TENSOR PRODUCTS OF SYMMETRIC FUNCTIONS OVER $\mathbb{Z}_2$

KARL HEINZ DOVERMANN AND JASON HANSON

Abstract. We calculate the homology and the cycles in tensor products of algebras of symmetric function over $\mathbb{Z}_2$.

1. Statement of Results

We calculate the homology and the cycles in tensor products of the differential graded algebra of symmetric functions over $\mathbb{Z}_2$, the integers modulo 2. The need for this calculation arises in a project in which we show that closed smooth manifolds with cyclic group actions have equivariant real algebraic models ([3] and [4]). There we need to calculate the ordinary equivariant cohomology of some classifying spaces with cyclic group action; the symmetric functions and the differential ∇ arise in a way explained in some detail in Section 5. Some special cases of our results can be extracted from [6] and [5].

Let $F_a := \mathbb{Z}_2[z_1, \ldots, z_a]$ be the polynomial ring in a variables of dimension 1. The natural differential ∇ on F_a is the sum of the partial derivatives. This derivative is obtained from the standard rules of differentiation: linearlity and Leibnitz' rule (the product rule), under the assumption that the derivative of each z_i is the constant function 1. Then $\nabla^2 = 0$, and (F_a, ∇) is a differential graded algebra.

In F_a we consider the subalgebra $\mathfrak{S}_a$ of symmetric functions. The i–th elementary symmetric function is denoted by σ_i. It is elementary to compute its derivative:

$$\nabla\sigma_i = (a - i + 1)\sigma_{i-1} \text{ for } 1 \leq i \leq a, \text{ and } \nabla\sigma_0 = 0. \tag{1.1}$$

The formula depends only on the parity of a.

Consider a sequence $\mathcal{A} = (a(0), \ldots, a(k))$ of nonnegative integers and set

$$\mathfrak{S}_{\mathcal{A}} = \mathfrak{S}_{a(0)} \otimes \cdots \otimes \mathfrak{S}_{a(k)}. \tag{1.2}$$

As a tensor product, $\mathfrak{S}_{\mathcal{A}}$ inherits a natural differential operator, which we still denote by ∇. We use an additional subscript to distinguish the factor to which an elementary symmetric function belongs. Specifically, $\sigma_{j,i}$ is the

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i–th elementary symmetric function in the j–th factor of the tensor product, where $0 \le j \le k$ and $0 \le i \le a(j)$. We set $a(j) = 2b(j)$ or $a(j) = 2b(j) + 1$, depending on whether $a(j)$ is even or odd. The letter Z denotes the cycles of the indicated DGA.

Theorem 1.1. *Suppose $\mathcal{A} = (a(0), \dots, a(k))$ and for a distinguished index t the associated entry $a(t)$ of $\mathcal{A}$ is odd. Then the differential graded algebra $\mathfrak{S}_{\mathcal{A}}$ is acyclic. Set*

$$\begin{aligned} D(\mathcal{A}) &= \{\sigma_{t,1} + \sigma_{j,1} \mid j \neq t,\ a(j) \text{ odd}\} \\ D^o(\mathcal{A}) &= \{\sigma_{j,2s},\ \sigma_{t,1}\sigma_{j,2s} + \sigma_{j,2s+1} \mid a(j) = 2b(j) + 1 \text{ and } 1 \le s \le b(j)\} \\ D^e(\mathcal{A}) &= \{\sigma_{j,2s-1},\ \sigma_{t,1}\sigma_{j,2s-1} + \sigma_{j,2s} \mid a(j) = 2b(j) \text{ and } 1 \le s \le b(j)\}. \end{aligned}$$

The cycles in $\mathfrak{S}_{\mathcal{A}}$ are $Z(\mathfrak{S}_{\mathcal{A}}) = \mathbb{Z}_2[\{\sigma_{t,1}^2\} \cup D(\mathcal{A}) \cup D^o(\mathcal{A}) \cup D^e(\mathcal{A})]$.

