ALGEBRAIC REALIZATION OF MANIFOLDS WITH GROUP ACTIONS

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Abstract. Let $G$ be a compact Lie group and $M$ a closed smooth $G$ manifold. With some restrictions on $G$ or on the isotropy groups of the action we show that $M$ is algebraically realized, i.e., there exists a non-singular real algebraic $G$ variety which is equivariantly diffeomorphic to $M$.

1. Introduction

A central question in real algebraic geometry is whether a given smooth situation can be realized algebraically. Results of Seifert [Se] and a program established by Nash [N] motivated the

Nash–Tognoli Theorem$^1$ [T]. Every closed smooth manifold (i.e., a compact manifold without boundary) is diffeomorphic to a non-singular real algebraic variety.

Subsequent work of Akbulut and King [AK1], [AK2] did not only provide a more conceptual proof of the Nash Conjecture, but it also opened the way to study the corresponding problem for manifolds which carry an additional structure. A very natural question of this kind is whether a vector bundle over a closed smooth manifold can be realized algebraically. The exact formulation and solution of this problem are due to Benedetti and Tognoli [BT]. Survey articles by Ivanov [I] and King [K2] and a book by Bochnak, Coste, and Roy [BCR] describe many of the results and questions in this program. It is important to observe that there are non-compact manifolds which are not diffeomorphic to real algebraic varieties (as examples one may use the manifolds constructed by Siebenmann in his thesis [Si], or even $\mathbb{R} \setminus \mathbb{Z}$), and that the algebraic structure on a closed smooth manifold is not unique. On any manifold of dimension $\geq 1$ there are uncountably many such structures [BK1], [BK2].

In our present work we impose as additional structure an action of a compact Lie group $G$. This may be viewed as a special case of the program described so far, as well as a

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$^1$The question which is answered by this theorem has often been referred to as ‘Nash Conjecture’, e.g., see [T], [K1]. As Akbulut and King point out, this name is not supported by Nash’s paper, and recently they started to use the word ‘Nash Conjecture’ with a different meaning [AK3], [AK4].
generalization of the program in its entirety. Substantial new problems arise, and we solve some of them in this paper. We will say that a smooth $G$ manifold is \emph{algebraically realized} if it is equivariantly diffeomorphic to a non-singular real algebraic $G$ variety (see Definition 1.1). The focus of this paper is the

\textbf{Algebraic Realization Conjecture.} \textit{Let $G$ be a compact Lie group. Every closed smooth $G$ manifold is algebraically realized.}\(^2\)

The following theorem confirms this conjecture for groups of odd order and for actions which satisfy some assumptions on the isotropy groups. An action of a group $G$ on a space $M$ is said to be \textit{semifree} if the isotropy group $G_x = \{g \in G \mid gx = x\}$ of $x$ is $\{1\}$ or $G$ for all $x \in M$.

\textbf{Theorem A.} \textit{Let $G$ be a compact Lie group. A closed smooth $G$ manifold $M$ is algebraically realized if one of the following assumptions holds.}

1. $G$ is of odd order.
2. The action of $G$ on $M$ is semifree.

For an exact understanding of the above we still have to define the concept of a real algebraic $G$ variety. Let $G$ be a compact Lie group and $\Omega$ an orthogonal representation of $G$. Here we think of an orthogonal representation as an underlying Euclidean space $\mathbb{R}^n$ together with an action of $G$ via orthogonal maps.

\textbf{Definition 1.1.} \textit{A real algebraic $G$ variety is the set of common zeros of polynomials $p_1, \ldots, p_m : \Omega \to \mathbb{R}$,}

$$V = \{x \in \Omega \mid p_1(x) = \cdots = p_m(x) = 0\},$$

\textit{which is invariant under the action of $G$. We also say that $G$ acts real algebraically on $V$.}

More exactly, such a variety should be called an affine variety, but as we are not considering any other varieties, such as projective ones, we omit this adjective throughout. An equivalent definition using algebraic groups and actions is given in Section 9.

There are two principles which, taken together, provide the proof of Theorem A. One of them is an approximation technique which reduces the algebraic realization problem to a bordism problem. The other one shows that, up to bordism, the given situation has an algebraic representative. We explain both in our context.

Two closed smooth $G$ manifolds $M_1$ and $M_2$ are said to be \textit{equivariantly cobordant} if there exists a compact smooth $G$ manifold whose boundary is the disjoint union of $M_1$ and $M_2$. The approximation result which we use in this paper was obtained in joint work with Ted Petrie.

\(^2\)Throughout this paper we follow the convention that a manifold has a dimension, i. e., all of its components have the same dimension. In part, this is motivated by the same convention for non-singular varieties. This conjecture and the results of this paper can be applied to manifolds with components of different dimension by applying them separately to the components of a fixed dimension, and then taking a disjoint union.
**Theorem 1.2** (see [DMP, Theorem 1.3]). Let $G$ be a compact Lie group. A closed smooth $G$ manifold is algebraically realized if it is equivariantly cobordant to a non-singular real algebraic $G$ variety. In particular, the boundary of a compact smooth $G$ manifold is algebraically realized.

In the non-equivariant situation this result constituted Tognoli’s break through in the proof of the Nash-Tognoli Theorem [T]. The only other fact needed was a previously known result of Milnor [M1]. It shows that every closed smooth manifold is cobordant to a non-singular real algebraic variety.

The other principle (to be proved in each of the individual cases of Theorem A) is that equivariant bordism classes are represented by manifolds which are made up from total spaces of projective bundles associated to equivariant vector bundles. In the cases listed in Theorem A we show that the relevant bundles are strongly algebraic $G$ vector bundles (for the definition see 5.1), and that the total spaces of their associated projective bundles are algebraically realized. We describe the application of these two principles in more detail.

The proof of one of the cases of Theorem A (case (2) if $G$ is not of odd order and $G$ acts freely) uses only the first principle. It is obtained as a generalization of a bordism theoretic result of Stong [S1]. We show that the actions bound equivariantly (see Proposition 4.1) and apply Theorem 1.2. With the same approach, any vanishing result in non-oriented equivariant bordism gives a solution to the Algebraic Realization Conjecture under the same assumptions. E.g., one may generalize another result of Stong [S2] (see also [Kh, Corollary 3.4; DKh, p. 59]) to obtain

**Proposition B.** Let $G$ be a compact Lie group. A closed smooth $G$ manifold $M$ is algebraically realized if the center of $G$ contains a non-trivial 2-torus $H = (\mathbb{Z}_2)^k$ and $M^H = \emptyset$.

