *algebraic model* of M. We call X a *strongly algebraic* model of M, if, in addition, all G–vector bundles over X are strongly algebraic. This means that, up to isomorphism, the bundles are classified by entire rational maps to equivariant Grassmannians with their canonical algebraic structure. Existing results lead us to believe

Conjecture 1.1. [12, p. 32] *Let G be a compact Lie group. Then every closed smooth G–manifold has a strongly algebraic model.*

Our principal result confirms the conjecture in a special case.

Theorem 1.2. *Let G be a finite cyclic group. Then every closed smooth G–manifold has a strongly algebraic model.*

1.1. **History.** Let us give a brief review of the history. J. Nash [17] posed the algebraic realization problem for closed smooth manifolds, and this problem has an affirmative answer, see Tognoli [22] and Akbulut-King [1], [2]. For cyclic groups G we showed that closed smooth G–manifolds are algebraically realized, see [10]. See [9] for other results on the equivariant algebraic realization problem.

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The tangent bundle of a nonsingular real algebraic variety is strongly algebraic, so instead of algebraically realizing a manifold together with one specific bundle, it is natural to try and algebraically realize the manifold with all of its bundles. That this is possible is Conjecture 1.1. Benedetti and Tognoli [3] proved this conjecture in the case where G is the trivial group. Conjecture 1.1 is true if G is a product of an odd order group with a 2–torus, see [12, Theorem B]. This implies Theorem 1.2 in the special case where G is a cyclic group whose order is twice an odd number. Conjecture 1.1 is also true if G is a compact Lie group and the action on M is semifree, see [12], or if G is cyclic and the action on the manifold has only one isotropy type, see [13]. Hanson [14] proved a result like Theorem 1.2 for $\mathbb{Z}_4$–manifolds with one vector bundle in addition to the tangent bundle. For a more extensive history see [13].

1.2. **Structure of the paper.** In Section 2 we will outline the proof of Theorem 1.2 and at the same time introduce notation that is used thoughout the paper. In Section 3 we collect results from the literature that we will employ. In Sections 4 – 7 we carefully develop the proof of our main result, Theorem 1.2. We relegate detailed proofs of some intermediate results to Sections 8 – 13.

Here is a roadmap to the proof of Theorem 1.2. The arrows stand for implications that are carried out in the indicated sections, and ‘P’d’ abbreviates ‘Proved”.

$$
\begin{tikzcd}
\text{Prop 2.7} \arrow[r, "\S\ 3"] & \text{Prop 2.3} \arrow[r, "\S\ 2.1"] & \boxed{\text{Theorem 1.2}} \\
\text{Prop 2.8} \arrow[u, "\S\ 2.5"] & \text{Prop 2.4 [13]} \arrow[u, "\S\ 3"] & \text{Prop 2.2} \arrow[u, "\S\ 2.1"'] \\
\text{Prop 2.9} \arrow[u, "\S\ 8"] & & \text{Prop 5.1 (P'd in } \S\ 5) \arrow[u, "\S\ 5"'] \\
\text{Prop 9.1} \arrow[u, "\S\ 9"] & & \\
\text{Prop 2.10 (P'd in } \S\ 7) \arrow[u, "\S\ 13"] & &
\end{tikzcd}
$$

2. Basic Definitions & Proof of Theorem 1.2

2.1. **Simplifying the combinatorial structure.** Blow-ups may be used to simplify the isotropy structure of a G–manifold [24], and the algebraic realization problem reduces to one for manifolds as characterized in the following definition.

Definition 2.1. Suppose G is cyclic. We call a G–manifold M *iso-special* if locally the action on M has one or two isotropy types. If, locally, the action has isotropy groups K and H, with $K \subset H$, then we require that the H–fixed set is of codimension 1 in the K–fixed set.

In Section 5 we will generalize the definition of being iso–special to actions of abelian groups (Wasserman [24] called iso–special actions *simple.*) and prove:

Proposition 2.2. *Let G be an abelian group and assume that all closed smooth iso-special G–manifolds have strongly algebraic models. Then all closed smooth G–manifolds have strongly algebraic models.*

In Section 3 we will derive our next result, Theorem 2.3, from Theorem 2.4 and Proposition 2.7. Generalizations of Theorems 2.3 will lead to generalizations of Theorem 1.2.

Theorem 2.3. *Let G be a cyclic group. Then every iso–special closed smooth G–manifold has a strongly algebraic model.*

Proof of Theorem 1.2. If G is cyclic, then Theorem 2.3 confirms the assumptions of Proposition 2.2, and with this Theorem 1.2 is proved. □

The following Theorem 2.4 for actions with one isotropy type is not only a special case of Theorems 1.2 and 2.3, but also an essential ingredient in the proof of Theorem 2.3. Suh [21] proved Theorem 2.4 if the isotropy groups are of odd index in G.

Theorem 2.4. [13] *Suppose G is a cyclic group and M a closed smooth G–manifold, such that locally the action has only one isotropy type. Then M has a strongly algebraic model.*

We will get to the two isotropy group case after some preparation.

2.2. **The Real Algebraic Category and Algebraic Models.** In the introduction we defined the concept of a real algebraic variety. The term *nonsingular* is used with its classical meaning as in [25] or [4, Section 3.3]. One may want to look at different concepts of morphisms. Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ be varieties and $f : X \to Y$. We call f *regular* if it extends to a map $F : \mathbb{R}^m \to \mathbb{R}^n$ and each coordinate of F is polynomial:

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{F} & \mathbb{R}^n \\ \subseteq\big\uparrow & & \big\uparrow\subseteq \\ X & \xrightarrow{f} & Y \end{array}$$

We call f *entire rational*, if it extends to $F = P/q$, so that each coordinate of P is polynomial, and q is a nowhere vanishing polynomial.

Let Y be a nonsingular real algebraic G–variety. We call $q : X \to Y$ an *algebraic map* if X is a nonsingular real algebraic G–variety and q is an equivariant entire rational map. Let $f : M \to Y$ be an equivariant map from a closed smooth G–manifold M to Y. We call an algebraic map (X, q) an *algebraic model* of (M, f) if there is an equivariant diffeomorphism $\Phi : X \to M$, so that $f \circ \Phi$ is equivariantly homotopic to q.

2.3. **Grassmannians and vector bundles.** Let Ω be a representation of G. Denote the Grassmannian of real subspaces of Ω of dimension d by $G_{\mathbb{R}}(\Omega, d)$. One may write down polynomial equations that describe $G_{\mathbb{R}}(\Omega, d)$ as a nonsingular real algebraic G–variety, see [4] or [9]. Pick a G–invariant inner product and an orthonormal basis for Ω. Identify a subspace V of Ω with the matrix of the orthogonal projection onto V. If $n = \dim \Omega$ and $\mathfrak{M}_{\mathbb{R}}^{n\times n}$ is the set of real $n \times n$ matrices, then

$$G_{\mathbb{R}}(\Omega, d) = \left\{ L \in \mathfrak{M}_{\mathbb{R}}^{n\times n} \mid L^2 = L,\ L^t = L, \text{ and } \operatorname{trace} L = d \right\}. \tag{2.1}$$

For sufficiently large Ω, one may use $G_{\mathbb{R}}(\Omega, d)$ as a classifying space for G–vector bundles over G–CW complexes, i.e., isomorphism classes of G–vector bundles are in 1–1 correspondence with equivariant homotopy classes of maps to $G_{\mathbb{R}}(\Omega, d)$, see [23]. How large Ω needs to be, i.e., with which multiplicity each irreducible representation χ of G needs to occur as a summand of Ω, depends on the dimension of the base space for the bundle and its fibre. Still, if Ω is a summand of Ω' and Ω is sufficiently large for a specific situation, then so is Ω'. One crude estimate is that

$$\text{multiplicity of } \chi \text{ in } \Omega > \dim base + \dim fibre + 1.$$

Definition 2.5. A vector bundle is said to be *strongly algebraic* if its classifying map is entire rational, up to homotopy. At times we identify bundles with their classifying maps. Then *strongly algebraic bundle* just means *homotopic to an entire rational map.*

To simultaniously classify k G–vector bundles of dimensions $d_1, \ldots, d_k$ we set

$$\mathfrak{G} = G_{\mathbb{R}}(\Omega_1, d_1) \times \cdots \times G_{\mathbb{R}}(\Omega_k, d_k) \tag{2.2}$$

Convention 2.6. We say that $\mathfrak{G}$ is *sufficiently large* if each of its factors $G_{\mathbb{R}}(\Omega_j, d_j)$ classifies G–vector bundles of dimension d_j in a situation usually understood from context.

For notational as well as computational reasons, it is sometimes convenient to take Ω as a universe, e.g., a direct limit of an increasing number of copies of the regular representation of G. Then $G_{\mathbb{R}}(\Omega, d)$ ceases to be a variety. In this case we still say that a map to $G_{\mathbb{R}}(\Omega, d)$ is entire rational if it factors entire rationally through $G_{\mathbb{R}}(\Omega_0, d)$ for a finite dimensional summand Ω_0 of Ω.

Consider a closed smooth G–manifold M together with equivariant vector bundles $\xi_1, \ldots \xi_k$ over M. We say that $(M, \xi_1, \ldots, \xi_k)$ has an algebraic model if there exists a nonsingular real algebraic G–variety X and an equivariant diffeomorphism $\Phi : X \to M$, so that $\Phi^*\xi_1, \ldots, \Phi^*\xi_k$ are strongly algebraic.

We call $(X, \Phi^*\xi_1, \dots, \Phi^*\xi_k)$ the algebraic model for $(M, \xi_1, \dots, \xi_k)$. Alternatively, we can ask for an algebraic model for a manifold together with a map to a sufficiently large $\mathfrak{G}$ as in (2.2).

2.4. **The Two Isotropy Group Case.** We focus on iso–special manifolds with two isotropy groups, denoted by H and K as in Definition 2.1.

Let $\mathcal{S}(G)$ be the set of all subgroups of G and $\mathcal{H} \subseteq \mathcal{S}(G)$. We say that a G–manifold M is of *type* $\mathcal{H}$ if the isotropy groups G_x belong to $\mathcal{H}$ for all $x \in M$. If we insist that the domains of representatives of bordism classes in $\mathcal{N}_r^G(Z)$ as well as bordisms in between them are of type $\mathcal{H}$, then we express this by writing $\mathcal{N}_r^G[\mathcal{H}](Z)$. We add a subscript c and write $\mathcal{N}_{r,c}^G[\{H,K\}](Z)$ to indicate that the codimension of the H–fixed point set in the K–fixed point is one. In Section 3 we will deduce the two isotropy group case of Theorem 2.3 from

Proposition 2.7. *Let G, H, K, and $\mathfrak{G}$ be as above, and $\mathfrak{G}$ sufficiently large. Then all classses in $\mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G})$ have algebraic representatives.*

2.5. **Reducing the Bordism Problem.** Our specialized bordism group $\mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G})$ fits into a long exact Conner–Floyd sequence:

$$\begin{aligned} \cdots \longrightarrow \mathcal{N}_r^G[\{K\}](\mathfrak{G}) \xrightarrow{i_c} \mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G}) \\ \xrightarrow{j_c} \mathcal{N}_{r,c}^G[\{H,K\},\{K\}](\mathfrak{G}) \xrightarrow{\partial_c} \mathcal{N}_{r-1}^G[\{K\}](\mathfrak{G}) \longrightarrow \cdots \end{aligned} \tag{2.3}$$

According to Theorem 2.4, classes in $\mathcal{N}_r^G[\{K\}]$ have algebraic representatives. We use this to deduce Proposition 2.7 from:

Proposition 2.8. *Let G, H, K, and $\mathfrak{G}$ be as above, and $\mathfrak{G}$ sufficiently large. Then elements in the kernel of*

$$\mathcal{N}_{r,c}^G[\{H,K\},\{K\}](\mathfrak{G}) \xrightarrow{\partial_c} \mathcal{N}_{r-1}^G[\{K\}](\mathfrak{G})$$

are images of algebraically represented classes in $\mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G})$ under the map j_c.

Deduce Proposition 2.7 *from Proposition* 2.8. If $\mathcal{A} \in \mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G})$, then $j_c(\mathcal{A}) \in \ker(\partial_c)$. According to Proposition 2.8 there exists an algebraically represented class class $\mathcal{B}$ in $\mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G})$ so that $j_c(\mathcal{A}) = j_c(\mathcal{B})$. Due to the exactness of the sequence in (2.3), there is a class $\mathcal{C} \in \mathcal{N}_r^G[\{K\}](\mathfrak{G})$ so that $i_c(\mathcal{C}) = \mathcal{A} + \mathcal{B}$, and $\mathcal{C}$ as well as $i_c(\mathcal{C})$ is represented by a map so that all points in its domain have isotropy type K. Algebraic representability is compatible with sums (disjoint union), and it follows that $\mathcal{A} = \mathcal{B} + i_c(\mathcal{C})$ has an algebraic representative. $\square$

We will employ bordism theoretic methods to disect the algebraic representation problem in Proposition 2.8. At its core we find Proposition 2.9 (restated as Proposition 8.12 in Section 8 after some preparation), from which we deduce Proposition 2.8 in Section 8.

Proposition 2.9. *Let G, H, K, and $\mathfrak{G}$ be as above, $C = G/K$, and $\mathfrak{G}$ sufficiently large. Then elements in the kernel of*

$$\mathcal{N}_{r-1}(EC \times_C \mathfrak{F}^e) \xrightarrow{\partial_c} \mathcal{N}_{r-1}(EC \times_C \overline{\mathfrak{F}^e})$$

are images of algebraically represented classes in $\mathcal{N}^G_{r,c}[\{H,K\}](\overline{\mathfrak{F}^e})$ under the map $L \circ j_c$.

In Proposition 2.9, $\mathfrak{F}$ is a component of $\mathfrak{G}^H$ and $\overline{\mathfrak{F}}$ is a component of $\mathfrak{G}^K$, so that $\mathfrak{F} \subseteq \overline{\mathfrak{F}}$, while $\mathfrak{F}^e$ is an *essential* factor of $\mathfrak{F}$ and $\overline{\mathfrak{F}^e}$ is the corresponding factor of $\overline{\mathfrak{F}}$, see Sections 8.4 and 8.5. The isomorphism L is defined in Section 8.3 and $EC \times_C -$ is the Borel construction.