In our next two results, $\mathcal{A} = (2b(0), \dots, 2b(k))$ is a sequence of even nonnegative integers. The first result describes the homology of the DGA $(\mathfrak{S}_{\mathcal{A}}, \nabla)$, and is an immediate consequence of Corollary 4.2.

Corollary 1.2. *Set $T(\mathcal{A}) = \{\sigma_{j,2s}^2 \mid 0 \le j \le k,\ 1 \le s \le b(j)\}$. Then $H_*(\mathfrak{S}_{\mathcal{A}}, \mathbb{Z}_2) = \mathbb{Z}_2[T(\mathcal{A})]$.*

Next, we describe the cycles in $(\mathfrak{S}_{\mathcal{A}}, \nabla)$ as a module over the ring

$$\Lambda_{\mathcal{A}} = \mathbb{Z}_2[\{\sigma_{j,2s-1}, \sigma_{j,2s}^2 \mid 0 \le j \le k,\ 1 \le s \le b(j)\}] \subseteq \mathfrak{S}_{\mathcal{A}}$$

Set $\mathfrak{b}'_{\mathcal{A}} = \{\sigma_{j,2s} \mid 0 \le j \le k,\ 1 \le s \le b(j)\}$. Let $\mathfrak{B}'_{\mathcal{A}}$ consist of all elements ∇b, where b is the product of at least two elements of $\mathfrak{b}'_{\mathcal{A}}$. Set $\mathfrak{B}_{\mathcal{A}} = \mathfrak{B}'_{\mathcal{A}} \cup \{1\}$. We define $\mathfrak{A}_{\mathcal{A}}$ to be the module over $\Lambda_{\mathcal{A}}$ with generating set $\mathfrak{B}_{\mathcal{A}}$. Without changing $\mathfrak{A}_{\mathcal{A}}$, we can and will assume that the factors of b are distinct.

Proposition 1.3. *If $\mathcal{A} = (2b(0), \dots, 2b(k))$, then $Z(\mathfrak{S}_{\mathcal{A}}) = \mathfrak{A}_{\mathcal{A}}$.*

We prove this proposition in Section 4. In Remark 4.3 we discuss the structural difference between Theorem 1.1 and Proposition 1.3.

2. Some preliminary remarks

The formula in (1.1) for the derivatives of the elementary symmetric functions in a variables can be written as:

$$\nabla\sigma_{2i} = \begin{cases} \sigma_{2i-1} & \text{if } a \text{ is even} \\ 0 & \text{if } a \text{ is odd} \end{cases} \quad \text{and} \quad \nabla\sigma_{2i+1} = \begin{cases} 0 & \text{if } a \text{ is even} \\ \sigma_{2i} & \text{if } a \text{ is odd.} \end{cases}$$

Remark 2.1. The homology of $\mathfrak{S}_{a(0)} \otimes \cdots \otimes \mathfrak{S}_{a(k)}$ is trivial if any one of the $a(t)$ is odd. For suppose $a(t)$ is odd, and f is any cycle, then $f = \nabla(\sigma_{t,1} f)$ is a boundary.

Remark 2.2. In precise terms, $\sigma_k \in \mathfrak{S}_a$ is the k–th symmetric function in a variables. Throughout we will work only with the symmetric functions and use the formulas for the derivatives. In this sense, a statement which we prove for $\mathfrak{S}_a$ will also hold for the algebra $\mathfrak{S}'_a$ generated by the elementary symmetric functions of degree $\leq a$ in $a + 2r$ variables. To avoid the introduction of further notation, we identify $\mathfrak{S}_a$ and $\mathfrak{S}'_a$.

3. Proof of Theorem 1.1

We prove the theorem by induction. The starting point of the induction is the following special case Theorem 1.1.