We prove the underlying bordism result in Proposition 4.3 as none of the references contains it quite in the required generality. As another example, vanishing results for actions of cyclic groups and $S^1$ (see [Os; Ko, Theorem 1, Lemma 6 and Lemma 7]) provide a proof of the Algebraic Realization Conjecture under the assumptions made in these references.

To explain the more general situation we need some preparation. Let $G$ be a compact Lie group, and $\Xi$ an orthogonal representation of $G$. The Grassmannian $G_R(\Xi, k)$ is a finite approximation of the classifying space for $k$-dimensional $G$ vector bundles. It is also a real algebraic $G$ variety. Let $\gamma_R(\Xi, k)$ denote the universal $G$ vector bundle over $G_R(\Xi, k)$. See Section 5 for details.

**Definition.** Let $G$ be a compact Lie group and $\xi$ a $k$-dimensional real $G$ vector bundle over a smooth $G$ manifold $B$. We say that $\xi$ is algebraically realized if there exists a non-singular real algebraic $G$ variety $X$ and an equivariant diffeomorphism $\phi : X \to B$ such that the classifying map of $\phi^*(\xi)$ is equivariantly homotopic to an equivariant entire rational map.

To simplify language, we also allow the case where $B$ consists of several components (which may have different dimensions) and the dimension of the fibre depends on the component. In this case we require that $\xi$ is algebraically realized if we restrict it to a collection of components of $B$ of the same dimension. Globally, we suppose that the total space of the bundle has a well defined dimension.
The important property for algebraically realized $G$ vector bundles is

**Proposition 1.3.** Let $G$ be a compact Lie group, $\xi = (E, p, B)$ a $G$ vector bundle, and $\mathbb{R}P(E)$ the total space of the associated projective bundle. If $\xi$ is algebraically realized, then $\mathbb{R}P(E)$ is algebraically realized.

This proposition will be proved as an immediate consequence of Proposition 5.2.

We return to the proof of Theorem A. In case (1) we interpret a calculation of the equivariant bordism ring by Costenoble (see [C]). The generators are constructed from total spaces of projective bundles associated to strongly algebraic $G$ vector bundles (see Theorem 6.3). With a little bit more work we could base our proof also on older work of Stong [S1].

In the remaining cases of Theorem A (2) the approach is as follows. We use a localization principle to reduce the problem. Its proof is given in Section 7.

**Proposition 1.4.** Let $G$ be a compact Lie group, and $M$ a closed smooth $G$ manifold. Let $\tau$ be a central element of order two in $G$, and let $\nu(M^\tau)$ be the $G$ normal bundle of the $\tau$ fixed set. Then $M$ is algebraically realized if and only if $\nu(M^\tau)$ is algebraically realized.

It remains to show that $\nu(M^\tau)$ is algebraically realized. We are only concerned with the case where the action on the base space of the bundle is trivial, and this case is easily solved using ideas of Segal [Sg] and Benedetti and Tognoli [BT] (see Proposition 7.2).

There are some applications of the results in this paper to other questions in transformation groups. As an application of Theorem A (2) we can give a completely different proof of an old result of Conner and Floyd. We do so in Section 8.

**Theorem** (see [CF, 31.3]). There is no smooth action of a 2-torus on a connected closed smooth manifold of positive dimension with exactly one fixed point.

For the second application we consider the

**Fixed Point Conjecture** (see [DMP, p. 50]). A compact Lie group acts real algebraically and without a fixed point on a variety diffeomorphic to a Euclidean space if and only if it acts smoothly and without a fixed point on a disk.

According to [DMP] this conjecture is true for alternating groups. Some other cases also follow from the results of that paper. In addition we have

**Theorem.** The Fixed Point Conjecture holds for odd order abelian groups.

The necessity part ($\implies$) has been shown by Petrie and Randall [PR1], [P2]. We show the converse for abelian groups of odd order. The proof which we provide in Section 8 is much simpler than the one which has been given previously in [DKS].

The paper is organized as follows. In Section 2 we recall some basic concept which we hope helps one or the other reader. In Section 3 we discuss induction, i. e., ways in which to get $G$ actions from $H$ actions, $H \subset G$. In Section 4 we prove the elementary cases of Theorem A. These are the cases which do not require the theory of equivariant strongly algebraic vector bundles. In Section 5 we discuss equivariant strongly algebraic vector bundles. In Section 6 we prove Theorem A (1), i. e., the case in which the acting group is of odd order. In Section 7 we prove the remaining cases of Theorem A (2). In Section
8 we provide the proofs of the applications mentioned just before. Finally, in Section 9 we provide more algebraically minded definitions for the set-up of the paper and show that they are equivalent to those used throughout this introduction and paper.

We would like to thank Dong Youp Suh for several useful discussions on the topics of this paper. He suggested a stronger form of Theorem 6.3. His contributions will be contained in joint papers on the general topic of algebraic approximation of equivariant smooth situations which will continue the project started in the present paper.

2. Basic Concepts

As expressed in the Algebraic Realization Conjecture, this paper deals with the relation between smooth and real algebraic transformation groups. As the reader may be unfamiliar with one or the other category, we recall the basic definitions in both, i.e., objects, morphisms, and isomorphism classes of objects. In addition, we recall a few facts which clarify the basic concepts.

In the introduction we defined real algebraic varieties. It remains to discuss non-singularity. Let \( p_1, \ldots, p_m : \mathbb{R}^n \to \mathbb{R} \) be polynomials and

\[
V = \{ x \in \mathbb{R}^n \mid p_1(x) = \cdots = p_m(x) = 0 \}.
\]

As a topological space we will consider \( V \) with two topologies, the subspace topologies induced by the Euclidean topology on \( \mathbb{R}^n \) and the one induced by the Zariski topology. Most of the time we will use the Euclidean topology without mentioning this explicitly. Whenever we use the Zariski topology we will say so.

**Definition 2.1.** The variety \( V \subset \mathbb{R}^n \) is said to be non-singular at \( x \in V \) if there are polynomials \( q_1, \ldots, q_s : \mathbb{R}^n \to \mathbb{R} \) which vanish on \( V \) and a Zariski open neighbourhood \( U \) of \( x \) in \( \mathbb{R}^n \) such that

\[
\begin{align*}
(1) & \quad V \cap U = U \cap q_1^{-1}(0) \cap \cdots \cap q_s^{-1}(0) \\
(2) & \quad \text{the gradients} (\nabla q_i)_x \text{ are linearly independent for } i = 1, \ldots, s.
\end{align*}
\]

We say that \( V \) is non-singular if \( V \) is non-singular at each point \( x \in V \), and all connected components of \( V \) have the same dimension.