2.6. **The homology kernel.** There is a commutative diagram

$$\begin{array}{ccc} \mathcal{N}_*(EC \times_C \mathfrak{F}^e) & \xrightarrow{\partial_c} & \mathcal{N}_*(EC \times_C \overline{\mathfrak{F}^e}) \\ \mu\big\downarrow & & \big\downarrow\mu \\ H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2) & \xrightarrow{\Phi_*} & H_*(EC \times_C \overline{\mathfrak{F}^e}, \mathbb{Z}_2) \end{array} \tag{2.4}$$

where $\Phi = \mathrm{Id} \times_C \phi : EC \times_C \mathfrak{F}^e \to EC \times_C \overline{\mathfrak{F}^e}$ is induced by the inclusion

$$\phi : \mathfrak{F}^e \hookrightarrow \overline{\mathfrak{F}^e}. \tag{2.5}$$

The vertical maps μ in (2.4) are Thom homomorphisms. They are functorial epimorphism, see [7, p.14]. In (9.3) we will recover the kernel of ∂_c from the kernel of Φ_*. In Theorem 11.1 we compute $\ker(\Phi_*)$ from its dual, the cokernel of Φ^*. The understanding of $\ker(\Phi_*)$ combined with classical results from [19] and [5] will allow us to reduce Proposition 2.9 to a special case of its homological reformulation, Proposition 2.10, see Section 13.

In the following proposition $BO(2)$ has a $\mathbb{Z}_4$ action, so that $\mathbb{Z}_2 \subset \mathbb{Z}_4$ acts trivially and $BO(2)^{\mathbb{Z}_4} = BU(1)$. We denote the inclusion of the fixed set by $\phi : BU(1) \to BO(2)$. After applying the Borel construction we obtain

$$\Phi = \mathrm{Id} \times_C \phi : EC \times_C BU(1) \longrightarrow EC \times_C BO(2). \tag{2.6}$$

We will prove the following proposition in Section 7. The proof will borrow from [14].

Proposition 2.10. *Let $G = H = \mathbb{Z}_4$, $K = \mathbb{Z}_2$, and $C = G/K = \mathbb{Z}_2$. Elements in the kernel of*

$$\Phi_* : H_*(EC \times_C BU(1), \mathbb{Z}_2) \to H_*(EC \times_C BO(2), \mathbb{Z}_2) \tag{2.7}$$

are images of algebraically represented classes in $\mathcal{N}^G_{*,c}[\{H,K\}](BO(2))$ under the map $\mu \circ L \circ j_c$.

In Section 7, see Proposition 7.1, we write out a basis for the kernel of the map in (2.7) and prove Proposition 2.10 by showing that the elements in a modified (twisted) basis are images of algebraically represented classes in $\mathcal{N}^G_{*,c}[\{H,K\}](BO(2))$ under the map $\mu \circ L \circ j_c$. With this the proof of Theorem 1.2 will be complete.

3. Background Material from the Literature

The results in this section hold when G is a compact Lie group. The first one says that algebraic realization problems reduce to bordism problems.

Theorem 3.1. [12, Theorem C] *An equivariant map from a closed smooth G–manifold to a nonsingular real algebraic G–variety has an algebraic model if and only if its cobordism class has an algebraic representative.*

The second result reduces the strong algebraic realization problem to one for finite collections of bundles.

Proposition 3.2. [12, Proposition 2.13] *A closed smooth G–manifold M has a strongly algebraic model if and only if, for any finite collection $\{\xi_1, \dots, \xi_k\}$ of G–vector bundles over M, there is an algebraic model for $(M, \xi_1, \dots, \xi_k)$.*

Recalling the definition of an algebraic model from Subsection 2.2, this means that there is a nonsingular real algebraic G variety X and an equivariant diffeomorphism $\Phi : X \to M$ that pulls back $\xi_1, \dots, \xi_k$ back to strongly algebraic bundles over X, bundles whose classifying maps are homotopic to entire rational maps.

The proof of the proposition uses that the equivariant K-theory of M is finitely generated as a module over the representation ring $R(G)$, and that one may apply the basic constructions of direct sum $\oplus$, tensor product $\otimes$, and taking orthogonal complements $\perp$ to strongly algebraic vector bundles and obtain a strongly algebraic vector bundle as result.

Suppose that $\mathfrak{G} = G_{\mathbb{R}}(\Omega_1, d_1) \times \cdots \times G_{\mathbb{R}}(\Omega_k, d_k)$ is sufficiently large for manifolds of dimension m, see (2.2) and Convention 2.6. In view of Proposition 3.2 we use a map $\chi : M \to \mathfrak{G}$ so that χ is the product of classifying maps for a set of bundles that generate the equivariant K–theory of M and apply Theorem 3.1 to (M, χ). As a consequence we find

Proposition 3.3. *A closed smooth G–manifold M of dimension m has a strongly algebraic model if for any sufficiently large $\mathfrak{G}$ and any equivariant map $\chi : M \to \mathfrak{G}$ its bordism class $[M, \chi] \in \mathcal{N}^G_m(\mathfrak{G})$ has an algebraic model.*

We will occasionally refer to another useful observation:

Proposition 3.4. *If M is the disjoint union of a finite number of closed smooth G–manifolds and each of them has a strongly algebraic model, then so does M.*

Proof of Theorem 2.3. We like to show that iso–special G–manifolds have strongly algebraic models. According to Proposition 3.4, we may assume that M consists of a single G–component, i.e., M is the union of a component of M together with its translates under the action of G. Being iso–special, M will have one or two isotropy groups.

If M has one isotropy type, then M has a strongly algebraic model according to Theorem 2.4, and our proof is complete.

Suppose that M is iso–special and of dimension m, and that we have the isotropy types H and K, where $K \subset H$ is of index 2. Let $\mathfrak{G}$ be as above and $\chi : M \to \mathfrak{G}$ any equivariant map. According to Proposition 2.7, the bordism class of (M, χ) in $\mathcal{N}^G_{m,c}[\{H, K\}](\mathfrak{G})$ has an algebraic representative. Proposition 3.3 implies that M has a strongly algebraic model. So, our proof is complete also in this second case. $\square$

For the next result, recall the construction of the projective bundle. Let $\xi = (E \to B)$ be a G–vector bundle with classifying map $\beta : B \to G_{\mathbb{R}}(\Xi, k)$. The total space of the associated projective bundle is:

$$\mathbb{R}P(\xi) = \big\{(x, T) \in B \times G_{\mathbb{R}}(\Xi, 1) \mid (\mathrm{Id} - \beta(x))T = 0\big\}. \tag{3.1}$$

The definition of $\mathbb{R}P(\xi)$ is the familiar one, with subspaces being replaced by orthogonal projection onto these spaces. The equation $(\mathrm{Id} - \beta(x))T = 0$ expresses that $\mathrm{im}\, \beta(x) \supseteq \mathrm{im}\, T$. The line onto which T projects is contained in the space that $\beta(x)$ projects onto.

Proposition 3.5. [9, Proposition 5.2] *If ξ is a strongly algebraic G–vector bundle, then its projectivization $\mathbb{R}P(\xi)$ is a projective G–fibre bundle with a real algebraic G-variety as total space and an entire rational projection. If the base space B is nonsingular, then so is the total space $\mathbb{R}P(\xi)$ of the bundle.*

4. Projectivization and Blow-ups

Let G be a compact Lie group, M a closed smooth G manifold, and N a G–submanifold of M. We denote the blow–up of M along N by $B(M, N)$. The principal conclusion of this section is:

Proposition 4.1. *If N and $B(M, N)$ have strongly algebraic models, then so does M.*

We recall the construction of a blow–up. Let M be a closed smooth G–manifold with a collection $\xi_1, \ldots, \xi_k$ of G–vector bundles over it. Let N be a G–invariant submanifold of M with normal bundle ν, and let $\underline{\mathbb{R}}$ denote the product bundle with fibre $\mathbb{R}$. We obtain bundles $\overline{\xi_1}, \ldots, \overline{\xi_k}$ over $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}})$ by first restricting $\xi_1, \ldots \xi_k$ over N and then pulling back the restricted bundles over $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}})$.

We may identify $(M, \xi_1, \ldots, \xi_k)$ and $(\mathbb{R}P(\nu \oplus \underline{\mathbb{R}}), \overline{\xi_1}, \ldots, \overline{\xi_k})$ along a neighbourhood of N that is contained in M and $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}})$. The result is commonly called the *blow-up* of $(M, \xi_1, \ldots, \xi_k)$ along N. It is denoted by $B((M, \xi_1, \ldots, \xi_k), N)$. By construction,

$$B((M, \xi_1, \ldots, \xi_k), N) \sim (M, \xi_1, \ldots, \xi_k) \sqcup (\mathbb{R}P(\nu \oplus \underline{\mathbb{R}}), \overline{\xi_1}, \ldots, \overline{\xi_k}), \tag{4.1}$$

where $\sim$ indicates a cobordism.

Proposition 4.2. *Let $B(M, \xi_1, \ldots, \xi_k), N)$ and N be as above. If N has a strongly algebraic model, then there is an algebraic model for $(M, \xi_1, \ldots, \xi_k)$ if and only if there is an algebraic model for $B((M, \xi_1, \ldots, \xi_k), N)$.*

Proof. We identify N with its strongly algebraic model. Proposition 3.5 tells us that $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}})$ has an algebraic model and that the projection map $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}}) \to N$ is entire rational. This means that the bundles that we denoted by $\overline{\xi_1}, \dots, \overline{\xi_k}$ are strongly algebraic. With this, the assertion of the proposition is a consequence of the cobordism relation in (4.1) and Theorem 3.1. □

Proof of Proposition 4.1. Consider a collection $\xi_1, \dots, \xi_k$ of G–vector bundles over M. Denote the corresponding collection of bundles over $B(M, N)$ by $\xi_1', \dots, \xi_k'$. According to our assumption, $B(M, N)$ has a strongly algebraic model, and hence $(B(M, N), \xi_1', \dots, \xi_k')$ has an algebraic model. Proposition 4.2 tells us that $(M, \xi_1, \dots, \xi_k)$ has an algebraic model. The argument works for every finite collection $\xi_1, \dots, \xi_k$ of G–vector bundles over M so that our assertion follows from Proposition 3.2. □

For reference purposes we state:

Proposition 4.3. *Let H be a subgroup of a cyclic group G. If there is a one–dimensional non–trivial representation of H, then we denote it by $\mathbb{R}_-$ and its kernel by K.*

1. *If $\mathbb{R}_-$ does not occur in the normal fibre to any component of M^H in M, then $B(M, M^H)^H = \emptyset$.*
2. *Otherwise $B(M, M^H)^H \neq \emptyset$, and the codimension of $B(M, M^H)^H$ in $B(M, M^H)^K$ is 1.*

5. Simplifying Isotropy Structures and the Proof of Proposition 2.2

In this section we simplify isotropy structures via blow-ups. We will generalize the concept of being iso–special to finite abelian group in Definition 5.2.

Proposition 5.1. *Let G be a finite abelian group and M a closed smooth G manifold. There exists a finite sequence of equivariant blow-ups*

$$M_0 = M,\ M_1 = B(M_0, A_0),\ \dots,\ M_k = B(M_{k-1}, A_{k-1}) \tag{5.1}$$

so that M_k is iso–special. If G is cyclic, then the manifolds A_i may be chosen to be iso–special, $0 \leq i \leq k-1$.

Wasserman proved the first assertion of the proposition, see [24]. Because it is quick and of importance to this paper, we will give a proof of the proposition if G is cyclic. Before anything else, we catch up with the

Proof of Proposition 2.2. Consider a closed smooth G–manifold M of dimension m. Inductively, assume that all manifolds of dimension strictly less than m have strongly algebraic models. If M is iso–special, then M has a strongly algebraic model. If M is not iso–special, then there is a blow–up sequence as in (5.1), so that M_k is iso–special, thus strongly algebraic. The A_i, $0 \leq i \leq k-1$, are also strongly algebraic because they are of dimension less than $\dim M$. It follows from Proposition 4.1 that M_{k-1}, …, M_1, and $M = M_0$ have strongly algebraic models. This proves our assertion. □

Definition 5.2. [24] Let G be an abelian group, M a smooth G–manifold, and M_0 a component of M with principal isotropy group $K \subseteq G$. Let x be a point in M_0, H its isotropy group, and N the component of M_0^H that contains x. We call the action on M *iso–special* (or *simple* in the language of [24]) if, for every choice of M_0 and x, H/K is a vector space over $\mathbb{Z}_2$, and

$$\dim_{\mathbb{Z}_2} H/K = \operatorname{codim}(N, M_0).$$

Proof of Proposition 5.1 *if G is cyclic.* Being iso–special is a local property, and it suffices to consider the case where M consists of a single G–component, i.e., it consists of a component and its translates under the action of G. We proceed by induction over isotropy types. Let H be an isotropy group for the action on M, so that there is no isotropy group that properly contains H. Observe that all points in M^H have isotropy type H. This means that M^H is iso–special.

Suppose that $\dim M^H = \dim M$. Then M is iso–special, and the proof is complete.

Otherwise, if $\dim M^H < \dim M$, then M^H will consist of components M_1^H, …, M_k^H, and $\dim M_j^H < \dim M$ for all $1 \le j \le k$. Denote the blow-up of M along M^H by $\overline{M} = B(M^H, M)$. The components M_j^H of M^H give rise to components $\overline{M}_j^H$ of $\overline{M}^H$. If the representation $\mathbb{R}_-$ of H (defined in Section 4) does not occur in the fibre of the normal bundle $\nu(M_j^H, M)$, then the component M_j^H will disappear in the blow–up ($\overline{M}_j^H = \emptyset$), see Proposition 4.3. If $\overline{M}^H = \emptyset$, then our inductive step is complete. We are done with H and continue the proof with $\overline{M}$ instead of M.

Suppose, there are components M_j^H of M^H, so that the irreducible representation $\mathbb{R}_-$ of H does occur in the fibre of the normal bundle $\nu(M_j^H, M)$. Then there is an index two subgroup K of H and $\operatorname{codim}(\overline{M}^H, \overline{M}^K) = 1$, and $\overline{M}^K$ is iso–special. If $\dim M = \dim M^K$, then $M = M^K$, and $\overline{M}$ is iso–special. In this case the proof is complete as well.

If $\dim M > \dim M^K$, then we blow up $\overline{M}$ along the iso–special submanifold $\overline{M}^K$ to obtain $\widehat{M} = B(\overline{M}, \overline{M}^K)$. Then $\widehat{M}^H = \emptyset$ because there are no lines that are fixed under the action of H in the fibre of the normal bundle $\nu(\overline{M}^K, \overline{M})$. Our inductive step is complete, we are done with H, and we continue the proof with $\widehat{M}$ instead of M.