Proposition 3.1. *Suppose $\mathcal{A} = (1, \ldots, 1)$ is a sequence of $(k + 1)$ ones. Then the DGA $(\mathfrak{S}_{\mathcal{A}}, \nabla)$ is acyclic and its cycles are the subalgebra*

$$Z(\mathfrak{S}_{\mathcal{A}}, \nabla) = \mathbb{Z}_2[\{z_0^2\} \cup \{z_0 + z_j \mid 1 \leq j \leq k\}] \subseteq \mathbb{Z}_2[z_0, \ldots, z_k].$$

We should start out with a sequence $\mathcal{A}$ consisting of zeros and ones, but $\mathfrak{S}_0$ is trivial, and so we ignore the zeros and suppress the corresponding trivial factors in the tensor product. In Theorem 1.1, t denotes a distinguished position in which $\mathcal{A}$ has an odd entry. Without loss of generality, this position is $t = 0$ in the proposition. Now $\mathfrak{S}_{a(j)} = \mathfrak{S}_1 = \mathbb{Z}_2[z_j]$ and $\sigma_{j,1} = z_j$, so that we may write

$$\mathfrak{S}_{\mathcal{A}} = \mathfrak{S}_1 \otimes \cdots \otimes \mathfrak{S}_1 = \mathbb{Z}_2[z_0, \ldots, z_k].$$

We note that $D^o(\mathcal{A}) = D^e(\mathcal{A}) = \emptyset$. This establishes the proposition as a special case of Theorem 1.1.

We expressed $Z(\mathfrak{S}_{\mathcal{A}}, \nabla)$ as a polynomial ring, and this means that the variables (or generators) need to be algebraically independent. This is easy to check for the given set of generators, because each new generator involves a new variable.

Proof. Let $\mathfrak{B}$ denote the algebra generated by z_0^2 and $z_0 + z_j$ for $1 \leq j \leq k$. Apparently $\nabla(z_0^2) = 0$ and $\nabla(z_0 + z_j) = 0$, so that $\mathfrak{B} \subseteq Z(\mathfrak{S}_{\mathcal{A}}, \nabla)$.

Conversely, we show that $Z(\mathfrak{S}_{\mathcal{A}}, \nabla) \subseteq \mathfrak{B}$. Remark 2.1 tells us that $(\mathfrak{S}_{\mathcal{A}}, \nabla)$ is acyclic. It suffices to show that the boundary of any monomial $q = z_0^{m_1} \cdots z_k^{m_k}$ in $(\mathfrak{S}_{\mathcal{A}}, \nabla)$ belongs to $\mathfrak{B}$, i.e., that $\nabla(q) \in \mathfrak{B}$.

Consider the quotient $\mathbb{Z}_2[z_0, \ldots, z_k]/\mathfrak{B}$. In this quotient we identify each z_j with z_0 because $z_0 + z_j$ belongs to the generators of $\mathfrak{B}$. We may also reduce the exponent of z_0 modulo 2 because z_0^2 is a generator of $\mathfrak{B}$. In conclusion, $\mathbb{Z}_2[z_0, \ldots, z_k]/\mathfrak{B} \cong \mathbb{Z}_2$, and the nonzero class is $[z_0]$. Here and below we indicate equivalence classes in the quotient by square brackets.

To conclude our argument, we show that $[\nabla(q)] = 0 \in \mathbb{Z}_2[z_0, \ldots, z_k]/\mathfrak{B}$. After identifying variables, we may suppose that $q = z_0^m$. If m is even, then $\nabla(q) = 0$. If q is of odd degree, then $\nabla(q)$ is of even degree and belongs to $\mathfrak{B}$. In either case $[\nabla(q)]$ vanishes and our argument is complete. □

In preparation for our inductive proof of Theorem 1.1, we study what happens to the cycles of the DGA when we increase one entry in the sequence $\mathcal{A} = (a(0), \ldots, a(k))$ by two.

Suppose $\mathcal{A}$ is a sequence of nonnegative integers and its r–th term is $a(r) = 2b + 1$. Let $\mathcal{A}'$ be the sequence whose r–th entry is $2b + 3$, and which agrees with $\mathcal{A}$ in all other places. To avoid double indexing, we just write σ_i for $\sigma_{r,i}$.