A subset \( M \subset \mathbb{R}^n \) is said to be a smooth submanifold of dimension \( m \) if for all \( x \in M \) there exists a neighbourhood \( U \) of \( x \) in \( \mathbb{R}^n \) and a smooth map \( \phi : U \to \mathbb{R}^{n-m} \) such that \( M \cap U = U \cap \phi^{-1}(0) \) and the derivative \( (D\phi)_x \) of \( \phi \) at \( x \) has rank \( n-m \). By definition, a non-singular variety is a smooth submanifold of \( \mathbb{R}^n \), though the converse does not hold. There are smooth submanifolds of \( \mathbb{R}^2 \) which are varieties, but they are singular (see [M2, p. 12, Example C]).

In a different approach, one defines the concept of a smooth atlas for a paracompact topological space \( M \), which then defines a smooth structure on \( M \). Together with the chosen structure (if one exists), \( M \) is said to be a smooth manifold. With the help of the implicit function theorem one may define a unique smooth structure on a smooth submanifold of \( \mathbb{R}^m \). With the help of Whitney’s embedding theorem one can realize every smooth manifold as a smooth submanifold of some \( \mathbb{R}^m \).
A smooth action of a compact Lie group $G$ on a smooth manifold $M$ is a smooth map $\mu : G \times M \to M$, which is also an action of $G$ on $M$. We call $M$ together with the action a smooth $G$ manifold.

We turn our attention to the concept of a morphism. In the smooth category we generally consider smooth maps. A smooth map between smooth manifolds is said to be a diffeomorphism if it has a smooth inverse. Diffeomorphic smooth manifolds are generally treated as the same. In the presence of an action of a group $G$ we consider equivariant maps. An equivariant smooth map between smooth $G$ manifolds is said to be an equivariant diffeomorphism if it has a smooth inverse. It is automatic that the inverse is equivariant. Again, equivariantly diffeomorphic manifolds are treated as the same.

In the real algebraic category we consider maps which are either polynomial or rational. Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be real algebraic varieties. A map $f : V \to W$ is said to be regular if it extends to a map $F : \mathbb{R}^n \to \mathbb{R}^m$ such that each component of $F$ is a polynomial. We say that $f$ is entire rational\(^3\) if there are regular maps $p : \mathbb{R}^n \to \mathbb{R}^m$ and $q : \mathbb{R}^n \to \mathbb{R}$ such that $f = p/q$ on $V$ and $q$ does not vanish on $V$. Occasionally, we even call a regular map a polynomial.

Before we give the equivariant equivalents we discuss the process of averaging a map (compare e. g. [B]). Let $\Omega$ and $\Xi$ be representations of a compact Lie group $G$, and $f : \Omega \to \Xi$. Denote the Haar measure of $G$ by $dg$, and let $x$ be a point in $\Omega$. Then

$$A(f)(x) = \int_G g^{-1}f(gx) \, dg.$$  

We collect some standard facts concerning the averaging operator $A$.

**Lemma 2.2.** With the notation from above:

1. $A(f)$ is equivariant, and $A(f) = f$ if $f$ is equivariant.
2. If $f$ is a polynomial, then so is $A(f)$.

Let $V \subset \Omega$ and $W \subset \Xi$ be real algebraic $G$ varieties. An equivariant map $f : V \to W$ is said to be an equivariant regular map if it is regular. Given any polynomial extension $F : \Omega \to \Xi$ of $f$ we get an equivariant polynomial extension of $f$ using $A(F)$. This follows from Lemma 2.2. An equivariant map is said to be an equivariant entire rational map if it is entire rational. Why we need not impose any stronger restriction is obvious from our next proposition.

**Proposition 2.3.** Let $V \subset \Omega$ and $W \subset \Xi$ be real algebraic $G$ varieties, and $f : V \to W$ an equivariant entire rational map. There exist equivariant polynomials $P : \Omega \to \Xi$ and $Q : \Omega \to \mathbb{R}$ such that $f = P/Q$, restricted to $V$, and $Q$ does not vanish on $V$.

**Proof.** Express $f$ as a quotient $p/q$, where $p : \Omega \to \Xi$ and $q : \Omega \to \mathbb{R}$. Without loss of generality we may assume that $q$ is non-negative. If $f = p_0/q_0$ we can always write it as $f = p_0q_0/(q_0)^2$ instead. Rewrite the equation for $f$ as

$$(*) \quad q \cdot f = p.$$  

\(^3\)Not all authors agree what a rational and an entire rational map should be. In this paper it is essential that maps are defined everywhere on their domain, and this seems to be assumed by all authors when they use the adjective ‘entire rational’.
We average the left hand side of this equation

\[
A(q \cdot f)(x) = \int_G g^{-1}(q(gx) \cdot f(gx)) \, dg
= \int_G q(gx) \cdot (g^{-1} f(gx)) \, dg
= \int_G (g^{-1} q(gx)) \cdot f(x) \, dg
= (A(q) \cdot f)(x).
\]

Note that \(A(q)(x) \neq 0\) for \(x \in V\) because \(q(x)\) is positive on \(V\). Averaging (*) and dividing both sides of the resulting equation by \(A(q)\) we obtain \(f = A(p)/A(q)\). Set \(P = A(p)\) and \(Q = A(q)\), then the claim of the proposition is obvious. □

We note that Proposition 2.3 fails in the context of complex algebraic transformation groups [Sp].

There are two additional remarks of interest at this point.

**Proposition 2.4.** Every real algebraic \(G\) variety can be expressed as the zero set of a single equivariant polynomial.

**Proof.** Let \(V \subset \Omega\) be a real algebraic \(G\) variety. The ideal \(I(V)\) of all polynomials \(p : \Omega \to \mathbb{R}\) which vanish on \(V\) is finitely generated, say by polynomials \(p_1, \ldots, p_k\). The average \(\overline{p} = A(p_1^2 + \cdots + p_k^2)\) is equivariant and \(\overline{p}^{-1}(0) = V\). □

**Proposition 2.5.** Let \(V \subset \Omega\) and \(W \subset \Xi\) be real algebraic \(G\) varieties, and \(f : V \to W\) an equivariant entire rational map. We may express \(f\) as a quotient \(p/q\) of equivariant polynomials such that \(q\) does not vanish anywhere on \(\Omega\).