This concludes the induction and completes the proof. □

6. Adjustment tool

In [12, Section 4] we constructed a bundle that will be instrumental is two places in this paper. Once we use it to adjust fibres of bundles, once we use it to twist bundles. As usual, G is a cylic group, H a subgroup, and K a subgroup of index 2 in H. Let $\rho = (E \to B)$ be a G–vector bundle classified by $\beta : B \to G_{\mathbb{R}}(\Xi, k)$. The total space of the associated projective bundle (see (3.1)) is:

$$\mathbb{R}P(E) = \{(x, T) \in B \times G_{\mathbb{R}}(\Xi, 1) \mid (\operatorname{Id} - \beta(x))T = 0\}.$$

Projection on the second factor defines a map $\widetilde{\beta} : \mathbb{R}P(E) \to G_{\mathbb{R}}(\Xi, 1)$. Use $\widetilde{\beta}$ to pull back the canonical line bundle $\gamma_{\mathbb{R}}(\Xi, 1)$ over $G_{\mathbb{R}}(\Xi, 1)$. Its total space is

$$Q(E) = \{(x, T, v) \in B \times G_{\mathbb{R}}(\Xi, 1) \times \Xi \mid (\operatorname{Id} - \beta(x))T = 0 \ \&\ Tv = v\}.$$

The result is a G–line bundle

$$\mathcal{L}(\rho) = (Q(E) \to \mathbb{R}P(E)). \tag{6.1}$$

Lemma 6.1. [12, Lemma 4.1] *If ρ is a strongly algebraic vector bundle, then so is $\mathcal{L}(\rho)$. The fibre of $\mathcal{L}(\rho)$ over a point $(x, T) \in \mathbb{R}P(E_\rho)$ is*

$$\mathcal{L}(\rho)_{(x,T)} = \widetilde{\beta}(x, T) = T \subseteq \beta_x. \tag{6.2}$$

Proof. That $\mathcal{L}(\rho)$ is strongly algebraic is true by construction. The containment in (6.2) should be understood on the space level, namely that

$$T(\Xi) \subseteq \beta_x(\Xi) = \rho_x,$$

and it holds by construction as well. □

Let us place $\mathcal{L}(\rho)$ into the context of iso–special G actions. As usual, $\underline{\mathbb{R}}$ denotes the product bundle with $\mathbb{R}$ as fibre. The following lemma holds essentially by inspection. We will use it to adjust fibres of bundles, see Subsection 8.4. It is customary to refer to B_0 as the canonical zero section and to B_∞ as the section at infinity of the projective bundle.

Lemma 6.2. *Let M be an iso–special G–manifold with isotropy groups H and K, and let $\rho = \underline{\mathbb{R}} \oplus \nu$ where $\nu = \nu(M^H, M)$ is the normal bundle of $B = M^H$ in M. Then*

$$\mathbb{R}P(E_\rho)^H = B_0 \sqcup B_\infty = \{(b, \mathbb{R} \oplus 0) \mid b \in B\} \sqcup \{(b, 0 \oplus \nu_b) \mid b \in B\}.$$

Furthermore, $\mathcal{L}(\rho)_{|B_0} = \underline{\mathbb{R}}$ and the fibre of $\mathcal{L}(\rho)$ restricted over B_∞ is the representation $\mathbb{R}_-$ of H.

In the upcoming construction $\eta = (\Gamma \to N)$ is an H vector bundle over a space N with trivial action. Let γ^s be the canonical line bundle over $\mathbb{R}P^s$. We use the trivial action of H on $\mathbb{R}P^s$. We suppose that the fibre of γ^s is the non-trivial representation $\mathbb{R}_-$ of H. Set $\rho = \gamma \oplus \underline{\mathbb{R}}$.

We have projections

$$\begin{tikzcd} & \mathbb{R}P(\pi_1^*(\rho)) \arrow[d, "\pi"'] & \\ \mathbb{R}P^s & \mathbb{R}P^s \times N \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & N \end{tikzcd} \tag{6.3}$$

We observe that

$$\mathbb{R}P(\pi_1^*(\rho))^H = B_0 \sqcup B_\infty \quad \text{with} \quad B_0 = B_\infty = \mathbb{R}P^s \times N.$$

For the restriction of $\mathcal{L}(\pi_1^*(\rho))$ over the fixed point components we have:

$$\mathcal{L}(\pi_1^*(\rho))_{|B_0} = \underline{\mathbb{R}} \qquad \& \qquad \mathcal{L}(\pi_1^*(\rho))_{|B_\infty} = \pi_1^*(\gamma^s). \tag{6.4}$$

We define a bundle over $\mathbb{R}P(\pi_1^*(\rho))$

$$\mathbb{T}^s(\eta) = \gamma \boxtimes \eta = \mathcal{L}(\pi_1^*(\rho)) \otimes (\pi_2 \circ \pi)^*(\eta) \tag{6.5}$$

Proposition 6.3. *Using the notation from above, we have an iso–special H–manifold $\mathbb{R}P(\pi_1^*(\rho))$ of type $\{H, K\}$ and a bundle $\pi^*(\pi_2^*(\eta))$ over it, so that*

(1) *If N is a non–singular real algebraic variety and η is a strongly algebraic bundle, then $\mathbb{R}P(\pi_1^*(\rho))$ is a nonsingular real algebraic H–variety and $(\pi_2 \circ \pi)^*(\eta))$ is strongly algebraic.*

(2) $\mathbb{R}P(\pi_1^*(\rho))^H = B_0 \sqcup B_\infty$ *where* $B_0 = B_\infty = \mathbb{R}P^s \times N$*. The normal bundles are*

$$\nu(B_0, \mathbb{R}P(\pi_1^*(\rho))) = \nu(B_\infty, \mathbb{R}P(\pi_1^*(\rho))) = \pi_1^*(\gamma^s).$$

(3) *The restrictions of $\rho \boxtimes \eta$ over the H–fixed set are*

$$\mathbb{T}^s(\eta)_{|B_0} = \pi_2^*(\eta) \quad \textit{and} \quad \mathbb{T}^s(\eta)_{|B_1} = \pi_1^*(\gamma^s) \otimes \pi_2^*(\eta).$$

Proof. All assertions follow straight forward from the construction. One may note that we are using $\mathbb{R}P^s$ with its natural algebraic structure. This makes the canonical line bundle over $\mathbb{R}P^s$ strongly algebraic. □

7. Proof of Proposition 2.10

In this section we prove Proposition 2.10. We deduce it from Proposition 7.1, which is also proved in this section. Throughout this section we use the setting for Proposition 2.10, where $C = \mathbb{Z}_2$ and C acts on $BO(2)$ with fixed point set $BU(1)$. The map $\Phi : EC \times_C BU(1) \to EC \times_C BO(2)$ is the one induced by the inclusion $BU(1) \hookrightarrow BO(2)$, see (2.6).

Consider the canonical line bundles γ over $\mathbb{R}P^a$ and η over $\mathbb{C}P^b$ and the projections π_1 and π_2 from $\mathbb{R}P^a \times \mathbb{C}P^b$ onto $\mathbb{R}P^a$ and $\mathbb{C}P^b$. Let

$$\kappa_0(a, b) : \mathbb{R}P^a \times \mathbb{C}P^b \longrightarrow B\mathbb{Z}_2 \times BU(1) \simeq EC \times_C BU(1)$$

be the product of the classifying maps of $\pi_1^*(\gamma)$ and $\pi_2^*(\eta)$ and

$$\kappa(a, b) : \mathbb{R}P^a \times \mathbb{C}P^b \longrightarrow B\mathbb{Z}_2 \times BU(1) \tag{7.1}$$

be the product of the classifying maps of the bundle $\pi_1^*(\gamma)$ and $\pi_1^*(\gamma) \otimes \pi_2^*(\eta)$. We will use the abbreviations

$$\kappa_0[a, b] = \kappa_0(a, b)_*[\mathbb{R}P^a \times \mathbb{C}P^b] \quad \& \quad \kappa[a, b] = \kappa(a, b)_*[\mathbb{R}P^a \times \mathbb{C}P^b] \tag{7.2}$$

for the images of the fundamental classes under the induced maps in $\mathbb{Z}_2$ homology. We have

Proposition 7.1. *The set* $\mathfrak{B} = \{\kappa_0[a, 2n+1] \mid a, n \geq 0\}$ *is a vector space basis of* $\ker\left[\Phi_* : H_*(EC \times_C BU(1), \mathbb{Z}_2) \to H_*(EC \times_C BO(2), \mathbb{Z}_2)\right]$*, and the maps*

$$\kappa_0(a, 2n+1) : \mathbb{R}P^a \times \mathbb{C}P^{2n+1} \longrightarrow B\mathbb{Z}_2 \times BU(1) \simeq EC \times_C BU(1)$$

are Steenrod representatives of the elements in $\mathfrak{B}$*. They also form a set of* $\mathcal{N}_*$*–generators of the kernel of*

$$\mathcal{N}_*(EC \times_C BU(1)) \xrightarrow{\partial_c} \mathcal{N}_*(EC \times_C BO(2)). \tag{7.3}$$

Proof. With $\mathbb{Z}_2$ cofficients:

$$H^*(EC \times_C BU(1)) \cong H^*(BC) \otimes H^*(BU(1)) \cong \mathbb{Z}_2[w_1] \otimes \mathbb{Z}_2[c_1],$$

where w_1 and c_1 are the first universal Stiefel–Whitney and Chern classes. The duals of the classes $\kappa[a, b]$ with a, $b \geq 0$ form a vector space basis of $H^*(BC) \otimes H^*(BU(1))$.

As a special case of Theorem 11.1, we find

$$\operatorname{coker} \Phi^* \cong H^*(B\mathbb{Z}_2, \mathbb{Z}_2) \otimes \left(\mathbb{Z}_2[c_1]/\mathbb{Z}_2[c_1^2]\right).$$

For a vector space basis we only need tensor products where the second factor is an odd power of c_1. Observe that $\ker \Phi_*$ is dual to $\operatorname{coker} \Phi^*$ and for a basis of the homology kernel of Φ_* we may use the classes $\kappa[a, b]$, where b is odd. This establishes that $\mathfrak{B}$ is a basis of $\ker \Phi_*$.

It follows from the Künneth formula ([7, p. 21], see also (9.3)) that the Steenrod representatives

$$\kappa_0(a, 2n+1) : \mathbb{R}P^a \times \mathbb{C}P^{2n+1} \longrightarrow B\mathbb{Z}_2 \times BU(1)$$

of the basis elements $\kappa[a, 2n+1]$ of $\ker \Phi_*$ form a set of $\mathcal{N}_*$ module generators of $\ker \partial_c$ in (7.3). □

Proof of Proposition 2.10. Apply the construction $\mathbb{T}$ in (6.5) with $\rho = \gamma^a \oplus \underline{\mathbb{R}}$ and $\eta = \eta^b$, where γ^a is the canonical line bundle over $\mathbb{R}P^a$ and η^b is the canonical line bundle over $\mathbb{C}P^b$. The result is a strongly algebraic vector bundle $\mathbb{T}^a(\eta^b)$. Its classifying map is an algebraic representative of a class $\mathcal{A}$ in the bordism group $\mathcal{N}^{\mathbb{Z}_4}_{a+2b+1,c}[\{\mathbb{Z}_4, \mathbb{Z}_2\}](BO(2))$. As we calculated in Proposition 6.3 (3):

$$(\mu \circ L \circ j_c)(\mathcal{A}) = \kappa_0[a, b] + \kappa[a, b].$$

Our next formula appears in [14, p. 65], and it holds if b is even. We will supply a proof below. As usual $\binom{\cdot}{\cdot}$ denotes the binomial coefficients and δ the Kronnecker symbol.

$$\kappa_0[a-2, b+1] = \kappa[a, b] + \kappa_0[a, b] + \sum_{n \geq 0} \sum_{j=2}^{n} \binom{n}{j} \delta_{b,n-j} \kappa_0[a-2j, b+j]. \tag{7.4}$$

According to Proposition 7.1, to show Proposition 2.10, it suffices to find algebraically represented classes in $\mathcal{N}^G_{*,c}\{[H, K]\}(BO(2))$ that map to

$\kappa_0(*, 2n+1)$ under the map $\mu \circ L \circ j_c$. To do so, we use the first coordinate to grade the classes $\kappa_0[*,*]$ and proceed by induction.

In the base cases ($a = 2$ and $a = 3$) the double sum in (7.4) is empty and $\kappa_0[a-2, b+1] = \kappa[a,b] + \kappa_0[a,b] = (\mu \circ L \circ j_c)(\mathcal{A})$, where $\mathcal{A}$ has the classifying map of $\mathbb{T}^a(\eta^b)$ as an algebraic representative.

In the inductive step we treat the summand $\kappa[a,b] + \kappa_0[a,b]$ as in the base case and in addition we use that the terms in the double sum are of lower grading than the term on the left hand side in (7.4). More precisely, the inductive assumption applies to terms $\kappa_0[a-2j, b+j]$, where j, and with this $b+j$, is odd. Treating (7.4) as an equation in $\ker(\Phi_*)$, we may ignore summands for even values of j. If the factor $\mathbb{C}P^{b+j}$ has an even exponent, then it corresponds under duality to a term with only even powers of c_1 in $H^*(EC \times_C BU(1), \mathbb{Z}_2)$. These are the classes that divided out when we formed $\operatorname{coker}(\Phi^*) \cong \ker(\Phi_*)$. $\square$

Proof of (7.4) *according to* [14]. As before, we denote the canonical real and complex line bundles by γ and η, the first Stiefel–Whitney class by w_1, the first Chern class modulo 2 by c_1, and the projection from $BO(1) \times BU(1)$ onto the first and second factor by π_1 and π_2.