Proposition 3.2. *For $\mathcal{A}$ and $\mathcal{A}'$ as above*

$$Z(\mathfrak{S}_{\mathcal{A}'}) = Z(\mathfrak{S}_{\mathcal{A}}) \otimes \mathbb{Z}_2[\sigma_{2b+2}, \sigma_1\sigma_{2b+2} + \sigma_{2b+3}]$$

Proof. We use the abbreviation $\mathfrak{A} = Z(\mathfrak{S}_{\mathcal{A}}) \otimes \mathbb{Z}_2[\sigma_{2b+2}, \sigma_1\sigma_{2b+2} + \sigma_{2b+3}]$. Apparently, $\nabla\sigma_{2b+2} = 0$ and $\nabla(\sigma_1\sigma_{2b+2} + \sigma_{2b+3}) = 0$, so that $\mathfrak{A} \subseteq Z(\mathfrak{S}_{\mathcal{A}'})$.

We will show $Z(\mathfrak{S}_{\mathcal{A}'}) \subseteq \mathfrak{A}$. Consider an element $f \in \mathfrak{S}_{\mathcal{A}'}$ and express it in the form

$$f = \sum_{n,m\geq 0} \left(g_{n,2m}\sigma_{2b+2}^{n}\sigma_{2b+3}^{2m} + g_{n,2m+1}\sigma_{2b+2}^{n}\sigma_{2b+3}^{2m+1}\right),$$

where $g_{*,*} \in \mathfrak{S}_{\mathcal{A}}$. If f is a cycle, then

$$0 = \sum_{n,m\geq 0} \nabla(g_{n,2m})\sigma_{2b+2}^{n}\sigma_{2b+3}^{2m} + \nabla(g_{n,2m+1})\sigma_{2b+2}^{n}\sigma_{2b+3}^{2m+1} + g_{n,2m+1}\sigma_{2b+2}^{n+1}\sigma_{2b+3}^{2m},$$

and we need to show that f belongs to $\mathfrak{A}$. A comparison of coefficients gives us the equations:

$$\nabla g_{0,2m} = 0 \quad \text{and} \quad \nabla g_{n-1,2m+1} = 0 \quad \text{and} \quad \nabla g_{n,2m} + g_{n-1,2m+1} = 0$$

for $n > 0$ and $m \geq 0$. In the following it will be useful to observe that

$$(\sigma_1\sigma_{2b+2} + \sigma_{2b+3})^2 + \sigma_1^2\sigma_{2b+2}^2 = \sigma_{2b+3}^2 \in \mathfrak{A}.$$

Cycles of the first kind: Since $g_{0,2m} \in Z(\mathfrak{S}_{\mathcal{A}})$, the summands $g_{0,2m}\sigma_{2b+3}^{2m}$ of f are in $\mathfrak{A}$.

Cycles of the second kind: For a pair (n, m) with $n > 0$ we look at a pair of summands of f:

$$\begin{aligned} h &= g_{n,2m}\sigma_{2b+2}^{n}\sigma_{2b+3}^{2m} + g_{n-1,2m+1}\sigma_{2b+2}^{n-1}\sigma_{2b+3}^{2m+1} \\ &= \sigma_{2b+2}^{n-1}\sigma_{2b+3}^{2m}(g_{n,2m}\sigma_{2b+2} + g_{n-1,2m+1}\sigma_{2b+3}). \end{aligned}$$