**Proof.** Let \(h\) be an equivariant polynomial, such that \(h^{-1}(0) = V\). Then \(pq/(q^2 + h^2)\) coincides with \(f\) on \(V\), and \(q^2 + h^2\) does not vanish on \(\Omega\). □

Let \(V \subset \Omega\) and \(W \subset \Xi\) be real algebraic varieties. An equivariant regular map \(f : V \to W\) is said to be an **equivariant regular isomorphism** (of these real algebraic \(G\) varieties) if it has an equivariant regular inverse \(g : W \to V\), i.e., \(g \circ f\) is the identity on \(V\) and \(f \circ g\) is the identity on \(W\). An **equivariant entire rational isomorphism** is defined similarly. The results of this paper do not depend on the type of isomorphism we use. Whenever we refer to an isomorphism of varieties we will mean an entire rational isomorphism. As for manifolds, we identify real algebraic \(G\) varieties if they are equivariantly isomorphic. We will often talk about real algebraic \(G\) varieties, leaving it up to the reader to remember that (for any representative in the equivariant isomorphism class) there is a given representation of \(G\) in which the variety is the zero set of a family of polynomials.

The following is an easy exercise.

**Proposition 2.6.**

1. The disjoint union of two (non-singular) real algebraic \(G\) varieties (of the same dimension) is a (non-singular) real algebraic \(G\) variety.
2. The cartesian product of two (non-singular) real algebraic \(G\) varieties is a (non-singular) real algebraic \(G\) variety.
3. Induction

Let $H$ be a subgroup of a group $G$, and let $X$ be an $H$ space. The induced $G$ space $\text{Ind}_H^G X$ is defined as the orbit space $G \times_H X$. More explicitly, elements in $G \times_X X$ are equivalence classes of elements in $G \times X$, where $(g, x)$ is equivalent to $(gh, h^{-1}x)$ for all $h \in H$. If $X$ is a (closed) smooth $H$ manifold and $H$ is a closed subgroup of a compact Lie group $G$, then $\text{Ind}_H^G X$ is a (closed) smooth $G$ manifold. This procedure is natural. Given an $H$ equivariant map $f : X \rightarrow Y$ of $H$ spaces, the map $\text{Id} \times f : G \times X \rightarrow G \times Y$ induces a $G$ map $\text{Ind}_H^G f : \text{Ind}_H^G X \rightarrow \text{Ind}_H^G Y$ of orbit spaces. If $X$ and $Y$ are smooth $H$ manifolds and $f$ is smooth, then so is $\text{Ind}_H^G f$. We shall prove a similar result for real algebraic varieties.

**Proposition 3.1.** Suppose $G$ is a compact Lie group, and $H$ is a closed subgroup of finite index. If $X$ is a (non-singular) real algebraic $H$ variety, then there is a natural procedure to define a (non-singular) real algebraic $G$ variety structure on $\text{Ind}_H^G X$. Given two real algebraic $H$ varieties $X$ and $Y$ and an $H$ equivariant regular (resp. entire rational) map $f : X \rightarrow Y$, there is a naturally defined $G$ equivariant regular (resp. entire rational) map $\text{Ind}_H^G f : \text{Ind}_H^G X \rightarrow \text{Ind}_H^G Y$.

The proof is based on an induction process for representations.

**Proposition 3.2.** Let $G$ be a compact Lie group, and $H$ a closed subgroup of finite index in $G$. Let $\Gamma$ be an orthogonal representation of $H$. There exists a representation $\tilde{\Gamma}$ of $G$ with the following properties.

1. As a real vector space $\tilde{\Gamma}$ is isomorphic to $\Gamma \times \cdots \times \Gamma$, the $s$-fold cartesian product of $\Gamma$. (If $H$ is normal, then the restricted $H$ action on $\tilde{\Gamma}$ will induce an $H$ action on each of the factors, and if $H$ is in the center of $G$, then each of the factors will be isomorphic to $\Gamma$ as a representation of $H$.)

2. There is a $G$ equivariant map $\gamma : G \times_H \Gamma \rightarrow \tilde{\Gamma}$ such that for any $x \in \Gamma$,
   \[ \gamma(G \times_H x) = (Hx, 0, \ldots, 0) \cup (0, Hx, 0, \ldots, 0) \cup \cdots \cup (0, \ldots, 0, Hx). \]

   Here, as a point set, $\tilde{\Gamma}$ is understood as $\Gamma \times \cdots \times \Gamma$ in the sense of (1). For the purpose of description we imposed the action of $H$ on each of the factors $\Gamma$ of $\tilde{\Gamma}$.

3. Given two representations $\Gamma$ and $\Omega$ of $H$, and an $H$ equivariant linear (resp. regular or entire rational) map $f : \Gamma \rightarrow \Omega$. There is a naturally defined $G$ equivariant linear (resp. regular or entire rational) map $\tilde{f} : \tilde{\Gamma} \rightarrow \tilde{\Omega}$, which has the form
   \[ f \times \cdots \times f : G \times \cdots \times \Gamma \rightarrow \Omega \times \cdots \times \Omega. \]

   Again, as a point set $\tilde{\Gamma}$ and $\tilde{\Omega}$ are understood as $\Gamma \times \cdots \times \Gamma$ and $\Omega \times \cdots \times \Omega$ in the sense of (1).

4. For $\Gamma$, $\Omega$, $f$, and $\gamma$ as in (2) and (3), the diagram
   \[
   \begin{array}{ccc}
   G \times_H \Gamma & \xrightarrow{\gamma} & \tilde{\Gamma} \\
   \text{Ind}_H^G f \downarrow & & \downarrow \tilde{f} \\
   G \times_H \Omega & \xrightarrow{\gamma} & \tilde{\Omega}
   \end{array}
   \]

   commutes.
Proof of Proposition 3.1. Suppose $\Gamma$ is an orthogonal representation of $H$ and $X$ is realized as the zero set of the polynomial $p : \Gamma \to \mathbb{R}$. We suppose that $0 \in \Gamma$ is not in $X$. This is no restriction as we could add $\mathbb{R}$ to $\Gamma$ and shift $X$ in the direction of $\mathbb{R}$. Let $\tilde{\Gamma}$ and $\gamma$ be as in 3.2. By 3.2 (2)

$$\gamma(G \times_H X) = (X, 0, \ldots, 0) \cup (0, X, 0, \ldots, 0) \cup \cdots \cup (0, \ldots, 0, X)$$

in $\tilde{\Gamma} = \Gamma \times \cdots \times \Gamma$. This describes $\gamma(G \times_H X)$ as a disjoint union of varieties, which by 2.6 is again a (non-singular) variety. It follows from the equivariance of $\gamma$ that $\gamma(G \times_H X)$ is $G$ invariant. Hence $\gamma(G \times_H X)$ is realized as (non-singular) real algebraic $G$ variety. Our claim about an equivariant regular (resp. entire rational) map follows by an equally easy argument from 3.2 (3) and (4). □

Many readers may be familiar with the induction construction applied to a representation $\Gamma$ of $H$ in Proposition 3.2. Only some may be familiar with all of its properties given in the proposition. For this reason we at least review the essential arguments.

