STRONGLY ALGEBRAIC REALIZATION OF DIHEDRAL GROUP ACTIONS

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ABSTRACT. Let D_{2q} be the dihedral group with $2q$ elements and suppose that q is not divisible by 4. Let M be a closed smooth D_{2q}–manifold. Then there exists a nonsingular real algebraic D_{2q}–variety X which is equivariantly diffeomorphic to M and all D_{2q}–vector bundles over X are strongly algebraic.

1. INTRODUCTION

Suppose G is a compact Lie group and Ω is an orthogonal representation of G, also called a real G–module. A *real algebraic G–variety X*, see Definition 3.1 or [8], is a G–invariant common set of zeros of a finite collection of polynomials. The action on X is given as the restriction of the action on Ω. We use the term *nonsingular* with its classical meaning, see [24] or [4, Section 3.3]. If M is a closed smooth G–manifold and X is a nonsingular real algebraic G–variety that is equivariantly diffeomorphic to M, then we say that M is *algebraically realized* and that X is an *algebraic model* of M. We call X a *strongly algebraic* model of M if, in addition, all G–vector bundles over X are strongly algebraic. This means that the bundles are classified, up to equivariant homotopy, by equivariant entire rational maps to equivariant Grassmannians with their canonical algebraic structure, see Section 3.3. Existing results motivate

Conjecture 1.1. [9, p. 32] *Let G be a compact Lie group. Then every closed smooth G–manifold has a strongly algebraic model.*

Our main result verifies the conjecture in a special case:

Theorem 1.2. *Every closed smooth D_{2q} manifold, q not divisible by* 4*, has a strongly algebraic model.*

Nash [15] had asked whether every closed smooth manifold has an algebraic model, and this was confirmed by Tognoli [21]. Benedetti and Tognoli [3] showed that every closed smooth manifold has a strongly algebraic model. See also the work of Akbulut and King, [2] and [1].

We have confirmed Conjecture 1.1 in special cases. They include the case where G is the product of an odd order group and a 2–torus [9, Theorem B]

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and the case where G is cyclic, see [10], [7], and [11]. The new difficulty that we face in this paper is that Sylow 2 subgroups of dihedral groups are not central, and being able to deal with this adds credance to the conjecture. With the goal of proving Conjecture 1.1 in greater generality, we are developing ideas and tools to overcome difficulties in its proof.

To prove Proposition 2.3 we combine results from representation theory with extensions of the process of simplifying isotropy structures via blow–ups as applied in [23] and [11]. To prove Theorem 2.4, resp. its bordism theoretic reformulation as Theorem 3.7, we make creative use of the existing literature.

Algebraic realization problems translate to unoriented bordism problems. One expects 2–Sylow subgroups to play a crucial role. If $G = D_{2q}$ and q is not divisible by 4, then any of its 2–Sylow subgroups G_2 is at most of order 4, and for such groups Conjecture 1.1 has been verified. If 4 divides q, then G_2 is dihedral and has at least 8 elements. The proof of Conjecture 1.1 for this group may require extensive equivariant bordism computations. The concept of being iso-special (see Definition 2.1) will be inadequate. Locally we will have to accept three isotropy groups.

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2. Outline of Proof

Throughout, until the very last section, we assume that $G = D_{2q}$ and q is odd. In the last section we deduce Theorem 1.2 for $G = D_{2q}$ when 2 divides q, but not 4, from the case when q is odd.

Definition 2.1. A smooth G–manifold is said to be *iso–special* if locally the action has at most two isotropy groups. If there are two isotropy groups, say H and K with $K \subset H$, then we assume that $[H : K] = 2$ and the codimension of the H–fixed point set in the K–fixed point set is 1.

Blow–ups (see [18, p. 41] and [12, p. 175f]) were used in [23] to simplify the isotropy structure of a manifold, at least in case of abelian group actions. Our term iso–special corresponds to the term *nonsingular* in [23, Definition 18]. We will review the process of a blow–up in Section 4. If M is a smooth G–manifold and N is a G–invariant submanifold, then we denote the blow–up of M along N by $B(M, N)$. Previously we have shown:

Proposition 2.2. [11, Section 4] *If N and $B(M, N)$ have strongly algebraic models, then so does M.*

In this paper we will show the following two assertions:

Proposition 2.3. *Let M be a closed smooth D_{2q} manifold, q odd. Then there exists a finite sequence of equivariant blow-ups*

$$M_0 = M,\ M_1 = B(M_0, A_0),\ \ldots,\ M_k = B(M_{k-1}, A_{k-1}) \tag{2.1}$$

so that M_k and each A_i, $0 \le i \le k - 1$, are iso–special.

Theorem 2.4. *Every iso–special closed smooth D_{2q} manifold, q odd, has a strongly algebraic model.*

The definition of being iso–special is designed so that in combination with the blow-up procedure the proof of Conjecture 1.1 reduces to the special case of iso–special manifolds. Proposition 2.3 is proved in Section 4. In Section 3.5 we deduce Theorem 2.4 from a bordism theoretic assertion, Theorem 3.7. The proof of Theorem 3.7 occupies the later sections of the paper.