Each of the three factors of h, and hence h itself, is a cycle. We would like to show that $h \in \mathfrak{A}$. The first two factors of h are in $\mathfrak{A}$. It remains to be shown that the third factor

$$h' = g_{n,2m}\sigma_{2b+2} + g_{n-1,2m+1}\sigma_{2b+3}$$

is in $\mathfrak{A}$. Observe that $g_{n-1,2m+1} \in Z(\mathfrak{S}_{\mathcal{A}})$, and $g_{n-1,2m+1}(\sigma_1\sigma_{2b+2} + \sigma_{2b+3}) \in \mathfrak{A}$. Showing that $h' \in \mathfrak{A}$ is equivalent to showing that the cycle

$$h'' = h' + g_{n-1,2m+1}(\sigma_1\sigma_{2b+2} + \sigma_{2b+3}) = (g_{n,2m} + g_{n-1,2m+1}\sigma_1)\sigma_{2b+2}$$

is in $\mathfrak{A}$. Observe that $(g_{n,2m} + g_{n-1,2m+1}\sigma_1) \in Z(\mathfrak{S}_{\mathcal{A}})$ and $\sigma_{2b+2} \in \mathfrak{A}$. It follows that $h'' \in \mathfrak{A}$, and so $h \in \mathfrak{A}$.

Taken together, our cycles of the first and second kind make up all the summands of f, thus $f \in \mathfrak{A}$. This completes the proof. □

As an immediate consequence of Theorem 3.1 and Proposition 3.2 we obtain the computation of the cycles in $\mathfrak{S}_{\mathcal{A}}$ if all entries in $\mathcal{A}$ are odd. If the length of $\mathcal{A}$ is one, and this one nonzero entry is $2b + 1$, then we have

Corollary 3.3. *The differential graded algebra* $(\mathfrak{S}_{2b+1}, \nabla)$ *is acyclic and* $Z(\mathfrak{S}_{2b+1}) = \mathbb{Z}_2[\sigma_1^2, \sigma_2, \dots, \sigma_{2b}, \sigma_1\sigma_2 + \sigma_3, \dots, \sigma_1\sigma_{2b} + \sigma_{2b+1}]$.

Suppose that $\mathcal{A} = (a(1), \dots, a(k))$ is a sequence of nonnegative integers, $a(t)$ is odd, and $a(r) = 2b$ is even. Let $\mathcal{A}'$ be identical to $\mathcal{A}$, with the one exception that the entry in position r is $2b+2$. To avoid double indexing, we write σ_i for $\sigma_{r,i}$.

Proposition 3.4. *For* $\mathcal{A}$ *and* $\mathcal{A}'$ *as above*

$$Z(\mathfrak{S}_{\mathcal{A}'}) = Z(\mathfrak{S}_{\mathcal{A}}) \otimes \mathbb{Z}_2[\sigma_{2b+1}, \sigma_{t,1}\sigma_{2b+1} + \sigma_{2b+2}].$$

Proof. The proof is the same as the one of Proposition 3.2, except for a shift in grading by 1. □

Proof of Theorem 1.1. We compute $Z(\mathfrak{S}_{\mathcal{A}})$ inductively, starting out with a sequence $\mathcal{A}_o$ which has a 1 in those places where $\mathcal{A}$ has an odd entry and a 0 in those places where $\mathcal{A}$ has an even entry. For $\mathcal{A} = \mathcal{A}_o$, Theorem 1.1 specializes to Proposition 3.1 (as explained after the statement of the proposition), and this proposition we proved already. Propositions 3.2 and 3.4 describe the effect on the cycles of the differential graded algebra when an odd, resp. even, entry is increased by two. Either of these increases adds two algebra generators to a basis for the cycles, exactly as it is described in the assertion of Theorem 1.1. □

4. Symmetric functions of an even number of variables

Throughout this section $\mathcal{A} = (a(0), \dots, a(k)) = (2b(0), \dots, 2b(k))$ is a sequence of even nonnegative integers.

We specify a position r in this sequence. To simplify notation we set $a(r) = 2b(r) = 2b$. Let $\mathcal{A}'$ be the sequence whose r–th entry is $2b+2$, and which agrees with $\mathcal{A}$ in all other places. To avoid double indexing, we just write σ_i for $\sigma_{r,i}$.