Proof of Proposition 3.2. Denote the set of all maps from $G$ to $\Gamma$ by Map($G, \Gamma$), and the subset of all $H$ equivariant maps by Map$_H(G, \Gamma)$, i.e., maps $\mu \in \text{Map}(G, \Gamma)$ for which $h\mu(g) = \mu(hg)$ for all $g \in G$ and $h \in H$.

One may also consider an action of $H$ on Map($G, \Gamma$) defined by

$$\mu^h(g) = h^{-1}\mu(hg) \text{ for } \mu \in \text{Map}(G, \Gamma), \ h \in H, \ \text{and } g \in G.$$ 

The $H$ fixed point set of this action is Map$_H(G, \Gamma)$. On Map($G, \Gamma$) we have an action of $G$ defined by

$$(g\mu)(g') = \mu(g'g) \text{ for } \mu \in \text{Map}(G, \Gamma) \text{ and } g, g' \in G.$$ 

It induces an action of $G$ on Map$_H(G, \Gamma)$. With the obvious definition of addition and scalar multiplication on Map$_H(G, \Gamma)$ obtained from the operations on $\Gamma$, we see that Map$_H(G, \Gamma)$ is a representation of $G$. We define an $H$ map

$$\alpha : \Gamma \to \text{Map}_H(G, \Gamma) \text{ with } \alpha(x)(g) = \begin{cases} gx & \text{if } g \in H \\ 0 & \text{otherwise} \end{cases}$$

and extend $\alpha$ to a $G$ map (here $G$ acts by left multiplication on the first factor)

$$\tilde{\alpha} : G \times \Gamma \to \text{Map}_H(G, \Gamma) \text{ setting } \tilde{\alpha}(g, x) = g\alpha(x).$$

It induces a $G$ map

$$\overline{\alpha} : G \times_H \Gamma \to \text{Map}_H(G, \Gamma).$$

One may stop at this point, declare Map$_H(G, \Gamma)$ to be $\tilde{\Gamma}$ and $\overline{\alpha}$ to be $\gamma$. Matters become more transparent if one continues as follows.
Let \( \Gamma = \Gamma \times \cdots \times \Gamma \) denote the \( s \)-fold cartesian product of \( \Gamma \), considered for the moment only as a real vector space. Here \( s \) is the index of \( H \) in \( G \). Let \( g_1, \ldots, g_s \) be representatives of the classes in \( H \backslash G \). Classes are of the form \( Hg \), and \( g \) and \( g' \) are in the same class if \( g'g^{-1} \in H \). We define a map

\[
\beta : \text{Map}_H(G, \Gamma) \to \Gamma \times \cdots \times \Gamma \text{ by } \beta(\mu) = (\mu(g_1), \ldots, \mu(g_s)).
\]

This map defines an isomorphism of real vector spaces, and via \( \beta \) we induce on \( \tilde{\Gamma} \) the structure of a representation of \( G \). Obviously, the choice of the \( g_i \) is not natural, and \( \beta \) depends on it. To get the naturality statement in the proposition we need to chose the \( g_i \) once and for all, independent of \( \Gamma \).

With \( \tilde{\Gamma} \) defined above as a representation of \( G \) and \( \gamma = \beta \circ \tilde{\alpha} \) (see 3.2 (2)) all of the claims made in the proposition follow. The statement in (1) is immediate from the description of \( \tilde{\Gamma} \).

We check (2). Let \( (g, x) \in G \times \Gamma \), then

\[
\beta \tilde{\alpha}(g, x) = (\alpha(x)(g_1g), \ldots, \alpha(x)(g_s g)).
\]

There is only one \( i \) for which \( g_i g \in H \), or equivalently \( g_i \in H g^{-1} \). For \( j \neq i \) we have that \( \alpha(x)(g_j g) = 0 \), only \( \alpha(x)(g_i g) \) can be non-zero. Let \( p_i : \Gamma \times \cdots \times \Gamma \to \Gamma_i \) be the projection on the \( i \)-th factor. Then \( p_i \beta \tilde{\alpha}(g, x) = \alpha(x)(g_i g) = g_i g x \). Hence it is a point in \( H x \subset \Gamma_i \). Allowing \( g \) to vary over all elements of \( G \) such that \( g_i g \in H \) produces all points in \( H x \subset \Gamma_i \). As we vary \( g \) over all elements of \( G \), all \( i = 1, \ldots, s \) occur, hence the image of \( \gamma(G \times_H x) \) is as claimed.

The naturality claims in (3) and (4) follow because the construction \( \text{Map}_H(G, \cdot) \) is covariant and natural, and because the definitions of \( \tilde{\alpha} \) and \( \beta \), hence \( \gamma \), are natural. \( \square \)

### 4. Some Elementary Cases of Theorem A

In this section we prove the elementary cases of Theorem A and Proposition B, i.e., those cases which are obtained as immediate consequences of bordism theoretic results and Theorem 1.2.