Proof of Theorem 1.2*, q odd.* Suppose M has a blow-up sequence as in (2.1). Theorem 2.4 tells us that M_k and A_{k-1} have strongly algebraic models. Proposition 2.2 tells us that M_{k-1} has a strongly algebraic model. Proceeding inductively, we conclude that M has a strongly algebraic model. □

3. Notation, Definitions, and Background Material

The dihedral group D_{2q} is generated by two elements that we call a and b, subject to the relations $a^2 = b^q = e$ and $aba = b^{-1}$. We write T for the subgroup generated by a and $\mathbb{Z}_q$ for the subgroup generated by b. If $q = rq'$ then a and b^r generate a subgroup of D_{2q} that we denote by $D_{2q'}$. It has the subgroup T generated by a and a subgroup $\mathbb{Z}_{q'}$ generated by b^r.

3.1. **Representations of the dihedral groups.**

The dihedral group is ambivalent: its elements are conjugate to their inverses. For finite groups being ambivalent is equivalent to all characters being real [13, p. 31]. The number of real, as well as complex, irreducible representations is equal to the number of conjugacy classes of elements of the group. The complex irreducible representations are complexifications of real irreducible representations. This, and more, follows from the Frobenius–Schur indicator, see [17, p. 90ff].

Specifically, if q is odd, then there are $\frac{q+3}{2}$ conjugacy classes of elements and irreducible representations, of which 2 are of dimension 1 and $\frac{q-1}{2}$ are of dimension 2. If q is even, then there are $\frac{q+6}{2}$ irreducible representations, of which 4 are of dimension 1 and $\frac{q-2}{2}$ are of dimension 2. They are described in [17, p. 37f].

Suppose $q > 1$ is odd. The trivial representation, denoted by $\mathbb{R}$, is one of the real irreducible representations of dimension 1. We denote the other one by $\mathbb{R}_-$. The element $a \in D_{2q}$ acts by multiplication with -1, while b acts trivially. The remaining real irreducible representations are of dimension 2. The generators act by multiplication with the matrices

$$\theta(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \& \quad \theta(b) = \begin{pmatrix} \cos\left(\frac{2\pi j}{q}\right) & \sin\left(\frac{2\pi j}{q}\right) \\ -\sin\left(\frac{2\pi j}{q}\right) & \cos\left(\frac{2\pi j}{q}\right) \end{pmatrix} \tag{3.1}$$

for $1 \leq j \leq (q-1)/2$.

3.2. **Real algebraic varieties and entire rational maps.** Let G be a compact Lie group and Ω an orthogonal representation of G. We think of an orthogonal representation as an underlying Euclidean space $\mathbb{R}^n$ together with an action of G via orthogonal maps.

Definition 3.1. A *real algebraic G–variety* is a G–invariant, common set of zeros of a finite set of polynomials $p_1, \ldots, p_m : \Omega \to \mathbb{R}$:

$$V = \{x \in \Omega \mid p_1(x) = \cdots = p_m(x) = 0\}.$$

The action of G on Ω restricts to an action on V. We use the Euclidean topology on varieties and the term *'nonsingular'* with its standard meaning [24].

Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ be real algebraic varieties. A map $f : V \to W$ is said to be *regular* if it extends to a map $F : \mathbb{R}^n \to \mathbb{R}^m$ such that each of its coordinates F_i (i.e., $F_i = \delta_i \circ F : \mathbb{R}^n \to \mathbb{R}$ where $\delta_i : \mathbb{R}^m \to \mathbb{R}$ is the projection on the i–th coordinate) is a polynomial. We say that f is *entire rational* if there are regular maps $p : \mathbb{R}^n \to \mathbb{R}^m$ and $q : \mathbb{R}^n \to \mathbb{R}$, such that $f = p/q$ on V and q does not vanish anywhere on V.

These concepts generalize naturally to the equivariant setting.

3.3. **Grassmannians & classification of vector bundles.** A good reference is [4, §3.4]. Let Λ stand for $\mathbb{R}$ or $\mathbb{C}$. Let Ξ be a representation of G over Λ. Its underlying space is Λ^n for some n. We assume that the action of G preserves the standard bilinear form on Λ^n. Let $\mathrm{End}_\Lambda(\Xi)$ denote the set of endomorphisms of Ξ over Λ. It is a representation of G with the action given by

$$G \times \mathrm{End}_\Lambda(\Xi) \to \mathrm{End}_\Lambda(\Xi) \quad \text{with} \quad (g, L) \mapsto gLg^{-1}.$$

Let d be a natural number. We set

$$\begin{aligned}
(3.2)\quad G_\Lambda(\Xi, d) &= \{L \in \mathrm{End}_\Lambda(\Xi) \mid L^2 = L,\ L^* = L, \operatorname{trace} L = d\} \\
(3.3)\quad E_\Lambda(\Xi, d) &= \{(L, u) \in \mathrm{End}_\Lambda(\Xi) \times \Xi \mid L \in G_\Lambda(\Xi, d),\ Lu = u\} \\
(3.4)\quad \gamma_\Lambda(\Xi, d) &= (p : E_\Lambda(\Xi, d) \to G_\Lambda(\Xi, d))
\end{aligned}$$