Setting $w_1 = \pi_1^* w_1(\gamma)$ and $c_1 = \pi_2^* c_1(\eta)$, there is a natural identification

$$H^*(BO(1) \times BU(1), \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, c_1]. \tag{7.5}$$

Using the formalism in [16, p. 87]) and Proposition 6.3 (3), we find

$$w\left((\mathbb{T}^a(\eta))_{|\mathbb{R}P^a \times \mathbb{C}P^b} \right) = w(\pi_1^*(\gamma) \times \pi_2^*(\eta)) = 1 + w_1^2 + c_1.$$

As in (7.1), $\kappa : \mathbb{R}P^a \times \mathbb{C}P^b \to BO(1) \times BU(1)$ classifies $\mathbb{T}^a(\eta)$ and we calculate:

$$\begin{aligned}
\left\langle w_1^{a'} c_1^{b'}, \kappa[a,b] \right\rangle &= \left\langle [\kappa^* w_1]^{a'} [\kappa^* c_1]^{b'}, [\mathbb{R}P^a] \otimes [\mathbb{C}P^b] \right\rangle \\
&= \left\langle [\pi_1^* w_1(\gamma)]^{a'} [\pi_1^* w_1^2(\gamma) + \pi_2^* c_1(\eta)]^{b'}, [\mathbb{R}P^a] \otimes [\mathbb{C}P^b] \right\rangle \\
&= \sum_{j=0}^{b'} \binom{b'}{j} \left\langle \pi_1^* w_1^{a'+2j}(\gamma)\ \pi_2^* c_1^{b'-j}(\eta), [\mathbb{R}P^a] \otimes [\mathbb{C}P^b] \right\rangle \\
&= \delta_{a,a'} \delta_{b,b'} + \sum_{j=1}^{b'} \binom{b'}{j} \delta_{a,a'+2j} \delta_{b,b'-j}
\end{aligned}$$

If b is even, then $\binom{b+1}{1} = b+1 \equiv 1 \pmod 2$, and we find

$$\left\langle w_,^{a'} c_1^{b'}, \kappa[a,b] \right\rangle = \delta_{a,a'}\, \delta_{b,b'} + \delta_{a,a'+2}\, \delta_{b,b'-1} + \sum_{j=2}^{b'} \binom{b'}{j} \delta_{a,a'+2j}\, \delta_{b,b'-j}. \tag{7.6}$$

We may write the twisted class $\kappa[a,b]$ as a linear combination of non–twisted classes $\kappa_0[\cdot,\cdot]$:

$$\kappa[a,b] = \sum_{n\geq 0} C_n\, \kappa_0[a+2b-2n, n].$$

We use that $\kappa_0[a',b']$ is dual to $w_1^{a'} c_1^{b'}$ and set $a' = a+2b-2n$ and $b' = n$ in (7.6) to calculate the coefficients C_n. In the last equation we note that the second factor δ is 1 if and only if the first one is.

$$\begin{aligned} C_n &= \left\langle w_1^{a+2b-2n}\, c_1^n, \kappa[a,b]\right\rangle \\ &= \delta_{a,a+2b-2n}\,\delta_{b,n} + \delta_{a,a+2b-2n+2}\,\delta_{b,n-1} + \sum_{j=2}^{n} \binom{n}{j} \delta_{a,a+2b-2n+2j}\,\delta_{b,n-j} \\ &= \delta_{b,n} + \delta_{b,n-1} + \sum_{j=2}^{n} \binom{n}{j} \delta_{b,n-j}. \end{aligned}$$

In particular, $C_b = C_{b+1} = 1$. For other values of n, C_n will be the sum in our last expression for C_n. In the following computation, the first equality is due to the definition of C_n and the second one due to the our expression for C_n. To justify the last equality, we observe that $\delta_{b,n-j} = 1$ if and only if $b = n-j$. This allows us to replace $\kappa_0[a+2b-2n,n]$ by $\kappa_0[a-2j,b+j]$.

$$\begin{aligned} \kappa[a,b] &= \sum_{n\geq 0} C_n \kappa_0[a+2b-2n,n] \\ &= \kappa_0[a,b] + \kappa_0[a-2,b+1] \\ &\qquad + \sum_{n\geq 0}\sum_{j=2}^{n} \binom{n}{j} \delta_{b,n-j}\kappa_0[a+2b-2n,n] \\ &= \kappa_0[a,b] + \kappa_0[a-2,b+1] \\ &\qquad + \sum_{n\geq 0}\sum_{j=2}^{n} \binom{n}{j} \delta_{b,n-j}\kappa_0[a-2j,b+j]. \end{aligned}$$

After moving two terms from one to the other side of the equation, we obtain the formula in (7.4). □

8. Deduce Proposition 2.8 from Proposition 2.9

After a fair amount of preparation we will give the precise formulation of Proposition 2.9, see Proposition 8.12, and apply it to prove Proposition 2.8. Throughout, $K \subset H \subseteq G$, and K is of index 2 in H. We set $C = G/K$ and suppose that $\mathfrak{G}$ is as in (2.2).

8.1. **Bordism theoretic reductions.** The following diagram will come in handy. The decoration of the bordism groups spefies the permissable isotropy groups on domains, and the c is the subscript is a reminder of the codimension 1 assumption is the iso–special settings, see Subsection 2.4.

The exact sequence in the first row decomposes as a direct sum of exact sequences as in the second row, with one summand for each component $\overline{\overline{\mathfrak{F}}}$ of $\mathfrak{G}^K$. In the middle term we restricted the codomain to $(\overline{\overline{\mathfrak{F}}})^H = \overline{\overline{\mathfrak{F}}} \cap \mathfrak{G}^H$, the H–fixed points. This is possible because we may assume that the domains of representatives contract to the H–fixed point set and that maps factor through the H–fixed point set, up to equivariant homotopy. The component $\mathfrak{F}$ of $\overline{\overline{\mathfrak{F}}} \cap \mathfrak{G}^H$ is as in Proposition 8.3. The map s is a retraction defined in Proposition 8.5. The isomorphism L is defined in Subsection 8.3. In the transition to the second last row we divide out the inessential part of the action. In the last transition we apply the Borel construction.

$$
\begin{tikzcd}
\mathcal{N}^G_{r,c}[\{H,K\}](\mathfrak{G}) \arrow[r, "j_c"] & \mathcal{N}^G_{r,c}[\{H,K\},\{K\}](\mathfrak{G}) \arrow[r, "\partial_c"] & \mathcal{N}^G_{r-1}[\{K\}](\mathfrak{G}) \\
\mathcal{N}^G_{r,c}[\{H,K\}](\overline{\overline{\mathfrak{F}}}) \arrow[u, "\subseteq"'] \arrow[r, "j_c"] & \mathcal{N}^G_{r,c}[\{H,K\},\{K\}](\overline{\overline{\mathfrak{F}}}^H) \arrow[u, "\subseteq"'] \arrow[r, "\partial_c"] \arrow[d, "s"'] & \mathcal{N}^G_{r-1}[\{K\}](\overline{\overline{\mathfrak{F}}}) \arrow[u, "\subseteq"'] \arrow[d, "="] \\
 & \mathcal{N}^G_{r,c}[\{H,K\},\{K\}](\mathfrak{F}) \arrow[r, "\partial_{c,}"] \arrow[d, "L"', "\cong"] & \mathcal{N}^G_{r-1}[\{K\}](\overline{\overline{\mathfrak{F}}}) \arrow[d, "="] \\
 & \mathcal{N}^G_{r-1}[\{K\}](\mathfrak{F}) \arrow[r, "\partial_c"] \arrow[d, "\cong"'] & \mathcal{N}^G_{r-1}[\{K\}](\overline{\overline{\mathfrak{F}}}) \arrow[d, "\cong"'] \\
 & \mathcal{N}^C_{r-1}[\text{free}](\mathfrak{F}) \arrow[r, "\partial_c"] \arrow[d, "\cong"'] & \mathcal{N}^C_{r-1}[\text{free}](\overline{\overline{\mathfrak{F}}}) \arrow[d, "\cong"'] \\
 & \mathcal{N}_{r-1}(EC \times_C \mathfrak{F}) \arrow[r, "\partial_c"] & \mathcal{N}_{r-1}(EC \times_C \overline{\overline{\mathfrak{F}}})
\end{tikzcd}
$$

8.2. **The components of $\mathfrak{G}^H$.** We recall results from [9, §10]. Given a representation Ω of G and a representation V of $L \subseteq G$, we define

$$G_{\mathbb{R}}(\Omega, V) = \{U \in G_{\mathbb{R}}(\Omega, d) \mid U \cong V\}, \tag{8.1}$$

where $\cong$ means isomorphic as representations of L. In [9, §10] we spelled out polynomial equations that describe $G_{\mathbb{R}}(\Omega, V)$ as a real algebraic variety. There is a one–to–one correspondence between the representations of L that are summands of $\mathrm{Res}_L\,\Omega$ and the components of $G_{\mathbb{R}}(\Omega, d)^L$. We can decompose the components $G_{\mathbb{R}}(\Omega, d)^L$ as products. Let $\mathcal{E}$ be an index set for the irreducible representations α_ϵ of L, $\epsilon \in \mathcal{E}$. Express V as a sum of multiples of irreducible representations $V = \sum a_\epsilon \alpha_\epsilon$, with $\epsilon \in \mathcal{E}$. Let Ω_ϵ be the summand of Ω that restricts to a multiple of ϵ. Then

$$G_{\mathbb{R}}(\Omega, V) = \prod_{\epsilon \in \mathcal{E}} G_{\mathbb{R}}(\Omega_\epsilon, a_\epsilon \alpha_\epsilon).$$

More generally, let $\mathfrak{G} = G_{\mathbb{R}}(\Omega_1, d_1) \times \cdots \times G_{\mathbb{R}}(\Omega_k, d_k)$ be a product of Grassmannians as in (2.2) and $V_1, \ldots, V_k$ of representations of L with $\dim V_j = d_j$. Define coefficients $a_{j\epsilon}$ by writing each representation V_j as a sum of multiple of the irreducible representations of L:

$$V_j = \sum_{\epsilon \in \mathcal{E}} a_{j\epsilon} \alpha_\epsilon. \tag{8.2}$$

The collection of the V_j's determines a component $\mathfrak{F}$ of $\mathfrak{G}^H$:

$$\mathfrak{F} = \prod_{j=1}^{k} G_{\mathbb{R}}(\Omega_j, V_j) = \prod_{j=1}^{k} \prod_{\epsilon \in \mathcal{E}} G_{\mathbb{R}}(\Omega_{j,\epsilon}, a_{j\epsilon}\alpha_\epsilon). \tag{8.3}$$

We will suppress the dependence of $\mathfrak{F}$ on the coefficients $\{a_{j\epsilon}\}$.

8.3. **The isomorphism L.** We explain the construction of the map

$$L : \mathcal{N}^G_{r,c}[\{H, K\}, \{K\}](\mathfrak{F}) \longrightarrow \mathcal{N}^G_{r-1}[\{K\}](\mathfrak{F}).$$

The domain of $f : M \to \mathfrak{F}$, representing a class $\mathcal{A} \in \mathcal{N}^G_{r,c}[\{H, K\}, \{K\}](\mathfrak{F})$, may be restricted to the closed G invariant unit disk bundle in a tubular neighbourhood $\nu = \nu(M^H, M)$ of the H–fixed point set. The restriction of f to its unit sphere bundle $f_| : S(\nu) \to \mathfrak{F}$ is a representative of $L(\mathcal{A})$.

Proposition 8.1. *The map L is an isomorphism.*

Proof. We defined L on representatives of bordism classes. It is easy to see that L defines a map of bordism classes and that it is homomorphism.

We construct the inverse of L. The domain of a representative $f : N \to \mathfrak{F}$ of a class in $\mathcal{N}^G_{r-1}[\{K\}](\mathfrak{F})$ has a free involution, the action of $H/K = \mathbb{Z}_2$. Because H acts trivially on the codomain $\mathfrak{F}$ of f, we see that f factors through $N/\mathbb{Z}_2$ and extends to a map F_π over the mapping cylinder M_π of the quotient map $\pi : N \to N/\mathbb{Z}_2$. As representative of $L^{-1}[f : N \to \mathfrak{F}]$ we use the class of $F_\pi : M_\pi \to \mathfrak{F}$.

Let ν be a line bundle over B and $\pi : S(\nu) \to B$ the associated unit sphere bundle. Denote the mapping cylinder of π by M_π. Then $D(\nu) = M_\pi$, and it follows that the constructions for L and L^{-1} are inverses of each other. □

8.4. **Preferred fibres.** We will standarize the fibres of bundles.

Choice 8.2. Given two irreducible representations α and β of H. We say that $\alpha \sim_K \beta$ if $\operatorname{Res}_K \alpha \cong \operatorname{Res}_K \beta$. We pick one irreducible representation of H from each $\sim_K$ equivalence class and call it *preferred*.

We apply the adjective *"preferred"* to representations if they are sums of preferred irreducible representations, and to bundles if their fibres are preferred representations. If V is a preferred representation of H, then we call $G_{\mathbb{R}}(\Omega, V)$ preferred. This idea generalizes to components of $\mathfrak{G}^H$, which are products of spaces of the form $G_{\mathbb{R}}(\Omega, V)$, see (8.3).

There are two observations of interest to us. The first one is true by choice, the second one is easily checked by inspection.

Proposition 8.3. *Each component $\overline{\mathfrak{F}}$ of $\mathfrak{G}^K$ contains exactly one preferred component $\mathfrak{F}$ of $\mathfrak{G}^H$.*

Proposition 8.4. *Let α and β be irreducible representations of H, such that $\alpha \sim_K \beta$, and let $\mathbb{R}_-$ be the nontrivial irreducible representation of H. The either $\alpha \cong \beta$ or $\alpha \cong \beta \otimes \mathbb{R}_-$.*

Our next result shows that we can force bundle fibres to be preferred, module classes that come from algebraically represented ones. We make reference to the middle two rows from our diagram in the beginning of the section.

$$\begin{array}{ccccc}
\mathcal{N}^G_{r,c}[\{H,K\}](\overline{\mathfrak{F}}) & \xrightarrow{j_c} & \mathcal{N}^G_{r,c}[\{H,K\},\{K\}](\overline{\mathfrak{F}} \cap \mathfrak{G}^H) & \xrightarrow{\partial_c} & \mathcal{N}^G_{r-1}[\{K\}](\overline{\mathfrak{F}}) \\
 & & {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle =} \\
 & & \mathcal{N}^G_{r,c}[\{H,K\},\{K\}](\mathfrak{F}) & \xrightarrow{\partial_{c,\mathrm{prf}}} & \mathcal{N}^G_{r-1}[\{K\}](\overline{\mathfrak{F}}).
\end{array}$$

Temporarily, we expanded the subscript of ∂ so that we may distinguish the two connecting homomorphisms in this diagram.