As usual, we denote the set of bordism classes of closed smooth manifolds by \( \mathcal{N}_* \), and the bordism classes of closed smooth \( G \) manifolds by \( \mathcal{N}_*^G \). We also use bordism classes of maps to a space \( Y \), denoted by \( \mathcal{N}_*(Y) \). A closed manifold of dimension \( n \) represents a class in \( \mathcal{N}_n \). A group \( G \) is said to act freely on a space \( X \) if \( G_x = \{ g \in G \mid gx = x \} = \{1\} \) for all \( x \in X \). We then consider the bordism theory of closed manifolds with free \( G \) action. It is denoted by \( \mathcal{N}_*^G[\text{free}] \). Given any two closed smooth \( G \) manifolds \( M_1 \) and \( M_2 \) with free action, they are said to be freely cobordant (or in the same class in \( \mathcal{N}_*^G[\text{free}] \)) if there exists a compact smooth \( G \) manifold \( W \) with free action of \( G \) such that \( \partial W \) is the disjoint union of \( M_1 \) and \( M_2 \). As a generalization of a result of Stong [S1, Proposition 14.1] we prove later in this section:

**Proposition 4.1.** Let \( G \) be a compact Lie group not of odd order, and suppose that \( G \) acts smoothly and freely on a closed manifold \( M \). Then \( M \) bounds equivariantly.
Corollary 4.2. Let $G$ be a compact Lie group, and suppose that $G$ acts smoothly and freely on a closed manifold $M$. Then $M$ is algebraically realized.

Proof. If $G$ is not of odd order, then $M$ is the boundary of a compact smooth $G$ manifold. We apply Theorem 1.2 to conclude that $M$ is algebraically realized.

The case where $G$ is of odd order is covered by Theorem A (1), but the proof is so easy, we just give it. It is well known that $\text{Ind}^G_1 : \mathcal{N}_* \rightarrow \mathcal{N}^G_*$ is surjective [CF], [S1]. By a result of Milnor [M1] every class in $\mathcal{N}_*$ can be represented by a non-singular real algebraic variety. By Proposition 3.1 every class in $\mathcal{N}^G_*$ can be represented by a non-singular real algebraic $G$ variety. Theorem 1.2 implies that every closed smooth $G$ manifold with free action is algebraically realized. □

Next we prove another generalization of a result by Stong [S2, p. 779], see also [Kh, Corollary 3.4; DKh, p. 59], which we apply in the proof of Proposition 4.1 as well as the proof of Theorem A (3).

Proposition 4.3. Let $G$ be a compact Lie group with a non-trivial central $2$-torus $H \cong (\mathbb{Z}_2)^k$, and $M$ a closed smooth $G$ manifold such that the $H$ fixed point set $M^H$ is empty. Then $M$ bounds $G$ equivariantly.

Proof. Consider a sequence of subgroups of $H$

$$1 = H_0 < H_1 < \cdots < H_k = H.$$  

Let $\{t_1\}$ generate $H_1$, $\{t_1, t_2\}$ generate $H_2$, etc. We show that if the $H_j$ fixed point set of $M$ is empty, then $M$ is equivariantly cobordant to a smooth manifold for which the $H_{j-1}$ fixed point set is empty, $j \geq 2$. Consider the normal bundle $\nu(M^{H_{j-1}})$ of $M^{H_{j-1}}$ in $M$. It is a $G$ vector bundle because $H$ is central in $G$, and $t_j$ acts freely on it because $M^{H_j}$ is empty. Consider the diagram, $D(\cdot)$ denotes the disk bundle,

$$
\begin{array}{ccc}
D(\nu(M^{H_{j-1}})) & \longrightarrow & D(\nu(M^{H_{j-1}}))/t_j \\
\downarrow & & \downarrow \\
M^{H_{j-1}} & \longrightarrow & M^{H_{j-1}}/t_j.
\end{array}
$$

The columns are projection maps of bundles and the rows are projection maps of principal $\mathbb{Z}_2$ fibrations. Let $W$ be the total space of the $D^1$ bundle associated with the fibration in the top row. Then

$$\partial W = D(\nu(M^{H_{j-1}})) \cup S(\nu(M^{H_{j-1}})) \times_{t_j} D^1.$$  

Here $S(\cdot)$ denotes the sphere bundle, and $t_j$ acts antipodally on $D^1$. Consider the $G$ bordism

$$Z = M \times [0, 1] \cup_{D(\nu(M^{H_{j-1}}))} W$$

where we attach $W$ along $M \times \{1\}$. Then $Z$ provides a bordism between $M = M \times \{0\}$ and a smooth $G$ manifold on which the action of $H_{j-1}$ has no fixed point. After applying this
argument repeatedly we find a smooth \( G \) manifold \( M_1 \) which is equivariantly cobordant to \( M \), and on which the central element \( t_1 \) acts freely. The mapping cylinder of \( M_1 \to M_1/t_1 \) bounds \( M_1 \) equivariantly. As claimed, \( M \) is an equivariant boundary. □

Suppose \( H \) is a normal subgroup of \( G \). We say that the \( H \) fixed point structure of a closed smooth \( G \) manifold bounds if \( M^H \) is the boundary of a \( G \) manifold \( W \) and the normal bundle \( \nu(M^H) \) of \( M^H \) in \( M \) extends to a \( G \) vector bundle over \( W \). We restate Proposition B from the introduction in a slightly stronger form.

**Proposition 4.4.** Let \( G \) be a compact Lie group with a non-trivial central 2-torus \( H \cong (\mathbb{Z}_2)^k \), and \( M \) a closed smooth \( G \) manifold such that the \( H \) fixed point structure bounds. Then \( M \) is algebraically realized.

**Proof.** Because the \( H \) fixed point structure of \( M \) bounds, \( M \) is equivariantly cobordant to a closed smooth \( G \) manifold \( M' \) for which the \( H \) fixed point set is empty. By Proposition 4.3, \( M' \) bounds equivariantly, and so does \( M \). It follows from Theorem 1.2 that \( M \) is algebraically realized. □

**Proof of Proposition 4.1.** We follow Stong’s proof adding one more idea. Let \( T \) be a maximal torus of \( G \) with normalizer \( NT \) and Weyl group \( W = NT/T \). Let \( W_2 \) be the 2-Sylow subgroup of \( W \), and let \( N_2 T \) be the subgroup of \( NT \) given by the exact sequence

\[
1 \to T \to N_2 T \to W_2 \to 1.
\]

In the proof we use the following diagram. The last two squares commute, though the first one may not.

\[
\begin{array}{ccc}
H_*(BG, \mathbb{Z}_2) & \overset{\cong}{\to} & \mathbb{N}_*(BG) \\
\uparrow i_* & & \uparrow i_* \\
H_*(BN_2 T, \mathbb{Z}_2) & \overset{\cong}{\to} & \mathbb{N}_*(BN_2 T) \\
\uparrow i_* & & \uparrow \text{Ind} \\
\mathbb{N}_* \otimes \mathbb{Z}_2 H_*(BG, \mathbb{Z}_2) & \overset{\cong}{\to} & \mathbb{N}_* \otimes \mathbb{Z}_2 H_*(BN_2 T, \mathbb{Z}_2)
\end{array}
\]

Here \( d = \dim G \) and \( d_1 = \dim N_2 T \). The classifying space for principal \( G \) fibrations is denoted by \( BG \). There are two vertical induction maps denoted by \( \text{Ind} \), and two maps induced by the inclusion \( N_2 T \to G \). The first pair of horizontal isomorphisms is obtained from the Künneth formula in bordism. The second pair of isomorphisms is induced by the map which assigns to a free action of \( H \) on a manifold \( M \) the classifying map of the principal \( H \) fibration \( M \to M/H \), applied with \( H = G \) in the top row and \( H = N_2 T \) in the bottom row. The maps \( \alpha \) and \( \beta \) are forgetful maps. We want to show that \( \alpha = 0 \).