Here L^* denotes the adjoint of L. If one chooses an orthogonal (resp., unitary) basis of Ξ, then $\mathrm{End}_\Lambda(\Xi)$ is canonically identified with the set of $n \times n$ matrices, and L^* is obtained by transposing L and conjugating its entries. This description specifies $G_\Lambda(\Xi, d)$ and $E_\Lambda(\Xi, d)$ as real algebraic G–varieties. These varieties are nonsingular. The map in (3.4) is projection on the first factor, and $\gamma_\Lambda(\Xi, d)$ is an equivariant vector bundle. Its base and total space are nonsingular real algebraic varieties, and the projection map is regular, hence entire rational.

Proposition 3.2. *The variety $G_\Lambda(\Xi, d)$ is the Grassmannian consisting of real (resp. complex) subspaces of Ξ of real (resp. complex) dimension d.*

Proof. There is a bijection between subspaces of Ξ and orthogonal (resp. unitary) projections. To a projection one associates its image. $\square$

We may take larger and larger representations Ξ of G and form a direct limit. At the same time, we can take direct limits of $G_\Lambda(\Xi, d)$, $E_\Lambda(\Xi, d)$ and $\gamma_\Lambda(\Xi, d)$. We call Ξ a universe if it contains each irreducible representation of G an infinite number of times. If Ξ is a universe, then $G_\Lambda(\Xi, d)$ is a classifying space for G–vector bundles of dimension d over nice space, like finite G–CW complexes. There is a $1-1$ correspondence between isomorphism classes of G–vector bundles of dimension d over a G–CW complex X and equivariant homotopy classes from X to $G_\mathbb{R}(\Xi, d)$. The non–equivariant proof in [14, §5] generalizes easily. See also [16, §2] and [22].

In the context of our discussion of strongly algebraic vector bundles, we like $G_\mathbb{R}(\Xi, d)$ to be a variety, which is the case as long as Ξ is of finite dimension. Depending on the bundle classification problem, one may get away using a finite dimensional representation Ξ. E.g., for any G–CW complex X of dimension k, $G_\mathbb{R}(\Xi, d)$ classifies G–vector bundles of dimension d over X if each irreducible representation of G occurs with multiplicity at least $k + d + 1$ in Ξ.

3.4. **Strongly algebraic vector bundles.** In our setting the preferred concept of a vector bundle is the one of a strongly algebraic vector bundle. See also [4, §12.1]. One has this notion with real, $\Lambda = \mathbb{R}$, as well as complex, $\Lambda = \mathbb{C}$, coefficients.

Definition 3.3. A *strongly algebraic G–vector bundle* over a real algebraic G–variety is a bundle whose classifying map to $G_\Lambda(\Xi, d)$ is equivariantly homotopic to an equivariant entire rational map.

Occasionally, we think of G–vector bundles as equivariant maps to a Grassmannian $G_\Lambda(\Xi, d)$. Then we need to allow stabilization of Ξ.

3.5. **Results from the literature.** We will use:

Proposition 3.4. [9, Proposition 2.13] *Let G be a compact Lie group and M a closed smooth G–manifold. Suppose that for every finite collection of G–vector bundles over M there is an algebraic model X, such that each bundle in this collection, pulled back over X, is strongly algebraic. Then M has an algebraic model over which all G–vector bundles are strongly algebraic.*

Suppose Y is a nonsingular real algebraic G–variety. It is convenient to call $\mu : X \to Y$ an *algebraic map* if X is a nonsingular real algebraic G–variety and μ is equivariant and entire rational. Suppose M is a closed smooth G–manifold and $f : M \to Y$ is equivariant. We call an algebraic map $\mu : X \to Y$ an *algebraic model* of $f : M \to Y$ if there is an equivariant diffeomorphism $\Phi : M \to X$ so that f is equivariantly homotopic to $\mu \circ \Phi$.

Theorem 3.5. [9, Theorem C] *Let G be a compact Lie group. An equivariant map from a closed smooth G–manifold to a nonsingular real algebraic*

G–variety has an algebraic model if and only if its cobordism class has an algebraic representative.

3.6. **Bordism formulation.** Consider a finite product of Grassmannians:

$$\mathfrak{G} = G_{\mathbb{R}}(\Xi, d_1) \times \cdots \times G_{\mathbb{R}}(\Xi, d_k) \tag{3.5}$$

where Ξ is a sufficiently large representation of G and $d_1, \ldots, d_k$ is a sequence of natural numbers. Such a space is used as a classifying space for a collection of k bundles.

Let $\mathcal{S}(G)$ be the set of all subgroups of G and $\mathcal{H} \subseteq \mathcal{S}(G)$. We say that a G–manifold M is of *type* $\mathcal{H}$ if the isotropy groups G_x belong to $\mathcal{H}$ for all $x \in M$. Recall that $G_{gx} = gG_xg^{-1}$, so we should assume that $\mathcal{H}$ is invariant under conjugation. To avoid listing all elements in a conjugacy class:

Notation 3.6. We write $\mathcal{K}^\bullet$ for the closure of $\mathcal{K} \subseteq \mathcal{S}(G)$ under conjugation.