Proposition 8.5. *Let $\mathfrak{F}$ be the preferred component of $\mathfrak{G}^H$ contained in $\overline{\mathfrak{F}} \cap \mathfrak{G}^H$. Then there is a retraction*

$$s : \mathcal{N}^G_{r,c}[\{H,K\},\{K\}](\overline{\mathfrak{F}} \cap \mathfrak{G}^H) \to \mathcal{N}^G_{r,c}[\{H,K\},\{K\}](\mathfrak{F})$$

and elements in the kernel of $(s - \mathrm{Id})$ are images, under j, of algebraically represented classes in $\mathcal{N}^G_{r,c}[\{H,K\}](\overline{\mathfrak{F}})$.

Corollary 8.6. *Classes in the kernel of ∂_c are images of algebraically represented classes in $\mathcal{N}^G_{r,c}[\{H,K\}](\overline{\mathfrak{F}})$ under the map j_c if and only if classes in the kernel of $\partial_{c,\mathrm{prf}}$ are images of algebraically represented classes in $\mathcal{N}^G_{r,c}[\{H,K\}](\overline{\mathfrak{F}})$ under the map $s \circ j_c$.*

Proof of Proposition 8.5. Consider $[M,f] \in \mathcal{N}^G_{r,c}[\{H,K\},\{K\}](\overline{\mathfrak{F}} \cap \mathfrak{G}^H)$. We will construct an algebraic representative (P,F) of a class in $\mathcal{N}^G_{r,c}[\{H,K\}](\overline{\mathfrak{F}})$, such that its restriction to a neighbourhood of the H–fixed point set is of the form $(M,f) \sqcup (M'f')$, and $f' : M' \to \mathfrak{F}$ maps to the preferred component $\mathfrak{F}$. We set

$$s[f : M \to \overline{\mathfrak{F}} \cap \mathfrak{G}^H] = [f' : M' \to \mathfrak{F}].$$

It will be obvious from the construction that s is linear and that s is the identity on $\mathcal{N}^G_{r,c}[\{H,K\},\{K\}](\mathfrak{F})$.

Set $B = M^H$ and let $\chi : B \to BO(1)$ be a classifying map for the normal bundle ν of $B = M^H$ in M. Set

$$P = \mathbb{R}P(\underline{\mathbb{R}} \oplus \nu).$$

The map $f_{|B} : B \to \overline{\mathfrak{F}} \cap \mathfrak{G}^H \subseteq \mathfrak{G}^H$ classifies bundles $\xi_1, \ldots, \xi_k$ over B. As a manifold with exactly one orbit type, B has a strongly algebraic model, see Theorem 2.4. We may assume that B is a real algebraic variety and χ and $f_{|B}$ are entire rational. Each of the bundles decomposes as a direct sum $\xi_j = \bigoplus \xi_{j,\epsilon}$. The sum ranges over the index set $\mathcal{E}$ for the irreducible representations α_ϵ of H and the fibre of $\xi_{j,\epsilon}$ is a multiple of α_ϵ.

We may, and we will, assume that P is a nonsingular real algebraic G–variety, and that the projection $\pi : P \to B$ is entire rational, see Proposition 3.5. Set

$$\begin{aligned} \overline{\xi}_{j,\epsilon} &= \begin{cases} \pi^*(\xi_{j,\epsilon}) & \text{if } \epsilon \text{ is preferred} \\ \pi^*(\xi_{j,\epsilon}) \otimes \mathcal{L}(\underline{\mathbb{R}} \oplus \nu) & \text{if } \epsilon \text{ is not preferred} \end{cases} \\ \overline{\xi}_j &= \oplus \overline{\xi}_{j,\epsilon} \end{aligned}$$

where $\mathcal{L}$ is as in Section 6. Note also that the tensor product of strongly algebraic bundles is strongly algebraic, see [12, Proposition 2.11]. As assumed, π is entire rational, so that $\overline{\xi}_j$ and $\overline{\xi}_{j,\epsilon}$ are strongly algebraic. Let $F : P \to \overline{\mathfrak{F}} \cap \mathfrak{G}^H$ be the classifying map for the collection $\overline{\xi}_1, \dots, \overline{\xi_k}$ of bundles over P. By choice F is entire rational, up to equivariant homotopy. This concludes the construction of (P, F) as it was asserted in the proof.

Recall that $P^H = B_0 \sqcup B_\infty$. As before, B_0 stands for the zero-section in the projective bundle, while B_∞ stands for the section at infinity. As we stated in Lemma 6.2, the fibre of $\mathcal{L}(\underline{\mathbb{R}} \oplus \nu)$ over B_∞ is $\mathbb{R}_-$, and tensoring with $\mathbb{R}_-$ changes a non-preferred irreducible representation into a preferred one, see Proposition 8.4. Thus, by tensoring summands with $\mathcal{L}(\underline{\mathbb{R}} \oplus \nu)$ if necessary, we assured that over B_∞ the fibres of the bundles $\overline{\xi}_j$ are preferred.

We let M' be a H-invariant tubular neighbourhood of B_∞ in P and f' the restriction of F over M'. Up to homotopy, $f' = F_{|B_\infty} \circ r$, where r retracts M' to B_∞. It follows that f' maps to $\mathfrak{F}$, up to equivariant homotopy. This concludes the proof of the lemma. □

Proof of Corollary 8.6. " $\Longleftarrow$ ": Let $f : M \to \overline{\mathfrak{F}} \cap \mathfrak{G}^H$ represent a class in $\ker(\partial_c)$, and let (P, F) and $f' : M' \to \mathfrak{F}$ be as in the proof of Proposition 8.5. Then $[M', f'] \in \ker(s \circ \partial_{c,\mathrm{prf}})$ because

$$j_c[P, F] = [M, f] + [M', f'].$$

For this direction of the proof, we assumed that we have an algebraic representative (P', F') for a class in $\mathcal{N}^G_{r,c}[\{H, K\}](\overline{\mathfrak{F}})$ that maps to (M', f') under j_c. In other words, (M', f') is a neighbourhood of the H–fixed set in (P', F'). Glue (P, F) and (P', F') together along their common part of the neighbourhood (M', f') of the the H–fixed set, calling the result $(\widetilde{P}, \widetilde{F})$:

$$(\widetilde{P}, \widetilde{F}) = (P, F) \#_{(M' f')} (P', F').$$

Both, (P, F) and (P', F') have an algebraic model, and $(\widetilde{P}, \widetilde{F})$ is cobordant to $(P, F) \sqcup (P', F')$. Thus $(\widetilde{P}, \widetilde{F})$ has an algebraic model. When we apply $j_{c,\mathrm{prf}}$ to $(\widetilde{P}, \widetilde{F})$, then we obtain (M, f).

" $\Longrightarrow$ ": Naturally, $\ker(\partial_{c,\mathrm{prf}} \circ s) \subseteq \ker(\partial_c)$. The desired implication follows from the observation that s is a retraction. □

8.5. **Essential and inessential representations.** In this section $\mathfrak{F}$ denotes a preferred component of $\mathfrak{G}^H$. To reduce the complexity of the discussion, we will distinguish an essential and inessential factor of $\mathfrak{F}$. The inessential factor will have no bearing on our argument and we will be able to split it off in (8.11). That will leave us with the essential factor.

Definition 8.7. The *essential* representation α of a cyclic group H has $\mathbb{C}$ as its underlying space, and a generator h of H acts by multiplication with $\sqrt{-1}$ on it. All other irreducible representations are called inessential.

Remark 8.8. A cyclic group H will have an essential representation only if its order is divisible by 4. For an essential representation, the action of H extends to an action of S^1.

Recall the factorization of $\mathfrak{F}$ in (8.3) with the notation developed in Subsection 8.2. We break up the factorization into blocks, the essential one $\mathfrak{F}^e$ and an inessential one $\mathfrak{F}^i$.

$$\mathfrak{F} = \mathfrak{F}^e \times \mathfrak{F}^i = \prod_{j=1}^{k} G_{\mathbb{R}}(\Omega_{j,e}, a_{j,e}\alpha_e) \times \prod_{j=1}^{k} \prod_{\epsilon \in \mathcal{E}^i} G_{\mathbb{R}}(\Omega_{j,\epsilon}, a_{j,\epsilon}\alpha_\epsilon). \tag{8.4}$$

Being cute about our notation, we used e to also denote the index for the essential irreducible representation of H. We denoted the index set for the inessential irreducible representations of H by $\mathcal{E}^i$.

Let $\overline{\mathfrak{F}}$ be the component of $\mathfrak{G}^K$ that contains $\mathfrak{F}$. There is a factorization of $\overline{\mathfrak{F}}$ that corresponds to the one of $\mathfrak{F}$:

$$\overline{\mathfrak{F}} = \prod_{j=1}^{k} G_{\mathbb{R}}(\Omega_{j,e}, a_{j,e} \operatorname{Res}_K \alpha_e) \times \prod_{j=1}^{k} \prod_{\epsilon \in \mathcal{E}^i} G_{\mathbb{R}}(\Omega_{j,\epsilon}, a_{j,\epsilon} \operatorname{Res}_K \alpha_\epsilon). \tag{8.5}$$

We call the first factor $\overline{\mathfrak{F}^e}$ and the second one $\overline{\mathfrak{F}^i}$.

An important distinction between inessential and essential irreducible representation is as follows. If α is an inessential irreducible representation of H, then α and $\operatorname{Res}_K \alpha$ are of the same type. They are either both real or both complex. On the other hand, the essential representation is of complex type, while its restriction to K is of real type.

We describe the factors in (8.4) and (8.5) in classical terms. Our principal reference is [9, Section 10], augmented with special considerations from [13] for the cyclic groups action case.

For the essential factor and the naturally induced action of G (and of S^1) we have

$$\mathfrak{F}^e = \prod_{j=1}^{k} BU(a_{j,e}) \subseteq \overline{\mathfrak{F}^e} = \prod_{j=1}^{k} BO(2a_{j,e}) \quad \& \quad \mathfrak{F}^e = (\overline{\mathfrak{F}^e})^G = (\overline{\mathfrak{F}^e})^{S^1}. \tag{8.6}$$

Remark 8.9. One may describe finite approximations of $BO(2a)$ as homogeneous spaces, and the action of S^1 on these approximations is regular, see [11].

On the other hand, for inessential irreducible representations α_ϵ of H it follows that

$$G_{\mathbb{R}}(\Omega_{j,\epsilon}, a_{j,\epsilon}\alpha_\epsilon) = G_{\mathbb{R}}(\Omega_{j,\epsilon}, a_{j,\epsilon} \operatorname{Res}_K \alpha_\epsilon). \tag{8.7}$$

The reason is that both spaces are a $BO(a_{j,\epsilon})$ or a $BU(a_{j,\epsilon})$, depending on the type of α_ϵ. It follows that

$$\mathfrak{F}^i = \overline{\mathfrak{F}^i} = \prod_{j=1}^{k} \prod_{\epsilon\in\mathcal{E}} B\Lambda(a_{j,\epsilon}), \tag{8.8}$$

where $\Lambda = O$ or $\Lambda = U$, depending on whether α_ϵ is of real or complex type. The naturally induced action of G on these spaces is trivial.

Proposition 8.10. *Finite approximations of $\mathfrak{F}^i$ have totally algebraic homology, and bordism classes in $\mathcal{N}_*(\mathfrak{F}^i)$ have algebraic representatives.*

Proof. Throughout the argument we are using finite approximations. As stated in [1], $BO(a)$ has totally algebraic homology. The algebraic representatives of the homology classes are the Schubert cells. A similar argument shows that $BU(a)$ has totally algebraic homology. As a product of $BO(a)$'s and $BU(a)$'s, $\mathfrak{F}^i$ has totally algebraic homology. According to the classical theory, classes in $\mathcal{N}_*$ are algebraically represented. It follows that classes in $\mathcal{N}_*(\mathfrak{F}^i) \cong \mathcal{N}_* \otimes H_*(\mathfrak{F}^i, \mathbb{Z}_2)$ are algebraically represented. $\square$

Classes in $\mathcal{N}_*(\mathfrak{F}^i)$ have algebraic representatives. This follows from the nonequivariant theory because $\mathfrak{F}^i$ is a product of $BO(a)$'s and $BU(a)$'s, and these spaces have totally algebraic homology. The algebraic representatives of the homology classes are Schubert cells. The argument given in [1] for the orthogonal case also holds for the unitary case.

8.6. **Stripping off inessential fibres.** We continue the discussion from Subsection 8.5. Let

$$\phi : \mathfrak{F}^e \hookrightarrow \overline{\mathfrak{F}^e} \quad \text{and} \quad \Phi : EC \times_C \mathfrak{F}^e \hookrightarrow EC \times_C \overline{\mathfrak{F}^e} \tag{8.9}$$

be the inclusion and its Borel construction. These maps specialize to the maps with the same name in (2.5). Recall that the action of G (and C) on the inessential factors is trivial, and that $\mathfrak{F}^i = \overline{\mathfrak{F}^i}$. With this we obtain an equivariant commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{F}^e & \xrightarrow{\iota} & \mathfrak{F} = \mathfrak{F}^i \times \mathfrak{F}^e & \xrightarrow{\pi} & \mathfrak{F}^e \\
{\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id} \times \phi} & & \big\downarrow \\
\overline{\mathfrak{F}^e} & \xrightarrow{\iota} & \overline{\mathfrak{F}} = \overline{\mathfrak{F}^i} \times \overline{\mathfrak{F}^e} & \xrightarrow{\pi} & \overline{\mathfrak{F}^e}
\end{array}$$

Here ι denotes an inclusion, π a projection, and $\pi \circ \iota = \mathrm{Id}$.