Proposition 4.1. *Let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be as above. Then any cycle* f *in* $\mathfrak{S}_{\mathcal{A}'}$ *can be expressed in the form*

$$f = h + c_0 + c_2\sigma_{2b+2}^2 + \cdots + c_{2N}\sigma_{2b+2}^{2N},$$

where h *is a boundary in* $\mathfrak{S}_{\mathcal{A}'}$, *and* $c_0, \dots, c_{2N}$ *are cycles in* $\mathfrak{S}_{\mathcal{A}}$.

Corollary 4.2. *Let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be as above. Then*

$$H_*(\mathfrak{S}_{\mathcal{A}'}, \mathbb{Z}_2) = H_*(\mathfrak{S}_{\mathcal{A}}, \mathbb{Z}_2) \otimes \mathbb{Z}_2[\sigma_{2b+2}^2].$$

Proof of Proposition 4.1. Express $f \in \mathfrak{S}_{\mathcal{A}'}$ in the form

$$f = \sum_{n,m\ge 0} \left(g_{n,2m}\sigma_{2b+1}^n\sigma_{2b+2}^{2m} + g_{n,2m+1}\sigma_{2b+1}^n\sigma_{2b+2}^{2m+1} \right),$$

where the $g_{*,*}$ are in $\mathfrak{S}_{\mathcal{A}}$. Assuming that f is a cycle we find that

$$0 = \sum_{n,m\ge 0} \nabla(g_{n,2m})\sigma_{2b+1}^n\sigma_{2b+2}^{2m} + \nabla(g_{n,2m+1})\sigma_{2b+1}^n\sigma_{2b+2}^{2m+1} + g_{n,2m+1}\sigma_{2b+1}^{n+1}\sigma_{2b+2}^{2m}.$$

Comparison of coefficients provides us with the equations

$$\nabla g_{0,2m} = 0 \quad \text{and} \quad \nabla g_{n-1,2m+1} = 0 \quad \text{and} \quad \nabla g_{n,2m} + g_{n-1,2m+1} = 0$$

for $n > 0$ and $m \geq 0$.

Cycles: As $\nabla g_{0,2m} = 0$, we may set $c_{2m} = g_{0,2m}$.

Boundaries: For $n \geq 1$ we have

$$\begin{aligned} \nabla\left(g_{n,2m}\sigma_{2b+1}^{n-1}\sigma_{2b+2}^{2m+1}\right) &= \nabla(g_{n,2m})\sigma_{2b+1}^{n-1}\sigma_{2b+2}^{2m+1} + g_{n,2m}\sigma_{2b+1}^{n}\sigma_{2b+2}^{2m} \\ &= g_{n-1,2m+1}\sigma_{2b+1}^{n-1}\sigma_{2b+2}^{2m+1} + g_{n,2m}\sigma_{2b+1}^{n}\sigma_{2b+2}^{2m}. \end{aligned}$$

Together, these terms give us the summand h called for in the proposition.

As one may easily verify, the exhibited boundaries and cycles together make up all the summands of f, so that our proposition is proved. □

Proof of Proposition 1.3. The assertion is that $Z(\mathfrak{S}_\mathcal{A}) = \mathfrak{A}_\mathcal{A}$. By definition, $\mathfrak{A}_\mathcal{A}$ is a module over the ring $\Lambda_\mathcal{A}$, generated by the set $\mathfrak{B}_\mathcal{A}$. It is trivial to verify that all elements in $\Lambda_\mathcal{A}$ and of $\mathfrak{B}_\mathcal{A}$ belong to $Z(\mathfrak{S}_\mathcal{A})$, so that $\mathfrak{A}_\mathcal{A} \subseteq Z(\mathfrak{S}_\mathcal{A})$.

Next we show that $Z(\mathfrak{S}_\mathcal{A}) \subseteq \mathfrak{A}_\mathcal{A}$. The given basis elements for $H_*(\mathfrak{S}_\mathcal{A}, \mathbb{Z}_2)$ given in Corollary 1.2 belong to $\mathfrak{A}_\mathcal{A}$, and it remains to be shown that the boundaries $B(\mathfrak{S}_\mathcal{A})$ in $\mathfrak{S}_\mathcal{A}$ belong to $\mathfrak{A}_\mathcal{A}$.