We adopt the notation used in [18, §2] and [19]. We use $\mathcal{N}_k^G(Y)$ to denote G equivariant unoriented bordism classes of equivariant maps $f : M \to Y$ from closed G manifolds of dimension k to a G space Y. Given a family $\mathcal{F}$ of subgroups of G we write $\mathcal{N}_k^G[\mathcal{F}](Y)$ to indicate that the isotropy groups for the domain M of the map are assumed to be in $\mathcal{F}$. The same restriction on the isotropy groups of the domain applies to a bordism between two maps. In the iso–special case we add a subscript c and write $\mathcal{N}_{r,c}^G[\{H,K\}^\bullet](Z)$ to indicate that the codimension of the H–fixed point set in the K–fixed point is one. Eventually we will prove:

Theorem 3.7. *Let $G = D_{2q}$ be the dihedral group, where q is odd, and $\mathfrak{G}$ as in* (3.5).

(1) *If H is a subgroup of G, then all classes in $\mathcal{N}_*^G[\{H\}^\bullet](\mathfrak{G})$ have algebraic representatives.*
(2) *If H and K are two subgroups of G and $[H : K] = 2$, then all classses in $\mathcal{N}_{r,c}^G[\{H,K\}^\bullet](\mathfrak{G})$ have algebraic representatives.*

Deduce Theorem 2.4 *from Theorem* 3.7. Let M be a closed, smooth, and iso–special D_{2q} manifold. Consider a finite collection $\xi_1, \ldots, \xi_k$ of D_{2q} vector bundles over M. Classify it by a map χ into a product of Grassmannians $\mathfrak{G}$ as in (3.5). Then $\chi = \chi_1 \times \cdots \times \chi_k$ where the individual χ_j classify the bundles ξ_j. According to Theorem 3.7, the bordism class of $\chi : M \to \mathfrak{G}$ has an algebraic representative. According to Theorem 3.5, there is an algebraic model $\widehat{\chi} : X \to \mathfrak{G}$ for $\chi : M \to \mathfrak{G}$. Then $\widehat{\chi} = \widehat{\chi}_1 \times \cdots \times \widehat{\chi}_k$. The $\widehat{\chi}_j$ are entire rational, up to equivariant homotopy, and they classify strongly algebraic bundles. Being able to do this for every collection $\xi_1, \ldots, \xi_k$ of D_{2q} vector bundles over M implies, according to Proposition 3.4, that M has a strongly algebraic model. $\square$

4. Blow–ups

In this section we recall the definition of blow–ups and study their effect on the isotropy structure of a G manifold in the special case where $G = D_{2q}$ and q is odd.

Construction 4.1. Let M be a closed smooth G–manifold with a collection $\xi_1, \ldots, \xi_k$ of G–vector bundles over it. Let N be a G–invariant submanifold of M with normal bundle ν. Denote the trivial representation by $\mathbb{R}$ and the product bundle with fibre $\mathbb{R}$ by $\underline{\mathbb{R}}$. We may restrict $\xi_1, \ldots \xi_k$ over N and then use the projection $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}}) \to N$ to pull the bundles back over $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}})$. The resulting bundles are called $\overline{\xi_1}, \ldots, \overline{\xi_k}$.

We may identify $(M, \xi_1, \ldots, \xi_k)$ and $(\mathbb{R}P(\nu \oplus \underline{\mathbb{R}}), \overline{\xi_1}, \ldots, \overline{\xi_k})$ along a neighbourhood of N that is contained in M and $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}})$. The result is commonly called the *blow-up* of $(M, \xi_1, \ldots, \xi_k)$ along N. It is denoted by $B((M, \xi_1, \ldots, \xi_k), N)$. By construction,

$$B((M, \xi_1, \ldots, \xi_k), N) \sim (M, \xi_1, \ldots, \xi_k) \sqcup (\mathbb{R}P(\nu \oplus \underline{\mathbb{R}}), \overline{\xi_1}, \ldots, \overline{\xi_k}), \tag{4.1}$$

where $\sim$ indicates an equivariant cobordism that incorporates bundle data.

Proof of Proposition 2.3. The argument is inductive. We induct over the partial order on the divisors of q. First we blow up components of the $\mathbb{Z}_p$ fixed point set that do not contain points that are fixed under D_{2p}. Secondly we blow up components of the $\mathbb{Z}_p$ fixed point set that contain points that are fixed under D_{2p}.

(α) Let A_0 be the union of those components of $M^{\mathbb{Z}_q}$ that contain only points of isotropy type $\mathbb{Z}_q$. Clearly A_0 is a D_{2q}–invariant submanifold of M, and having only one isotropy type it is iso–special. Certainly, A_0 can have components of different dimensions. Let $A_0^{=0}$ be the part of A_0 that is of codimension 0 in M. In particular, A_0 consists of a D_{2q} invariant collection of components of M. For this part of M the assertion of Proposition 2.3 holds. We can exclude it from further consideration. Notationally it is easier to set it aside.