We apply the Borel construction and find the commutative diagram

$$\begin{array}{ccccc} EC \times_C \mathfrak{F}^e & \longrightarrow & \mathfrak{F}^i \times (EC \times_C \mathfrak{F}^e) = EC \times_C \mathfrak{F} & \longrightarrow & EC \times_C \mathfrak{F}^e \\ \downarrow & & \downarrow & & \downarrow \Phi \\ EC \times_C \overline{\mathfrak{F}^e} & \longrightarrow & \overline{\mathfrak{F}^i} \times (EC \times_C \overline{\mathfrak{F}^e}) = EC \times_C \overline{\mathfrak{F}} & \longrightarrow & EC \times_C \overline{\mathfrak{F}^e} \end{array}$$

Induced in bordism, we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{N}_*(\mathfrak{F}^i) \otimes_{\mathcal{N}_*} \mathcal{N}_*(EC \times_C \mathfrak{F}^e) & \xrightarrow{\cong} & \mathcal{N}_*(EC \times_C \mathfrak{F}) \\ \downarrow \mathrm{Id} \otimes \partial_c^e & & \partial_c \downarrow \\ \mathcal{N}_*(\overline{\mathfrak{F}^i}) \otimes_{\mathcal{N}_*} \mathcal{N}_*(EC \times_C \overline{\mathfrak{F}^e}) & \xrightarrow{\cong} & \mathcal{N}_*(EC \times_C \overline{\mathfrak{F}}). \end{array} \tag{8.10}$$

The horizontal isomorphisms are the naturally induced Künneth maps, see [7, Section 19]. The map ∂_c^e is the connecting homomorphism in the bordism sequence of the pair $EC \times_C (\overline{\mathfrak{F}^e}, \mathfrak{F}^e)$. Naturally, it is induced by the inclusion $\Phi : EC \times_C \mathfrak{F}^e \to EC \times_C \overline{\mathfrak{F}^e}$. The second vertical map is the last horizontal one in the diagram in Subsection 8.1. The tensor products are over the $\mathbb{Z}_2$–algebra $\mathcal{N}_*$. For the kernels of the vertical maps in (8.10) we obtain

$$\ker \partial_c \cong \mathcal{N}_*(\mathfrak{F}^i) \otimes_{\mathcal{N}_*} \ker \partial_c^e. \tag{8.11}$$

8.7. **Deduce Proposition 2.8 from Proposition 2.9.** Having established more precise notation, we restate Proposition 2.9. We will refer to a diagram like the one in Subsection 8.1, only that we go straight from the second to the last row, and we only consider the essential factor of $\mathfrak{F}$. For future reference (in Section 9) we add one row, where the vertical map μ is the Thom homomorphism.

$$\begin{array}{ccccc} \mathcal{N}^G_{r,c}[\{H,K\}](\overline{\mathfrak{F}}^e) & \xrightarrow{j_c^e} & \mathcal{N}^G_{r,c}[\{H,K\},\{K\}](\overline{\mathfrak{F}}^e) & \xrightarrow{\partial_c^e} & \mathcal{N}^G_{r-1}[\{K\}](\overline{\mathfrak{F}}^e) \\ & & L\circ s \downarrow & & \downarrow \\ & & \mathcal{N}_{r-1}(EC \times_C \mathfrak{F}^e) & \xrightarrow{\Phi_*} & \mathcal{N}_{r-1}(EC \times_C \overline{\mathfrak{F}}^e) \\ & & \mu \downarrow & & \downarrow \\ & & H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2) & \xrightarrow{\Phi_*} & H_*(EC \times_C \overline{\mathfrak{F}}^e, \mathbb{Z}_2) \end{array}$$

Remark 8.11. $\overline{\mathfrak{F}}^e$ has only one C–fixed component, which makes it somewhat redundant to write $L \circ s$ instead of simply L. Keeping the s may avoid some confusion.

Proposition 8.12. *Let $K \subset H \subseteq G$ be as before, and $C = G/K$. Let $\mathfrak{F}^e$ be the essential factor* (Section 8.5) *of a preferred component $\mathfrak{F}$* (Section 8.4) *of $\mathfrak{G}^H$ and let $\overline{\mathfrak{F}^e}$ be as in* (8.5) (see also (8.6))*. Any class in the kernel of*

$$\partial_c^e : \mathcal{N}_{r-1}(EC \times_C \mathfrak{F}^e) \longrightarrow \mathcal{N}_{r-1}(EC \times_C \overline{\mathfrak{F}^e})$$

is $(L \circ s \circ j_c^e)(\mathcal{A})$ *for some algebraically represented* $\mathcal{A} \in \mathcal{N}_{r,c}^G[\{H,K\}](\overline{\mathfrak{F}^e})$, *and* $j_c^e(\mathcal{A})$ *has a representative* $M \to (\overline{\mathfrak{F}^e})^H$ *factors through* $\mathfrak{F}^e$.

Remark 8.13. If 4 does not divide the order of H, then $\mathfrak{F}^e = \overline{\mathfrak{F}^e}$ is a point (see Definition 8.7 and 8.5) and Proposition 2.9 holds trivially.

Deduce Proposition 2.8 *from Proposition* 8.12. Proposition 2.8 asserts that elements in the kernel of ∂_c as in the first row of the diagram in Section 8.1 can be lifted back to algebraically represented classes in $\mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G})$. Points in the domains of representatives of classes in $\mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G})$ are in $\{H,K\}$, so that the maps factor through $\mathfrak{G}^K$. We may thus replace $\mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G})$ by $\mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G}^K)$. A decomposition of $\mathfrak{G}^K$ into components leads to a direct sum decomposition of $\mathcal{N}_{r,c}^G[\{H,K\}](\mathfrak{G}^K)$. Hence it suffices to prove Proposition 2.8 with $\mathfrak{G}$ being replaced by $\mathfrak{G}^K$. This gets us to the second row in the diagram in Subsection 8.1.

Based on Corollary 8.6, the desired assertion will follow if classes in the kernel of

$$\mathcal{N}_{r,c}^G[\{H,K\},\{K\}](\mathfrak{F}) \xrightarrow{\partial_{c,\mathrm{prf}}} \mathcal{N}_{r-1}^G[\{K\}](\overline{\mathfrak{F}})$$

can be lifted back to algebraically represented classes in $\mathcal{N}_{r,c}^G[\{H,K\}](\overline{\mathfrak{F}})$, so that the restriction to the fixed point set factors through $\mathfrak{F}$. It is only a bordism theoretic reformulation (as discussed in Subsection 8.1) to use instead elements in the kernel of

$$\mathcal{N}_{r-1}(EC \times_C \mathfrak{F}) \xrightarrow{\partial_c} \mathcal{N}_{r-1}(EC \times_C \overline{\mathfrak{F}}). \tag{8.12}$$

We compared the kernel of this map with the kernel of

$$\mathcal{N}_{r-1}(EC \times_C \mathfrak{F}^e) \xrightarrow{\partial_c^e} \mathcal{N}_{r-1}(EC \times_C \overline{\mathfrak{F}^e}) \tag{8.13}$$

and found that $\ker \partial_c \cong \mathcal{N}_*(\mathfrak{F}^i) \otimes_{\mathcal{N}_*} \ker \partial_c^e$, see (8.11). Classes in the factor $\mathcal{N}(\mathfrak{F}^i)$ have algebraic representatives, see Proposition 8.10.

A typical generator of a class in $\ker \partial_c \cong \mathcal{N}_*(\mathfrak{F}^i) \otimes_{\mathcal{N}_*} \ker \partial_c^e$ may be represented as a product

$$(f : N \to \mathfrak{F}^i \times (EC \times_C \mathfrak{F}^e)) = (f_1 : N_1 \to \mathfrak{F}^i) \times (f_2 : N_2 \to EC \times_C \mathfrak{F}^e).$$

In Proposition 8.12 we asserted that (N_2, f_2) is the image (under the map $L^e \circ s \circ j_c^e$) of an algebraically represented class $\mathcal{A} \in \mathcal{N}_{r,c}^G[\{H,K\}](\overline{\mathfrak{F}}^e)$. Denote an algebraic representative of $\mathcal{A}$ by $F : M \to \overline{\mathfrak{F}}^e$. As the action on N_1 is trivial, we see that $L^e \circ s \circ j_c^e$ maps the class of $(f_1 : N_1 \to \mathfrak{F}^i) \times (F : M \to \overline{\mathfrak{F}}^e)$ to the class of (N, f).

The cartesian product of two entire rational maps is another entire rational map. We conclude that $(F : M \to \overline{\mathfrak{F}}^e) \times (f_1 : N_1 \to \mathfrak{F}^i)$ is algebraic, up to bordism, and with this we have provided an algebraic representative of a class that maps to the one of (N, f). $\square$

9. Deduce Proposition 2.9 and 8.12 from Proposition 9.1 Reduction to homology

We will deduce Proposition 2.9 (reformulated as Proposition 8.12) from its homological variant, Proposition 9.1. This proposition will be proved in Section 13. Let $\Phi : EC \times_C \mathfrak{F}^e \to EC \times_C \overline{\mathfrak{F}^e}$ as in (8.9) and $\mu \circ L \circ s \circ j_c^e$ as in the diagram in Subsection 8.7.

Proposition 9.1. *Let $K \subset H \subseteq G$ and $C = G/K$ be as always and $\mathfrak{F}^e$ and $\overline{\mathfrak{F}^e}$ as in* (8.6)*. Any class in the kernel of*

$$\Phi_* : H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2) \longrightarrow H_*(EC \times_C \overline{\mathfrak{F}^e}, \mathbb{Z}_2)$$

is the image of an algebraically represented class in $\mathcal{N}_{r,c}^G[\{H,K\}](\overline{\mathfrak{F}^e})$ under the map $\mu \circ L \circ s \circ j_c^e$.

We recall the definition of the Thom homomorphism (see [7, p. 14]):

$$\mu : \mathcal{N}_*(EC \times_C \mathfrak{F}^e) \to H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2).$$

If $f : M \to EC \times_C \mathfrak{F}^e$ represents a class in $\mathcal{N}_n(EC \times_C \mathfrak{F}^e)$, then

$$\mu[M, f] = f_*[M] \in H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2),$$

where $[M]$ stands for the fundamental class of the manifold M. The Thom homomorphism is functorial and it commutes with the connecting homomorphism in the long exact bordism sequence of a pair.

Proof of Propositions 2.9 *&* 8.12. Set $\theta[M, f] = 1 \otimes \mu[M, f]$ and consider the diagram (a modification of the bottom square in the diagram in Subsection 8.7)

$$\begin{array}{ccc} \mathcal{N}_*(EC \times_C \mathfrak{F}^e) & \xrightarrow{\Phi_* = \partial_c^e} & \mathcal{N}_*(EC \times_C \overline{\mathfrak{F}^e}) \\ \theta \big\downarrow & & \big\downarrow \\ \mathcal{N}_* \otimes H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2) & \xrightarrow{1 \otimes \Phi_*} & \mathcal{N}_* \otimes H_*(EC \times_C \overline{\mathfrak{F}^e}, \mathbb{Z}_2) \end{array} \tag{9.1}$$

While the Thom homomorphism μ is an epimorphism, θ may not be onto. The top row in this diagram is the same as the map in Proposition 8.12.

To define a homorphism going upwards in (9.1) one needs to pick a basis of $H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2)$ consisting of homogeneous classes. If x is a chosen basis element, then one needs to represent it by a manifold M together with a map $f : M \to EC \times_C \mathfrak{F}^e$ so that $f_*[M] = x$. This is possible and called Steenrod representation. The induced map

$$\eta : \mathcal{N}_* \otimes H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2) \to \mathcal{N}_*(EC \times_C \mathfrak{F}^e) \tag{9.2}$$

is an isomorphism, see [7, p. 21]. The diagram in (9.1) with arrows going upwards does not commute due to the need of making choices. The composition of the two constructions

$$H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2) \to \mathcal{N}_*(EC \times_C \mathfrak{F}^e) \to H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2)$$

will be the identity.

In our choice for the construction of η we may start out with a basis of $\ker(\Phi_*)$ and expand it to a basis of $H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2)$. We may then lift the basis elements of $\ker(\Phi_*)$ (Steenrod representation) to classes in $\ker(\partial_c^e)$. With this choice the linear extension

$$\eta : \mathcal{N}_* \otimes \ker(\Phi_*) \xrightarrow{\cong} \ker(\partial_c^e) \tag{9.3}$$

will be an isomorphism.

With the isomorphism in (9.3) the proof of Proposition 8.12 is an immediate consequence of Proposition 9.1. □

10. Cohomology and homology of cyclic groups

Eventually we will need to understand the homology of a cyclic group from a real algebraic point of view. We recall the relevant result. Suppose that C is a cyclic group. In [19] Serre credits unnamed sources for

$$H^*(BC, \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } |C| \text{ is odd,} \\ \mathbb{Z}_2[x] & \text{if } |C| \text{ is twice an odd number,} \\ \mathbb{Z}_2[x,y]/\langle x^2 = 0\rangle & \text{if 4 divides } |C|. \end{cases} \tag{10.1}$$

Here x stands for a class in grading 1 and y for one in grading 2.

We write $\zeta^{(r)}$ for the unique nonzero class in $H^r(BC, \mathbb{Z}_2)$, and $\zeta_{(r)}$ for its dual. One may express BC as a direct limit of finite approximations B_sC, and each B_sC is a homogeneous space. As homogeneous space B_sC has a unique real algebraic structure (see [11]) which we will use thoughout. Making use of explicit constructions from [15, 3.4.7 LEMMA] we previously proved:

Proposition 10.1. [13] *For any $0 \neq \zeta_{(r)} \in H_r(BC, \mathbb{Z}_2)$ and for a sufficiently large s, there exists a nonsingular real algebraic variety Z of dimension r and an entire rational map $\kappa_r : Z_r \to B_sC$, such that $(\kappa_r)_*[Z_r] = \zeta_{(r)}$.*

Remark 10.2. If the order of C is divisible by 4, then the relation between even and odd dimensional classes is as follows. Consider the diagram for the principal C–bundle classified by the map in the bottom row:

$$\begin{array}{ccc} \widetilde{Z}_{2r} & \xrightarrow{\widetilde{\kappa}_{2r}} & E_sC \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_C} \\ Z_{2r} & \xrightarrow{\kappa_{2r}} & B_sC. \end{array} \tag{10.2}$$

View C as a subgroup of S^1. To the diagram in (10.2) we associate a diagram, which classifies the principal C–bundle in its first column:

$$\begin{array}{ccc} \widetilde{Z}_{2r+1} = S^1 \times_{\mathbb{Z}_2} \widetilde{Z}_{2r} & \xrightarrow{\widetilde{\kappa}_{2r+1}} & E_sC \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_C} \\ Z_{2r+1} = S^1/C \times Z_{2r} & \xrightarrow{\kappa_{2r+1}} & B_sC. \end{array} \tag{10.3}$$

The action of S^1 on EC induces an action of S^1/C on B_sC, and the bundle is classified by

$$\kappa_{2r+1} : (S^1/C) \times Z_{2r} \to B_sC \text{ where } \kappa_{2r+1}(t,x) = t\,\kappa_{2r}(x). \tag{10.4}$$

Furthermore, if κ_{2r} is regular, then so is κ_{2r+1}, because it is the composition of a regular map and the regular action of (S^1/C').

Eventually we will need a comparison.