Let us describe the boundaries in $\mathfrak{S}_\mathcal{A}$. Let h be a monomial in $\mathfrak{S}_\mathcal{A}$. We express it in the form

$$h = \prod_{j=0}^{k}\prod_{i=1}^{2b(j)} \sigma_{j,i}^{r(j,i)} = \prod_{j=0}^{k}\prod_{i=1}^{2b(j)} \sigma_{j,i}^{s(j,i)} \cdot \prod_{j=0}^{k}\prod_{i=1}^{2b(j)} \sigma_{j,i}^{\epsilon(j,i)} = \lambda \cdot \beta,$$

and break it up as a product $\lambda\cdot\beta$. In the first double product we collected all the factors of h that belong to $\Lambda_\mathcal{A}$, and we abbreviated it as λ. In particular, $s(j,i) = r(j,i)$ if i is odd, and $s(j,i)$ is the largest even summand of $r(j,i)$ if i is even. Consequently, $\epsilon(j,i)$ is 1 if i is even and $r(j,i)$ is odd, and is 0 otherwise. Because λ is a cycle, we have

$$\nabla h = \lambda \cdot \nabla\left(\prod_{j=0}^{k} \sigma_{j,1}^{\epsilon(j,1)} \cdots \sigma_{j,2b(j)}^{\epsilon(j,2b(j))}\right). \tag{4.1}$$

Each boundary is the sum of boundaries of monomials, and (4.1) tells us what they look like. We distinguish cases based on the number of factors of β. If β has no factor, then $\nabla h = 0$, and there is nothing to be proved. If β has exactly one factor, say $\sigma_{j,2s}$, then $\nabla h = \lambda\sigma_{j,2s-1} \in \Lambda_\mathcal{A} \subseteq \mathfrak{A}_\mathcal{A}$. If β has at least two factors, then $\nabla h = \lambda \cdot \nabla\beta$, and $\beta \in \mathfrak{B}'_\mathcal{A}$, so that $\nabla h \in \mathfrak{A}_\mathcal{A}$. In conclusion, all $\mathbb{Z}_2$-homology classes of $(\mathfrak{S}_\mathcal{A}, \nabla)$ are represented by elements in $\mathfrak{A}_\mathcal{A}$. The boundaries of all monomials in $(\mathfrak{S}_\mathcal{A}, \nabla)$, and hence the boundaries of all elements in $(\mathfrak{S}_\mathcal{A}, \nabla)$ are in $\mathfrak{A}_\mathcal{A}$. Hence $Z(\mathfrak{S}_\mathcal{A}) \subseteq \mathfrak{A}_\mathcal{A}$, and our proof is complete. □

Remark 4.3. If the sequence $\mathcal{A}$ has at least one odd entry, then we are able to write down a basis (algebraically independent generating set) of the algebra $Z(\mathfrak{S}_{\mathcal{A}})$. If all entries of $\mathcal{A}$ are even, then we are able to describe $Z(\mathfrak{S}_{\mathcal{A}})$ only as a module $\mathfrak{A}_{\mathcal{A}}$ over a ring $\Lambda_{\mathcal{A}}$ with a generating set $\mathfrak{B}_{\mathcal{A}}$. Still, even this generating set is not a basis, and we do not know whether the algebra $Z(\mathfrak{S}_{\mathcal{A}})$ has a basis, or if the module $\mathfrak{A}_{\mathcal{A}}$ over $\Lambda_{\mathcal{A}}$ has a basis. Even if $\mathcal{A} = (6)$, which consists of only one even entry, then $\mathfrak{B}_{\mathcal{A}}$ is not linearly independent. Here is a relation in between the module generators over $\Lambda_{\mathcal{A}}$:

$$\begin{aligned} 0 &= \sigma_5 \nabla(\sigma_2\sigma_4) + \sigma_3 \nabla(\sigma_2\sigma_6) + \sigma_1 \nabla(\sigma_4\sigma_6) \\ &= \sigma_5(\sigma_1\sigma_4 + \sigma_2\sigma_3) + \sigma_3(\sigma_1\sigma_6 + \sigma_2\sigma_5) + \sigma_1(\sigma_3\sigma_6 + \sigma_4\sigma_5). \end{aligned}$$