Let $A_0^{>0}$ be the union of those components of A_0 that are of positive codimension in M. Blow up $M \setminus A_0^{=0}$, the remaining part of M, along $A_0^{>0}$. Set $M_1 = B(M \setminus A_0^{=0}, A_0^{>0})$. Let ν_x stand for the normal slice at a point $x \in A_0^{>0}$. Because $\mathbb{Z}_q$ is of odd order and ν_x does not have the trivial representation $\mathbb{R}$ as a summand, there is no real $\mathbb{Z}_q$ invariant line in ν_x. Hence $\mathbb{R}P(\nu_x \oplus \mathbb{R})$ has exactly one $\mathbb{Z}_q$ fixed point. After the identification of $M \setminus A_0^{=0}$ and $\mathbb{R}P(\nu \oplus \mathbb{R})$ along a common neighborhood of $A_0^{>0}$ the common $\mathbb{Z}_q$ fixed set $A_0^{>0}$ has been eliminated. Our blow–up removes $A_0^{>0}$ from the $\mathbb{Z}_q$ fixed point set. Any remaining points of isotropy type $\mathbb{Z}_q$ belong to components that contain D_{2q}–fixed points.

Let q' be a maximal non–trivial proper divisor of q. We repeat above process with $\mathbb{Z}_q$ replaced by $\mathbb{Z}_{q'}$. Let A_1 be the union of the components of the $\mathbb{Z}_{q'}$ fixed point set, all of whose points are of isotropy type $\mathbb{Z}_{q'}$. Note that A_1 is iso–special. As before, set the codimension 0 components $A_1^{=0}$ aside. Blow up $M_1 \setminus A_1^{=0}$ along $A_1^{>0}$, the components of A_1 of positive codimension in M_1. The blow–up removes $A_1^{>0}$ from the $\mathbb{Z}_{q'}$ fixed point set. Any remaining points of isotropy type $\mathbb{Z}_{q'}$ belong to components that also contain $D_{2q'}$–fixed points.

We continue this process for all non–trivial divisors of q, partially ordered by divisibility, and end up with a manifold M_k, for some k. For some $1 \neq r \mid q$ we may have set aside a manifold B_r all of whose points have isotropy type $\mathbb{Z}_r$. Denote their union by B. We have a blow–up sequence that starts with M and ends with $M_k \sqcup B$, and all blow–ups are along iso–special submanifolds.

(β) We create a second blow–up sequence, starting with $\overline{M} = M_k$. Suppose that $\overline{M}^{\mathbb{Z}_q} \neq \emptyset$. Then $\overline{M}^{D_{2q}} \neq \emptyset$. Set $\overline{A}_0 = \overline{M}^{D_{2q}}$. This manifold is D_{2q}–invariant and all points have isotropy type D_{2q}, hence it is iso–special. Blow up along $\overline{A}_0$. We will show that $B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_q}$ is iso–special. Locally we have isotropy groups D_{2q} and $\mathbb{Z}_q$, and one is of index two in the other. We have to show that the codimension of the D_{2q} fixed set in the $\mathbb{Z}_q$ fixed set is 1.

The normal slice ν_x to $\overline{A}_0$ in $\overline{M}$ at $x \in \overline{A}_0$ is a representation of D_{2q}. It has no trivial irreducible representation as summand. It is of the form $\alpha\mathbb{R}_- \oplus \Omega$. As in Section 3.1, $\mathbb{R}_-$ is the non–trivial one dimensional irreducible representation of D_{2q}. We denote its multiplicity in ν_x by α. The two dimensional irreducible representations of D_{2q} were described in (3.1). Various values for j, reflecting different angles of rotation, may occur. We gather those that occur as part of ν_x, with their multiplicites, in the summand Ω. Note, as q is odd there are no D_{2q} invariant real lines in Ω.

A D_{2q} invariant real line in $\alpha\mathbb{R}_- \oplus \Omega \oplus \mathbb{R}$ is a line in $\alpha\mathbb{R}_-$ or it is the line $\mathbb{R}$. Use the symbol $\approx$ to denote a diffeomorphism. Then

$$\mathbb{R}P(\nu_x \oplus \mathbb{R})^{D_{2q}} = \mathbb{R}P(\alpha\mathbb{R}_- \oplus 0 \oplus 0) \sqcup \mathbb{R}P(0 \oplus 0 \oplus \mathbb{R}) \approx \mathbb{R}P^{\alpha-1} \sqcup \mathbb{R}P^0.$$

Any real line in $\alpha\mathbb{R}_- \oplus 0 \oplus \mathbb{R}$ is fixed under the action of $\mathbb{Z}_q$. Hence

$$\mathbb{R}P(\nu_x \oplus \mathbb{R})^{\mathbb{Z}_q} = \mathbb{R}P(\alpha\mathbb{R}_- \oplus 0 \oplus \mathbb{R}) \approx \mathbb{R}P^{\alpha}.$$

We can compute codimensions (cd) in the manifold by looking at codimensions in normal slices:

$$\mathrm{cd}(B(\overline{M}, \overline{A}_0)^{D_{2q}}, B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_q}) = \mathrm{cd}(\mathbb{R}P(\alpha\mathbb{R}_- \oplus 0 \oplus 0), \mathbb{R}P(\nu_x \oplus \mathbb{R})^{\mathbb{Z}_q}) = 1.$$

Having checked the codimension conditions, we deduce that $B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_q}$ is iso–special.