Proposition 10.3. *Consider a short exact sequence of cyclic groups*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow C \longrightarrow C' \longrightarrow 1$$

where 4 *divides* $|C|$. *Then the following induced maps are isomorphisms:*

$$H_{2N}(B\mathbb{Z}_2,\mathbb{Z}_2) \to H_{2N}(BC,\mathbb{Z}_2) \quad \& \quad H_{2N+1}(BC,\mathbb{Z}_2) \to H_{2N+1}(BC',\mathbb{Z}_2).$$

11. The Cokernel of Φ^* and kernel of Φ_*

We assume that C is of even order. In preparation for the proof of Proposition 9.1 we compute the cokernel of the map

$$\Phi^* : H^*(EC \times_C \overline{\mathfrak{F}^e}, \mathbb{Z}_2) \longrightarrow H^*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2),$$

induced by $\Phi : EC \times_C \mathfrak{F}^e \to EC \times_C \overline{\mathfrak{F}^e}$ (see (8.9)), where (compare (8.6))

$$\mathfrak{F}^e = BU(a_1) \times \cdots \times BU(a_k) \;\&\; \overline{\mathfrak{F}^e} = BO(2a_1) \times \cdots \times BO(2a_k). \tag{11.1}$$

The action of C on $\mathfrak{F}^e$ is trivial, see (8.6), and $\overline{\mathfrak{F}^e}^C = \mathfrak{F}^e$. We dropped the '$e$' from the subscript of the multiplicities, writing a_i instead of $a_{i,e}$, because we are only dealing with $\mathfrak{F}^e$ and $\overline{\mathfrak{F}^e}$.

According to a Künneth formula and the computation of the cohomology of $BU(a)$ (see [5]) applied to each factor of $\mathfrak{F}^e$, and with $\mathbb{Z}_2$–coefficients throughout,

$$\begin{aligned} H^*(EC \times_C \mathfrak{F}^e) &\cong H^*(BC) \otimes H^*(\mathfrak{F}^e) \\ &\cong H^*(BC) \otimes \mathbb{Z}_2[\{c_{j,i} \mid 1 \le j \le k,\ 1 \le i \le a_j\}], \end{aligned} \tag{11.2}$$

where $c_{j,i}$ denotes the i–th Chern class modulo 2 for the j–th factor $BU(a_j)$ of $\mathfrak{F}^e$.

Theorem 11.1. *With the notation set up so far:*

$$\operatorname{coker} \Phi^* = H^*(BC,\mathbb{Z}_2) \otimes \frac{\mathbb{Z}_2[\{c_{j,i} \mid 1 \le j \le k,\ 1 \le i \le a_j\}]}{\mathbb{Z}_2[\{c_{j,i}^2 \mid 1 \le j \le k,\ 1 \le i \le a_j\}]}. \tag{11.3}$$

Proof. In [13] we calculated $H^*(EC \times_C \overline{\mathfrak{F}^e}, \mathbb{Z}_2)$ using the Leray–Serre spectral sequence of the fibration

$$\overline{\mathfrak{F}^e} \to EC \times_C \overline{\mathfrak{F}^e} \to BC \tag{11.4}$$

The E^2–term is (note the overline in reference to $\overline{\mathfrak{F}^e}$)

$$\overline{E}_2^{*,*} = H^*(BC,\mathbb{Z}_2) \otimes H^*(\overline{\mathfrak{F}^e},\mathbb{Z}_2).$$

The transgression and the differential at the E_2–level are nontrivial.

Let us look at the second factor first. We have a subring

$$\mathbb{Z}_2[\{w_{j,2i}^2 \mid 1 \le j \le k,\ 1 \le i \le a_j\}] \subseteq H^*(\overline{\mathfrak{F}^e}). \tag{11.5}$$

The $w_{j,2i}$ are the Stiefel–Whitney classes of the real classifying bundles $\gamma_{\mathbb{R}}^{2a_j}$ over $BO(2a_j)$. By definition, $\gamma_{\mathbb{R}}^{2a_j}$ pulls back to the complex classifying bundle $\gamma_{\mathbb{C}}^{a_j}$ over $BU(a_j)$. With $\mathbb{Z}_2$ coefficients we have $\phi^*(w_{j,2i}) = c_{j,i}$.

For the purpose of calculating the cokernel of Φ^* it is convenient to compare spectral sequences. In addition to the fibration in (11.4) we consider the fibration

$$\mathfrak{F}^e \to EC \times_C \mathfrak{F}^e = BC \times \mathfrak{F}^e \to BC. \tag{11.6}$$

The E_2–term of its Leray–Serre spectral sequence is

$$E_2^{*,*} = H^*(BC, \mathbb{Z}_2) \otimes H^*(\mathfrak{F}^e, \mathbb{Z}_2).$$

The fibration is a product, so that the spectral sequence collapses at the E_2–level.

On the E_2–level the inclusion induces the map

$$E_2 = H^*(BC, \mathbb{Z}_2) \otimes H^*(\mathfrak{F}^e, \mathbb{Z}_2) \xleftarrow{1\otimes\phi^*} \overline{E}_2 = H^*(BC, \mathbb{Z}_2) \otimes H^*(\overline{\mathfrak{F}^e}, \mathbb{Z}_2).$$

There is a nonzero class $\zeta^{(i)} \in H^i(BC, \mathbb{Z}_2)$ in each degree i (see (10.1)). There is a differential ∇ on $H^*(\overline{\mathfrak{F}^e}, \mathbb{Z}_2)$, and $d_2^{p,q} : \overline{E}_2^{p,q} \to \overline{E}_2^{p+2;q-1}$ is given by

$$d_2^{p,q}(\zeta^{(i)} \otimes u) = \zeta^{(i+2)} \otimes \nabla u.$$

One may conclude from [8, Corollary 1.2] that

$$\nabla u = 0 \implies \phi^*(u) \in \mathbb{Z}_2[\{c_{j,i}^2 \mid 1 \le j \le k,\ 1 \le i \le a_j\}] \tag{11.7}$$

Having the identity on the first factor of the E_2–term of the spectral sequence, we conclude that

$$\operatorname{im} \Phi^* = H^*(BC, \mathbb{Z}_2) \otimes \mathbb{Z}_2[\{c_{j,i}^2 \mid 1 \le j \le k,\ 1 \le i \le a_j\}]. \tag{11.8}$$

The asserted formula for the cokernel of Φ^* is an immediate consequence. □

12. Representing Characteristic Classes

We will describe Steenrod representatives of homology classes that are dual to a standard basis of $H^*(BU(a), \mathbb{Z}_2)$. Theorem 12.1 describes the generators that we will use in the proof of XXXX.

Consider Borel's diagonal map (shuffling the coordinates)

$$\Delta : \underbrace{BU(1) \times \cdots \times BU(1)}_{a \text{ times}} \to BU(a). \tag{12.1}$$

Identify $BU(1)$ with $\mathbb{C}P^\infty$ and express elements in $BU(1)$ in homogeneous coordinates. Then Δ sends $([x_{10}, x_{11}, \dots], [x_{20}, x_{21}, \dots], \dots, [x_{a0}, x_{a1}, \dots])$

to

$$\begin{bmatrix} x_{10} & 0 & \cdots & 0 & x_{11} & 0 & \cdots & 0 & \dots \\ 0 & x_{20} & \cdots & 0 & 0 & x_{21} & \cdots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & x_{a0} & 0 & 0 & \dots & x_{a1} & \dots \end{bmatrix}$$

Borel [5] showed that the cohomology of $BU(a)$ is a polynomial ring in the universal Chern classes

$$H^*(BU(a), \mathbb{Z}_2) \cong \mathbb{Z}_2[c_1, \dots, c_a], \tag{12.2}$$

and the image of Δ^* consists of the symmetric functions in the degree 2 generators $z_1, \dots z_a$ of the cohomology of the $BU(1)$'s. Furthermore, $\Delta^*(c_i)$ is the i–th elementary symmetric function in the variables $\{z_1, \dots, z_a\}$.

Let $I = (i_1, \dots, i_a)$ be a multi–index. For each $1 \leq j \leq a$ we have the inclusion $\mu_j : \mathbb{C}P^{i_j} \hookrightarrow BU(1)$, where $BU(1)$ is the j–th factor in (12.1). For the multi–index I we have the product $\mu_I = \mu_{i_1} \times \cdots \times \mu_{i_a}$. Composed with Δ we get

$$\iota_I : \mathbb{C}P^I = \mathbb{C}P^{i_1} \times \cdots \times \mathbb{C}P^{i_a} \overset{\mu_I}{\hookrightarrow} BU(1) \times \cdots \times BU(1) \xrightarrow{\Delta} BU(a). \tag{12.3}$$

Following Milnor [16, pp. 186] we say that two monomials z^I and $z^{I'}$ are equivalent if one is derived from the other one by permuting variables. We write $z^I \sim z^{I'}$. Adding all $z^{I'}$ that are equivalent of z^I we get the symmetrization of z^I:

$$\mathfrak{S}(z^I) = \sum z^{I'}. \tag{12.4}$$

We may apply the process of symmetrization also to maps as in (12.3):

$$\mathfrak{S}\left(\iota_I : \mathbb{C}P^I \longrightarrow BU(a)\right) = \coprod_{I' \sim I} \Big(\iota_{I'} : \mathbb{C}P^{I'} \longrightarrow BU(a)\Big) \tag{12.5}$$

In the following the action of G or C on $BO(2a)$ is as in (8.6). In particular, the fixed point set is $BU(a)$.

Theorem 12.1. *Let $\kappa_r : Z_r \to BC$ be as in Proposition* 10.1*. Allow r to vary over all non–negative integers and I over all multi–indices of length a with at least one odd entry. The maps*

$$\kappa_r \times \mathfrak{S}(\iota_I) : Z_r \times \mathfrak{S}(\mathbb{C}P^I) \longrightarrow BC \times BU(a) \tag{12.6}$$

are Steenrod representatives of a set of $\mathbb{Z}_2$ vector space generators of

$$\ker\left(\Phi_* : H_*(EC \times_C BU(a), \mathbb{Z}_2) \to H_*((EC \times_C BU(2a), \mathbb{Z}_2)\right),$$

as well as a set of $\mathcal{N}$ module generators of the kernel of

$$\partial : \mathcal{N}_*(EC \times_C BU(a)) \longrightarrow \mathcal{N}_*(EC \times_C BU(2a)).$$

Proof. The second claim is an immediate consequence of the first one due to the Künneth formula, see (9.2).

We address the first claim. In Theorem 11.1 we calculated

$$\operatorname{coker}\left(\Phi^* : H^*((EC \times_C BU(2a), \mathbb{Z}_2) \to H^*(EC \times_C BU(a), \mathbb{Z}_2)\right).$$

It has a $\mathbb{Z}_2$ vector space basis consisiting of elements of the form $\zeta^{(r)} \otimes u$, where $\zeta^{(r)}$ is as in (10.1) and u is a monomial in Chern classes with at least one odd exponent.

The symmetrizations $\mathfrak{S}(z^I)$ of monomials of degree n, allowing one monomial from each equivalence class, form a basis of the $\mathbb{Z}_2$ vector spaces of all symmetric functions in the variables $\{z_1, \ldots z_a\}$. Working with Chern classes on one hand and symmetric functions on the other one allows us to explicitely translate conditions on characteristic classes into conditions for manifolds used to represent their duals.

Modulo squares, which are divided out in the cokernel of Φ^*, odd powers of Chern classes map under Δ^* to symmetrizations of polynomials in the variables z_j $(1 \leq j \leq a)$ with at least one odd exponent. The fundamental cohomology class of $\mathbb{C}P^I$ is

$$[\mathbb{C}P^I] = \mu^*(z^I) = \mu^*_{i_1}(z_1^{i_1}) \cdots \mu^*_{i_a}(z_a^{i_a}). \tag{12.7}$$

Having an odd exponent means that one of the entires of I is odd. Thus, Steenrod representatives for the second factor of the afore mentioned basis elements are linear combinations of $\mathfrak{S}\left(\iota_I : \mathbb{C}P^I \longrightarrow BU(a)\right)$. Crossing the Steenrod representatives for the first and second factor provides us with Steenrod representatives for the duals of a basis of $\mathrm{coker}(\Phi^*)$. $\square$

13. Proof of Proposition 9.1

This section is meaningful only if 4 divides the oder of H. Otherwise H does not have any essential representations, see Definition 8.7, $\mathfrak{F}^e$ and $\overline{\mathfrak{F}^e}$ are points, and Proposition 9.1 holds trivially. Note also that 2 divides the order of $C = G/K$ because K is of order 2 in $H \subseteq G$. After some preparation we will prove Proposition 9.1 in the case where 4 does not divide $|C|$. After some more preparation we prove we will prove the proposition in the case where 4 divides $|C|$. Recall also (see (8.6)) that $\mathfrak{F}^e = BU(a_1) \times \cdots \times BU(a_k)$.

Proposition 13.1. *Proposition* 9.1 *holds if it does so in the special case where* $\mathfrak{F}^e = BU(a)$.

Let us introduce some notation before we start the proof. Let Φ be as in (8.9) and $\ker(\Phi_*)$ as in Proposition 9.1. Define Φ_j as one of the factors.

$$\Phi_j : EC \times_C BU(a_j) \longrightarrow EC \times_C BO(2a_j).$$

Then we may consider (the hat indicates a term that is omitted)

$$\mathfrak{K}_j = \ker(\Phi_j)_* \otimes H_*(BU(a_1) \times \cdots \times \widehat{BU(a_j)} \times \cdots \times BU(a_k)) \subset \ker \Phi_*$$

A product maps to zero if one factor does so. This implies

Lemma 13.2. *The* $\mathfrak{K}_j$ *generate* $\ker \Phi_*$ *as* j *varies between* 1 *and* k.