5. Topological Motivation

In [4] we need to calculate certain equivariant singular bordism groups. Well known techniques, as established in [2] and [8], reduce the bordism calculation to the computation of the homology group

$$H^*(E(C) \times_C \mathfrak{F}, \mathbb{Z}_2) \tag{5.1}$$

where C is a cyclic group, $E(C)$ is the universal C–space (contractible with free action of C), and $\mathfrak{F}$ is a product of equivariant Grassmannians. In the calculation one uses the Leray–Serre spectral sequence of the fibration

$$\mathfrak{F} \to E(C) \times_C \mathfrak{F} \to B(C).$$

The local coefficient system $\mathcal{H}(\mathfrak{F}, \mathbb{Z}_2)$ turns out to be simple, so that

$$E_2^{p,q} = H^p(B(C), \mathbb{Z}_2) \otimes H^q(\mathfrak{F}, \mathbb{Z}_2).$$

The cohomology of each factor of $\mathfrak{F}$ is an algebra of symmetric functions ([1]). The cohomology of $B(C)$ can be found in [7]. The spectral sequence collapses at the E_2–level if the transgression is trivial, and this happens when the order of C is an odd number. So, in this case the calculation of (5.1) is easily completed.

If the order of C is twice an odd number, then C may act nontrivially on some of the factors of $\mathfrak{F}$. Denote the product of these factors by $\mathfrak{F}_b$. Consider the spectral sequence of the fibration

$$\mathfrak{F}_b \to E(C) \times_C \mathfrak{F}_b \to B(C).$$

The cohomology of $\mathfrak{F}_b$ is a tensor product of algebras of symmetric functions $\mathfrak{S}_{\mathcal{A}}$, as discussed earlier in the paper. The sequence $\mathcal{A}$ of indices depends on the factors of $\mathfrak{F}_b$. Let $x^{(n)}$ denote the nonzero element in $H^n(B(C), \mathbb{Z}_2)$. An element $u \in \mathfrak{S}_{\mathcal{A}}$ is a polynomial in variables of degree 1, and one shows that the transgression maps each of them to $x^{(2)}$. The algebra structure of the spectral sequence allows us to write down the formula for the differential at the E_2–level of the spectral sequence for a typical generator:

$$d_2(x^{(p)} \otimes u) = x^{(p+2)} \otimes \nabla u.$$

Here ∇ is the differential of the DGA discussed throughout the paper. This allows us to compute the E_3-term of the spectral sequence:

$$E_3^{p,q} \cong \begin{cases} \{x^{(p)} \otimes u \mid u \in Z_q(\mathfrak{S}_{\mathcal{A}})\} & \text{if } 0 \leq p \leq 1 \\ \{x^{(p)} \otimes u \mid u \in H_q(\mathfrak{S}_{\mathcal{A}})\} & \text{if } 2 \leq p. \end{cases} \tag{5.2}$$

The spectral sequence collapses at the E_3-level and we obtain the calculation of $H^*(E(C) \times_C \mathfrak{F}_b, \mathbb{Z}_2)$. In conclusion, our paper provides the calculation of $E_3^{p,q} = E_\infty^{p,q}$. The general form of the generators is needed in [3], and the precise calculation for the cycles is needed in [4], if at least one entry in the sequence $\mathcal{A}$ is odd.

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Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822, *visiting:* Department of Mathematics, Michigan State University, East Lansing, MI 48824

E-mail address: heiner@math.hawaii.edu

DigiPen Institute of Technology, Redmond, WA

E-mail address: jhanson@digipen.edu