Blow up along $B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_q}$. We obtain

$$\overline{M}_1 = B(B(\overline{M}, \overline{A}_0), B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_p}).$$

The normal slice ν_x to $B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_p}$ in $B(\overline{M}, \overline{A}_0)$ at a point $x \in B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_p}$ is a sum of irreducible representations as in (3.1). There is a single $\mathbb{Z}_q$

invariant line in $\nu_x \oplus \mathbb{R}$ and $\mathbb{R}P(\nu_x \oplus \mathbb{R})$ has a single $\mathbb{Z}_q$ fixed point. The latter disappears in the blow–up.

In summary, this pair of two blow–ups, each along an iso–special submanifold, removes the $\mathbb{Z}_q$ fixed points from $\overline{M}$.

As before, we perform this step for all non–trivial divisors of q, and we proceed inductively following the partial order on the set of divisors of q. Eventually, after repeated blow–ups along iso–special submanifolds, we end up with a manifold $\overline{M}_s$ whose isotropy types are the trivial group and/or the order 2 subgroup T of D_{2q}.

If 1 and T are the only isotropy types of the action on $\overline{M}_s$, we may still have to arrange the codimension 1 condition for $\overline{M}_s$ to be an iso–special manifold. To achieve this we blow up $\overline{M}_s$ along the iso–special submanifold $\overline{M}_s^T$. This is a special case of the first blow–up in (β) with $q = 1$ and $\Omega = 0$. The normal fibre ν_x is a representation of T. As we computed earlier, $\mathrm{cd}(B(\overline{M}_s, \overline{M}_s^T)^T, B(\overline{M}_s, \overline{M}_s^T)) = 1$, so that $B(\overline{M}_s, \overline{M}_s^T)$ is iso–special.

Combined with the first sequence of blow–ups, we have a sequence of blow–ups along iso–special submanifolds that starts with M and terminates with the iso–special manifold $\overline{M}_s \sqcup B$. The proposition asserted that this is possible, and we verified it. □

5. Proof of Theorem 3.7 (1): The one Isotropy Type Case

The assertion of Theorem 3.7 is that iso–special $G = D_{2q}$ manifolds (q odd) have strongly algebraic models, and in this section we prove the assertion if the manifold has only one isotropy type.

Some cases are easy to dispose of. If the single isotropy group is the trivial group, then D_{2q} acts freely and the assertion has been shown as Theorem B (2) in [9]. The same reference covers the case where D_{2q} acts trivially.

Next, suppose that the closed smooth D_{2q} manifold N has the single isotropy type $(D_{2q'})$. The index of $D_{2q'}$ in its normalizer is q/q', which is odd. Theorem C in [20] tells us that N together with the set of all D_{2q} vector bundles over it can be algebraically realized. Our expression is that the D_{2q} manifold N has a strongly algebraic model. Hence the assertion of the theorem is proved in this case as well. Setting $q' = 1$ this includes the case when $D_{2q'} = T$.

In our final case, suppose that M is a closed D_{2q} manifold and that $\mathbb{Z}_{q'}$ is its only isotropy group. Necessarily $1 \neq q'$ divides q. We combine ideas from [5, Section 2] and [8]. Applying Proposition 3.4 we need to show: Given any finite collection $\{\xi_1, \ldots, \xi_m\}$ of G vector bundles over M, there is an algebraic model X of M so that these G vector bundles pull back to strongly algebraic G vector bundles over X.

Let $\mathcal{E}$ be an indexing set for the irreducible representations of $\mathbb{Z}_{q'}$. The irreducible representation associated with $\epsilon \in \mathcal{E}$ is denoted by α_ϵ. Consider one bundle ξ in the collection. For each $\epsilon \in \mathcal{E}$ there is a unique largest D_{2q} subbundle $\xi(\alpha_\epsilon)$ of ξ whose fibre is a multiple of α_ϵ. This uses the fact that irreducible representations of $\mathbb{Z}_{q'}$ are restrictions of representations of D_{2q}. Following [5] there should be one subbundle $\xi(\alpha_\epsilon)$ for a each conjugacy class of irreducible representations. But, the conjugation action of D_{2q} on the set of real irreducible D_{2q} representations is trivial. Thus, in our context, a bundle is α_ϵ–isotypical in the sense of [5] if the fibre of the bundle is a multiple of α_ϵ.

There is a direct sum decomposition $\xi = \bigoplus_{\epsilon\in\mathcal{E}} \xi(\alpha_\epsilon)$. The direct sum of strongly algebraic bundles is strongly algebraic (see [9, Proposition 2.11]). That means, to prove Theorem 3.7 in our special case, we may assume that each of the bundles ξ_i in our collection is isotypical.