This is an immediate consequence of

1. \( \beta = 0 \)
2. \( \text{Ind}_f \) is surjective

To show (1) it suffices to show that there exists a central non-trivial element \( \tau \) of order two in \( N_2 T \). Then (1) follows from Proposition 4.3.

There is an action of \( W_2 \) on \( T \) given by

\[
W_2 \times T \to T \quad \text{with} \quad (g, t) \mapsto gtg^{-1}.
\]
As elements of order 2 in $T$ are again mapped to elements of order 2, there is an induced action of $W_2$ on the 2-torus $H = (\mathbb{Z}_2)^k$ in $T$. Here $k = \dim T$. As this action has the unit element $e \in H$ as a fixed point, there must be another, non-trivial $W_2$ fixed point in $H$. We call it $\tau$ and note that $\tau$ lies in the center of $N_2 T$. This concludes the proof of (1).

To see (2) we consider the fibration

$$G/N_2 T \to BN_2 T \to BG$$

and the composition

$$\text{H}_{\ast}(BG, \mathbb{Z}_2) \to \text{H}_{\ast}(BN_2 T, \mathbb{Z}_2) \to \text{H}_{\ast}(BG, \mathbb{Z}_2).$$

Here $tr$ denotes the Becker-Gottlieb transfer [G, Theorem C], [BG]. As $i_{\ast} \circ tr$ is multiplication with the Euler characteristic of $G/N_2 T$, and $\chi(G/NT) = 1$ and $|NT/N_2 T| = |W/W_2| \equiv 1 \pmod{2}$, it follows that $i_{\ast} \circ tr = Id$, and that $i_{\ast}$ is surjective.

Compare [CF, Section 17] for the following argument. The K"{u}nneth isomorphism depends on a choice of basis for the homology groups. For each $n$, choose a basis $\{b_j\}_{1 \leq j \leq m}$ for $H_n(BN_2 T, \mathbb{Z}_2)$ such that $\{i_{\ast}(b_j)\}_{1 \leq j \leq k}$ form a basis of $H_n(BG, \mathbb{Z}_2)$. There are closed manifolds $M^n_j$ of dimension $n$ and maps $f_j: M^n_j \to BN_2 T$ such that $(f_j)_{\ast}([M^n_j]) = b_j$. We use the $\{(M^n_j, f_j) \mid 1 \leq j \leq m\}$ to define $\mathfrak{H}_{\ast} \otimes_{\mathbb{Z}_2} H_n(BN_2 T, \mathbb{Z}_2) \to \mathfrak{H}_{\ast}(BN_2 T)$, and $\{(M^n_j, i \circ f_j) \mid 1 \leq j \leq k\}$ to define $\mathfrak{H}_{\ast} \otimes_{\mathbb{Z}_2} H_n(BG, \mathbb{Z}_2) \to \mathfrak{H}_{\ast}(BG)$. Let $B$ be the span of $\{b_j \mid 1 \leq j \leq k\}$. Restricted to $\mathfrak{H}_{\ast} \otimes_{\mathbb{Z}_2} B$ the first square in the diagram given above commutes. It follows that $i_{\ast} : \mathfrak{H}_{\ast}(BN_2 T) \to \mathfrak{H}_{\ast}(BG)$, hence also Ind$_f$ is onto. This completes the proof of (2) as well as the proof of the proposition. □

5. **Strongly Algebraic G Vector Bundles**

We discuss strongly algebraic $G$ vector bundles to the extent as we need them in the next section. In a later section we continue the discussion.

Let $\Lambda$ stand for $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let $\Xi$ be a representation of $G$ over $\Lambda$, in particular, its underlying space is $\Lambda^n$ for some $n$. We assume that the action of $G$ preserves the standard bilinear form on $\Lambda^n$ over $\Lambda$. Let $\text{End}_\Lambda(\Xi)$ denote the set of endomorphisms of $\Xi$ over $\Lambda$. It is a representation of $G$ with the action given by

$$G \times \text{End}_\Lambda(\Xi) \to \text{End}_\Lambda(\Xi) \quad \text{with} \quad (g, L) \mapsto gLg^{-1}.$$

Let $k$ be a natural number. We set

$$G_\Lambda(\Xi, k) = \{L \in \text{End}_\Lambda(\Xi) \mid L^2 = L, L^* = L, \text{ trace } L = k\}$$

$$E_\Lambda(\Xi, k) = \{(L, u) \in \text{End}_\Lambda(\Xi) \times \Xi \mid L \in G_\Lambda(\Xi, k), Lu = u\}.$$

Here $L^*$ denotes the adjoint of $L$. If one chooses an orthonormal (resp. unitary or symplectic) basis, then $\text{End}_\Lambda(\Xi)$ is canonically identified with the set of $n \times n$ matrices $\Lambda^n$, and $L^*$ is obtained by transposing $L$ and conjugating its entries. This description specifies $G_\Lambda(\Xi, k)$ and $E_\Lambda(\Xi, k)$ as real algebraic $G$ varieties. Define $p : E_\Lambda(\Xi, k) \to G_\Lambda(\Xi, k)$ as projection on the first factor. This defines a $G$ vector bundle, which is called the universal bundle over $G_\Lambda(\Xi, k)$, and which is denoted by $\gamma_\Lambda(\Xi, k)$. 