Proof of Proposition 13.1. In Proposition 9.1 it is asserted that we can lift back classes in $\ker \Phi_*$ to algebraically represented classes in $\mathcal{N}^G_{*,c}(\overline{\mathfrak{F}^e})$. According to the lemma, it suffices to do so for classes $\mathcal{A} \in \mathfrak{K}_j$. Set $\mathcal{A} = \mathcal{A}_j \otimes \mathcal{A}^c$

where $\mathcal{A}_j \in \ker(\Phi_j)_*$ (for some j) and

$$\mathcal{A}^c \in H_*\left(BU(a_1) \times \cdots \times \widehat{BU(a_j)} \times \cdots \times BU(a_k)\right).$$

Let $f_1 : M_1 \to BU(a_j)$ be an algebraic representative for $\mathcal{A}_j$ lifted back to a class in $\mathcal{N}^G_{r,c}[\{H,K\}](BO(2a_j))$. It exists due to the $k=1$ special case of Proposition 9.1, which we are assuming. Let

$$f_2 : M_2 \to BU(a_1) \times \cdots \times \widehat{BU(a_j)} \times \cdots \times BU(a_k)$$

be a Steenrod representative for the complementary factor $\mathcal{A}^c$. The group G acts trivially on this factor, and according to the classical theory we may assume that this map is algebraic. The product

$$f_1 \times f_2 : M_1 \times M_2 \to BU(a_j) \times BU(a_1) \times \cdots \times \widehat{BU(a_j)} \times \cdots \times BU(a_k),$$

composed with a reshuffling of the coordinates, provides an algebraic representative of $\mathcal{A}$ lifted back to a class in $\mathcal{N}^G_{*,c}(\overline{\mathfrak{F}^e})$. This is what we needed to show to prove our assertion. □

13.1. **The Case where 4 does not divide the order of C.**

Lemma 13.3. *If 4 does not divide the order of C, then Proposition 9.1 holds if it holds in the special case where $K = \mathbb{Z}_2$ and $H = G = \mathbb{Z}_4$.*

Proof. We will work with a commutative diagram (with coefficients for the homology throughout)

$$\begin{array}{ccccc}
\mathcal{N}^G_{*,c}[\{H,K\}](\overline{\mathfrak{F}^e}) & \xrightarrow{J} & H_*(EC \times_C \mathfrak{F}^e) & \xrightarrow{\Phi_*} & H_*(EC \times_C \overline{\mathfrak{F}^e}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{N}^{G'}_{*,c}[\{H',K'\}](\overline{\mathfrak{F}^e}) & \xrightarrow{J'} & H_*(EC' \times_{C'} \mathfrak{F}^e) & \xrightarrow{\Phi'_*} & H_*(EC' \times_{C'} \overline{\mathfrak{F}^e}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{N}^{G''}_{*,c}[\{H'',K''\}](\overline{\mathfrak{F}^e}) & \xrightarrow{J''} & H_*(EC'' \times_{C''} \mathfrak{F}^e) & \xrightarrow{\Phi''_*} & H_*(EC'' \times_{C''} \overline{\mathfrak{F}^e})
\end{array}$$

The first row is from Proposition 9.1, and J is an abbreviation for $\mu \circ L \circ s \circ j^e_c$. In the proposition we asserted that elements in the kernel of Φ_* are images of algebraically represented classes in $\mathcal{N}^G_{*,c}[\{H,K\}](\overline{\mathfrak{F}^e})$.

There is a subgroup L of index 2 in K which acts trivially on all spaces and fibres of bundles (encoded in $\mathfrak{F}^e$). The problem for G, H, and K is the same as the one for $G' = G/L$, $H' = H/L$ and $K' = K/L$. Note that $K' = \mathbb{Z}_2$ and $H' = \mathbb{Z}_4$. The group $C' = G'/K'$ is still C. We get from the first to the second row of the diagram by dividing out the ineffective action of L.

Secondly, G' has an odd order factor O' that acts freely on the domains of classes in $\mathcal{N}^{G'}_{*,c}[\{H',K'\}](\overline{\mathfrak{F}^e})$. The factor O' is also a factor of C'. Set $G'' = G'/O'$, $H'' = H'$, $K'' = K'$, and $C'' = G''/K'' = C'/O'$, and observe that $G'' = H'' = \mathbb{Z}_4$. The first vertical map between the second and third row is obtained by passing to orbit spaces with respect to the action of O'. The in homology are induced by quotient map $C' \to C''$.

The converses to the vertical maps between the second and third row are the induction map in bordism and the transfer in homology, both of which are onto in our situation. Use [20] for bordism and [6] for homology. Induction preserves algebraic representation [12]. In summary, if we can lift back classes in the kernal of Φ''_* to algebracially represented classes, then we can do so also for classes in the kernel of Φ_*, and this proves the assertion of the lemma. □

Proof of Proposition 9.1 *if* $4 \not| |C|$. If the order of C is twice an odd number, then, according to Lemma 13.3, we are essentially dealing with the situation in which $K = \mathbb{Z}_2$, $H = G = \mathbb{Z}_4$ and $C = \mathbb{Z}_2$. The assertion is that in

$$\mathcal{N}^G_{*,c}[\{H,K\}](\overline{\mathfrak{F}^e}) \xrightarrow{J} H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2) \xrightarrow{\Phi_*} H_*(EC \times_C \overline{\mathfrak{F}^e}, \mathbb{Z}_2),$$

elements in the kernel of Φ_* are images under $J = \mu \circ L \circ s \circ j^e_c$ of algebraically represented classes in $\mathcal{N}^G_{*,c}[\{H,K\}](\overline{\mathfrak{F}^e})$.

According to Theorem 12.1, a representative of a typical generator of the kernel is of the form

$$\kappa_r \times \mathfrak{S}(\iota_I) : Z_r \times \mathfrak{S}(\mathbb{C}P^I) \longrightarrow BC \times BU(a) \tag{13.1}$$

which is obtained from a map

$$\kappa_r \times \mu_I : Z_r \times \mathbb{C}P^{i_1} \times \cdots \times \mathbb{C}P^{i_a} \longrightarrow BC \times BU(1) \times \cdots \times BU(1) \tag{13.2}$$

after symmetrization and composition with Borel's diagonal approximation. The multi–index $I = (i_1, \ldots .i_a)$ has at least one odd entry, say i_j.

According to Proposition 2.10 (proved in Section 7)

$$\kappa_r \times i_j : Z_r \times \mathbb{C}P^{i_j} \longrightarrow BC \times BU(1)$$

is the image of an algebraically represented class in $\mathcal{N}^G_{r,c}[\{H,K\}](BO(2))$ under the map J. Multiply this data with the other factors in (13.2), the maps $i_t : \mathbb{C}P^{i_t} \to BU(1)$ for $t \neq i$. We can symmetrize the second factor of this data, including the lift back. Composed with Borel's diagonal map Δ we obtain an algebraic representative of a class in $\mathcal{N}^{\mathbb{Z}_4}_{*,c}[\{\mathbb{Z}_4, \mathbb{Z}_2\}](BU(a))$ that maps to the typical generator that we started out with in (13.1). With this we have deduced Proposition 9.1 from Proposition 2.10. □

13.2. **The Case where 4 divides the order of C.** Given a G equivariant function $f : M \to \overline{\mathfrak{F}^e}$, and making use of the action of S^1 on $\overline{\mathfrak{F}^e} = BO(2a)$ (see Remarks 8.8 and 8.9 and (8.6)), we set

$$\widetilde{f} : S^1 \times_H M \to \overline{\mathfrak{F}^e} \quad \text{setting} \quad \widetilde{f}[z,x] = zf(x). \tag{13.3}$$

If $f : M \to \overline{\mathfrak{F}^e}$ represents a class $\mathcal{A} \in \mathcal{N}^G_{*,c}[\{H,K\}](\overline{\mathfrak{F}^e})$, then we write $\Xi(\mathcal{A})$ for the class that is represented by $\widetilde{f} : S^1 \times_H M \to \overline{\mathfrak{F}^e}$. This defines an $\mathcal{N}_*$–module homomorphism

$$\Xi : \mathcal{N}^G_{*,c}[\{H,K\}](\overline{\mathfrak{F}^e}) \longrightarrow \mathcal{N}^G_{*+1,c}[\{H,K\}](\overline{\mathfrak{F}^e}). \tag{13.4}$$

Lemma 13.4. *If $\mathcal{A}$ is algebraically represented, then so is $\Xi(\mathcal{A})$.*

Proof. If M is a nonsingular real algebraic variety, then so is the balanced product $S^1 \times_H M$, see [18] or [11, Section 3]. If f is entire rational, then so is $\widetilde{f}$. To see the latter, one uses that the action of S^1 on $\overline{\mathfrak{F}^e}$ is regular (as observed in Remark 8.9) and the universal mapping property of the algebraic quotient, see [11, Section 3]. Taken together, if (M, f) is algebraic, then so is $(S^1 \times_H M, \widetilde{f})$. If $\mathcal{A}$ is algebraically represented, then so is $\Xi(\mathcal{A})$. □

Lemma 13.5. *Let $\mathcal{A} \in \mathcal{N}^G_{2r,c}[\{H, K\}](\overline{\mathfrak{F}^e})$ be a class in grading $2r$, then*

$$(\mu \circ L \circ s \circ j^e_c)(\Xi(\mathcal{A})) = [(\mu \circ L \circ s \circ j^e_c)(\mathcal{A})]^{+1}. \tag{13.5}$$

The superscript $+1$ in (13.5) indicates that we increase the grading on the first factor of $H_*(BC, \mathbb{Z}_2) \otimes H_*(\mathfrak{F}^e, \mathbb{Z}_2)$ by 1. In Section 10 we denoted the unique nonzero class in $H_i(BC, \mathbb{Z}_2)$ by $\zeta_{(i)}$ so that

$$\Big(\sum \zeta_{(m-i)} \otimes Y_i\Big)^{+1} = \sum \zeta_{(m-i+1)} \otimes Y_i$$

Proof of Lemma 13.5. We abbreviate $\mu \circ L \circ s \circ j^e_c$ in (13.5) as J. Set

$$J(\mathcal{A}) = \sum_i \zeta_{(2r-i)} \otimes Y_i \quad \& \quad J(\Xi(\mathcal{A})) = \sum_i \zeta_{(2r+1-i)} \otimes \overline{Y}_i. \tag{13.6}$$

Suppose $\mathcal{A}$ is represented by $f : M \to \overline{\mathfrak{F}^e}$. We may think of $j^e_c(\mathcal{A})$ and $j^e_c(\Xi(\mathcal{A}))$ as being represented by

$$\chi_\nu \times f^H : M^H \to B\mathbb{Z}_2 \times \mathfrak{F}^e \tag{13.7}$$

and

$$\chi_{\Xi(\nu)} \times \Xi(f)^H : (S^1/H) \times M^H \to B\mathbb{Z}_2 \times \mathfrak{F}^e. \tag{13.8}$$

The map $\chi_\nu : M^H \to \mathfrak{F}^e$ classifies the normal bundle $\nu(M^H, M)$ (a line bundle) of M^H in M, while the map $\chi_{\Xi(\nu)} : (S^1/H) \times M^H \to B\mathbb{Z}_2$ classifies $\nu(S^1 \times_H M^H, S^1 \times_H M)$ the normal bundle of $S^1 \times_H M^H = S^1/H \times M^H$ in $S^1 \times_H M$. A comparison of the construction underlying Ξ in (13.3) and the one in the transition from (10.2) to (10.3) reveals that they are the same. (The $\mathbb{Z}_2$ in (10.3) corresponds to $H/K = \mathbb{Z}_2$ in (13.3), where K is divided out as it acts trivially). The latter construction, the one in Remark 10.2, increases the grading of an even-dimensional class in $H_*(BC, \mathbb{Z}_2)$ by 1.

Note also that Y_i is nonzero in (13.6) only if i is even, because $\mathfrak{F}^e$ is a product of $BU(a)$'s. As $\mathcal{A}$ is assumed to be in even grading, we see that the $J(\mathcal{A})$ have terms $\zeta_{(2r-i)} \otimes Y_i$ only for even values of $2r - i$. Thus the effect of applying Ξ to $\mathcal{A}$ on these first terms is to increase their grading by 1.

Let us study the second factor of the map in (13.8):

$$\Xi(f)^H : (S^1/H) \times M^H \to \mathfrak{F}^e.$$

As an immediate consequence of the definition in (13.3) and the fact that S^1 acts trivially on $\mathfrak{F}^e$ (see (8.6)) we deduce that $\Xi(f)^H = f^H \circ \pi_2$, where

π_2 denotes the projection on the second factor. This implies that $Y_i = \overline{Y}_i$ in (13.6).

Taken together, the analysis of the effect of Ξ on the first and second factors of the summands of $J(\mathcal{A})$ in (13.6) verifies the formula in (13.5). □

Proof of Proposition 9.1 *if* 4 *divides the order of* C. The claim is that in

$$\mathcal{N}^G_{*,c}[\{H,K\}](\overline{\mathfrak{F}^e}) \xrightarrow{J} H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2) \xrightarrow{\Phi_*} H_*(EC \times_C \overline{\mathfrak{F}^e}, \mathbb{Z}_2)$$

elements in the kernel of Φ_* are images under $J = \mu \circ L \circ s \circ j^e_c$ of algebraically represented classes in $\mathcal{N}^G_{*,c}[\{H,K\}](\overline{\mathfrak{F}^e})$. Due to Proposition 13.1, we may assume that $\mathfrak{F}^e = BU(a)$, instead of being a product of $BU(a)$'s, and that $\overline{\mathfrak{F}^e} = BO(2a)$.

By definition

$$\ker_*(\Phi_*) \subseteq H_*(EC \times_C \mathfrak{F}^e, \mathbb{Z}_2) \cong H_*(BC, \mathbb{Z}_2) \otimes H_*(\mathfrak{F}^e, \mathbb{Z}_2).$$

We described these kernels in Theorem 12.1, and one sees that increasing the grading by 1 in the first factor induces an isomorphism

$$\ker_{2*}(\Phi_*) \to \ker_{2*+1}(\Phi_*).$$

It follows from Lemmata 13.4 and 13.5 that if we can lift back classes in $\ker_{2*}(\Phi_*)$ to algebraically represented classes in $\mathcal{N}^G_{*,c}[\{H,K\}](\overline{\mathfrak{F}^e})$, then we can do so also for classes in $\ker_{2*+1}(\Phi_*)$.

We turn our attention to classes in $\ker_{2*}(\Phi_*)$. Necessarily, they have their first factor in $H_{2*}(BC, \mathbb{Z}_2)$. Due to the isomorphism in Proposition 10.3, we may work with $\mathbb{Z}_2$ instead of C and use H as acting group. In this case the lift back to algebraically represented classes is as in the proof of Proposition 9.1 in the case, discussed earlier in this section, where 4 does not divide the order of C.

The map $H_{2N}(B\mathbb{Z}_2, \mathbb{Z}_2) \to H_{2N}(BC, \mathbb{Z}_2)$ is induced by the orbit map $B\mathbb{Z}_2 \to BC$, and that means geometrically that we use induction to get from $\mathbb{Z}_2$–data to C–data. Finite induction, i.e., crossing with $C \times_{\mathbb{Z}_2}$ works within the algebraic category. Applied to an algebraic map it results again in an algebraic map [9, Section 3]. Allowing for a kernel of the action, we obtain an action of G. This concludes our proof of Proposition 9.1 also in the second case where the order of C is divisible by 4. □

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