Let $\mathrm{Vect}_G(M)$ stand for the semi–group of G vector bundles over M and $\mathrm{Vect}_G(M, \alpha_\epsilon)$ for the sub–semi–group of α_ϵ–isotypical bundles. There is a G vector bundle L over M whose fibre is α_ϵ. Suppose now that ξ is α_ϵ–isotypical. The assignment that sends ξ to $\mathrm{Hom}_{\mathbb{Z}_{q'}}(L, \xi)$ defines an isomorphism $\mathrm{Vect}_G(M, \alpha_\epsilon) \longrightarrow \mathrm{Vect}_{G/\mathbb{Z}_{q'}}(M)$ (see [5, Lemma 2.2]). Its inverse sends a bundle $\eta \in \mathrm{Vect}_{G/\mathbb{Z}_{q'}}(M)$ to $L \otimes \eta$. Depending on the type of α_ϵ, the tensor product will be over $\mathbb{R}$ or $\mathbb{C}$. The tensor product of strongly algebraic bundles is strongly algebraic (see [9, Proposition 2.11]).

The group $G/\mathbb{Z}_{q'}$ acts freely on M and the bundles $\mathrm{Hom}_{\mathbb{Z}_{q'}}(L, \xi)$, resp. η, in the previous paragraph are $G/\mathbb{Z}_{q'}$ bundles over M. Our reduction says that we only need to prove the assertion of Theorem 3.7 in case the group acts freely on M. The latter holds according to [9, Theorem B (2)].

With this we have proved Theorem 3.7 if the action of D_{2q} has a single isotropy type.

6. Proof of Theorem 3.7 (2): The two Isotropy Type Case

In the following we will make use of the exact Conner–Floyd sequences. They were established in [6, §5]. Earlier, in the setup for Theorem 3.7, we recalled basic bordism theoretic notation. We need a little more. Given two families of subgroups, $\mathcal{F}$ and $\mathcal{F}'$ with $\mathcal{F}' \subseteq \mathcal{F}$, there is a relative bordism group $\mathcal{N}_k^G[\mathcal{F}, \mathcal{F}'](Y)$ [18]. In this setting we allow the domain M to be a compact manifold with boundary and the isotropy groups of ∂M are assumed to belong of $\mathcal{F}'$. Two maps are bordant if the $\mathcal{F}$ fixed points together with their normal data are bordant.

Suppose that there are two isotropy types. As before we denote them by H and K. It is assumed that $[H : K] = 2$. The order of G is twice an odd number, so it follows that K is of odd order and normal in $G = D_{2q}$. We abbreviate $\widetilde{G} := G/K$. We have the following commutative diagram of Conner–Floyd sequences. All bordism groups in the diagram should have $\mathfrak{G}$ as codomain. Due to the restrictions on the isotropy groups we may replace

$\mathfrak{G}$ by $\mathfrak{G}^K$. For reasons of space, we suppress this codomain altogether.

$$\begin{tikzcd}
\mathcal{N}_*^G[\{K\}] \arrow[r] \arrow[d, "\cong"'] & \mathcal{N}_{*c}^G[\{H,K\}^\bullet] \arrow[r, "j_G"] \arrow[d, "\cong"'] & \mathcal{N}_{*c}^G[\{H,K\}^\bullet,\{K\}] \arrow[r, "\partial_G"] \arrow[d, "\cong"'] & \mathcal{N}_{*-1}^G[\{K\}] \arrow[d, "\cong"'] \\
\mathcal{N}_*^{\widetilde{G}}[\{1\}] \arrow[r] & \mathcal{N}_{*c}^{\widetilde{G}}[\{T,1\}^\bullet] \arrow[r, "j_\sim"] & \mathcal{N}_{*c}^{\widetilde{G}}[\{T,1\}^\bullet,\{1\}] \arrow[r, "\partial_\sim"] & \mathcal{N}_{*-1}^{\widetilde{G}}[\{1\}] \\
\mathcal{N}_*^T[\{1\}] \arrow[r] \arrow[u, "\mathrm{Ind}'"] & \mathcal{N}_{*c}^T[\{T,1\}] \arrow[r, "j_T"] \arrow[u, "\mathrm{Ind}''"] & \mathcal{N}_{*c}^T[\{T,1\},\{1\}] \arrow[r, "\partial_T"] \arrow[u, "\mathrm{Ind}", "\cong"'] & \mathcal{N}_{*-1}^T[\{1\}] \arrow[u, "\mathrm{Ind}'"]
\end{tikzcd}$$

In the transition from the first to the second row we divide out K, the ineffective part of the action, and observe that $H/K \cong T$. The vertical maps are natural isomorphisms.

In the transition from the third to the second row we apply induction. If M is a T manifold, then $\mathrm{Ind}_T^{\widetilde{G}} M = \widetilde{G} \times_T M$ is a $\widetilde{G}$ manifold. It consists of equivalence classes of pairs $(g,x) \in \widetilde{G} \times M$, where $(gt,x) \sim (g,tx)$ when $t \in T$. If Y is a $\widetilde{G}$ space and $f : M \to Y$ is T equivariant, then $\mathrm{Ind}_T^{\widetilde{G}} f : \mathrm{Ind}_T^{\widetilde{G}} M \to Y$ is defined by setting $(\mathrm{Ind}_T^{\widetilde{G}} f)[g,x] = gf(x)$. Functoriality implies that the squares commute.

Restricting $\widetilde{G}$ actions and $\widetilde{G}$ equivariance to T actions and T equivariance defines the map

$$\mathrm{Res}_{\widetilde{G}}^T : \mathcal{N}_{*c}^{\widetilde{G}}[\{T,1\}^\bullet,\{1\}](\mathfrak{G}^K) \longrightarrow \mathcal{N}_{*c}^T[\{T,1\},\{1\}](\mathfrak{G}^K).$$

We study the outcome. Set $T^g = gTg^{-1}$. According to the definition of the bordism group, a representative of a class in $\mathcal{N}_{*c}^{\widetilde{G}}[\{T,1\}^\bullet,\{1\}](\mathfrak{G}^K)$ is completely determined by its restriction to a neighbourhood of the T^g fixed point sets for all $g \in \widetilde{G}$. In $\mathcal{N}_{*c}^T[\{T,1\},\{1\}](\mathfrak{G}^K)$, once we restricted the action of $\widetilde{G}$ to one of T, the class is completely determined by its restriction to a neighbourhood of the T fixed point set.

The induction map $\mathrm{Ind}_T^{\widetilde{G}} = \widetilde{G} \times_T$ restores the neighbourboods of all the T^g fixed point sets. By construction, $\mathrm{Res}_{\widetilde{G}}^T$ and $\mathrm{Ind}_T^{\widetilde{G}}$ are inverses of each other.

In [9, Proposition 5.2] we proved that

$$\mathrm{Ind}' : \mathcal{N}_*^T[\{1\}](\mathfrak{G}^K) \longrightarrow \mathcal{N}_*^{\widetilde{G}}[\{1\}](\mathfrak{G}^K)$$

is onto. The Five–Lemma implies that Ind'' is onto.

Classes in $\mathcal{N}_{*c}^T[\{T,1\}](\mathfrak{G}^K)$ have algebraic representatives, see [9, Proposition F]. If we apply $\mathrm{Ind}_T^{\widetilde{G}}$ we obtain algebraic representatives of the classes in $\mathcal{N}_{*c}^{\widetilde{G}}[\{T,1\}^\bullet](\mathfrak{G}^K)$. This tells us that all classes in $\mathcal{N}_{*c}^{\widetilde{G}}[\{T,1\}^\bullet](\mathfrak{G}^K)$ and in $\mathcal{N}_{*c}^G[\{H,K\}](\mathfrak{G}^K)$ have algebraic representatives. We have completed the proof of Theorem 3.7 also in this second case.

7. Proof of Theorem 1.2, $q = 2r$ and r odd

Let $G = D_{2q}$ where $q = 2r$ and r is odd. Let $a, b \in G$ be as in the beginning of Section 3. The element $\tau = b^r$ is central in G and of order 2. Set $G' = G/\langle\tau\rangle = D_{2r}$. In fact, D_{2q} is a direct product of $\langle\tau\rangle$ and $G' = D_{2r}$.

Let M be a closed smooth G manifold. According to [9, Proposition F] M has a strongly algebraic model if and only if the G manifold $N = M^\tau$ has such a model. We will show the latter.

As τ acts trivially on N, there is an induced action of G' on N. We have seen that, as a G' manifold, N has a strongly algebraic model. Call it X. At the same time X is a G equivariant algebraic model of N.

Let ξ be a G vector bundle over X. The action of τ on ξ induces one on the fibres ξ_x of the bundle, $x \in N$. Each fibre, as well as the bundle, decomposes as a direct sum of the fixed point set and its orthogonal complement, on which τ acts by multiplication with -1. We write $\xi = \xi^+ \oplus \xi^-$.

As τ acts trivially on ξ^+ this bundle is actually a G' bundle and strongly algebraic, also as a G vector bundle.

We have a real 1 dimensional representation σ of G. The element a acts trivially, while b and τ act by multiplication with -1. Clearly $\sigma \otimes_{\mathbb{R}} \sigma = \mathbb{R}$. Let $\underline{\sigma}$ be the product bundle with fibre σ. This bundle is classified by a constant map, which is entire rational. Hence the bundle is strongly algebraic. The action of τ on $\xi^- \otimes_{\mathbb{R}} \underline{\sigma}$ is trivial and this G' bundle is strongly algebraic, also as a G vector bundle. The tensor product of strongly algebraic bundles is strongly algebraic, and so is $\xi^- \otimes_{\mathbb{R}} \underline{\sigma} \otimes_{\mathbb{R}} \underline{\sigma} = \xi^-$.

The direct sum of strongly algebraic bundles is strongly algebraic, and so is $\xi = \xi^+ \oplus \xi^-$. Hence X is a strongly algebraic model of N as a G manifold. This is what we needed to show.

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