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Definition 5.1. A strongly algebraic $G$ vector bundle over $\Lambda$ is a pair $(X, \mu)$ where $X$ is a real algebraic $G$ variety and $\mu: X \to G_\Lambda(\Xi, k)$ is an equivariant entire rational map. Assuming that $\Xi$ is a summand of a representation $\Xi'$ of $G$, we have an embedding $i: G_\Lambda(\Xi, k) \to G_\Lambda(\Xi', k)$. In this sense we identify the strongly algebraic $G$ vector bundles $(X, \mu)$ and $(X, i\mu)$.

For the discussion of the non-equivariant case we refer the reader of [BT]. Generalizing the non-equivariant concept of an algebraic vector bundle [BT], [I], there is a canonical way to define algebraic vector bundles. It is elementary to check that for any strongly algebraic $G$ vector bundle $(X, \mu)$ the pull back $\mu^*(\gamma_\Lambda(\Xi, k))$ is an algebraic $G$ vector bundle. In particular, its total space has a canonical description as a real algebraic $G$ variety, and its projection map is polynomial.

Remark. There is no apparent ‘natural’ way to define the concept of isomorphism for strongly algebraic vector bundles. One might say that two strongly algebraic $G$ vector bundles $\mu_i: X \to G_\Lambda(\Xi, k)$ $(i = 0, 1)$ are isomorphic if there exists an equivariant map $\eta: X \times I \to G_\Lambda(\Xi, k)$ such that $\eta|_{X \times \{t\}} = \mu_i$, and $\eta|_{X \times t}$ is entire rational for each $t \in I$. This defines an equivalence relation. It is tempting to require that $\eta$ is defined on $X \times \mathbb{R}$, and that $\eta$ is entire rational, but then it is not clear that this defines an equivalence relation. Something interesting can still be said at this point. Suppose $(X, \mu_1)$ and $(X, \mu_2)$ are strongly algebraic $G$ vector bundles, and $\mu_1$ and $\mu_2$ are equivariantly homotopic. Then $\mu_1^*(\gamma_\Lambda(\Xi, k)) \text{ and } \mu_2^*(\gamma_\Lambda(\Xi, k))$ are not only isomorphic as real $G$ vector bundles, but they also equivariantly entire rationally isomorphic as algebraic $G$ vector bundles. This is an equivariant generalization of a result in [BCR, p. 265] proved by Kawakami [Ka].

Proposition 5.2. Let $(B, \alpha: B \to G_\mathbb{R}(\Xi, k))$ be a strongly algebraic $G$ vector bundle. The total space
$$\mathbb{R}P(E) = \{(x, T) \in B \times G_\mathbb{R}(\Xi, 1) \mid (Id - \alpha(x))T = 0\}$$
of the associated projective bundle is a real algebraic $G$ variety. If $B$ is non-singular, then $\mathbb{R}P(E)$ is non-singular.

We restate and prove

Proposition 1.3. Let $G$ be a compact Lie group, $\xi = (E, p, B)$ a $G$ vector bundle, and $\mathbb{R}P(E)$ the total space of the associated projective bundle. If $\xi$ is algebraically realized, then $\mathbb{R}P(E)$ is algebraically realized.

Proof. Let $F$ be the collection of components of $M$ of a given dimension $m$. There exists a non-singular $G$ variety $X$ and an equivariant diffeomorphism $\phi: X \to F$ such that the classifying map of $\phi^*(\xi)$ is equivariantly homotopic to an equivariant entire rational map $\mu$. By Proposition 5.2, $\mathbb{R}P(\mu^*(\gamma_\mathbb{R}(\Xi, k)))$ is a non-singular real algebraic $G$ variety. Here $k$ denotes the fibre dimension of $\Xi|_F$. The resulting equivariant diffeomorphism $\mathbb{R}P(\mu^*(\gamma_\mathbb{R}(\Xi, k))) \to \mathbb{R}P(E|_F)$ provides the algebraic realization of $\mathbb{R}P(E|_F)$. As the disjoint union of the $\mathbb{R}P(E|_F)$, as $F$ varies over the collections of components of $B$ of the different possible dimensions $m$, $\mathbb{R}P(E)$ is algebraically realized. \(\square\)

Proof of Proposition 5.2. Let $B$ be realized as a real algebraic $G$ variety in a representation $\Omega$ of $G$. We omit the subscript $\mathbb{R}$. To see the first claim, we write $\alpha(x) = \beta(x)/\gamma(x)$ where
\[ \beta : \Omega \to \text{End}(\Xi) \text{ and } \gamma : \Omega \to \mathbb{R} \text{ are regular and } \gamma \text{ does not vanish, and } x \in B. \text{ Then} \]

\[ (Id - \alpha(x))T = 0 \iff (\gamma(x) \cdot Id - \beta(x))T = 0 \]

and the latter condition defines a polynomial in \( x \) and \( T \) which exactly describes \( \mathbb{R} P(E) \) as a variety in \( B \times G(\Xi, 1) \). This verifies the first sentence of the proposition.

We show the non-singularity of \( \mathbb{R} P(E) \) at a point \((b, L)\). It is well known that \( G(\Xi, 1) \) is non-singular, hence \( B \times G(\Xi, 1) \) is non-singular. Let \( \tau_{(b, L)} \) be the tangent space of \( \Omega \times \text{End}(\Xi) \) at \((b, L)\), \( \tau_{(b, L)}^0 \) the subspace of vectors tangent to \( B \times G(\Xi, 1) \), and \( \pi : \tau_{(b, L)} \to \tau_{(b, L)}^0 \) the orthogonal projection. We describe \( \mathbb{R} P(E) \) as a subvariety of \( B \times G(\Xi, 1) \). As a smooth submanifold, its codimension is \( n - k \) where \( n = \text{dim } \Xi \). It suffices to find a Zariski open neighbourhood \( U \) of \((b, L)\) in \( \Omega \times \text{End}(\Xi) \) and polynomials \( t_i : \Omega \times \text{End}(\Xi) \to \mathbb{R} \), which vanish on \( \mathbb{R} P(E) \), \( 1 \leq i \leq n - k \), such that

1. \( \mathbb{R} P(E) \cap U = U \cap (B \times G(\Xi, 1)) \cap t_1^{-1}(0) \cap \cdots \cap t_{n-k}^{-1}(0) \)
2. \( \pi((\nabla t_i)_{(b, L)}) \) are linearly independent.

We define

\[ U = \{(x, T) \in \Omega \times \text{End}(\Xi) \mid \text{rk}[(Id - \alpha(b))(Id - \bar{\alpha}(x))] \geq n - k, \ TL \neq 0\}. \]

Here we denoted the extension of \( \alpha \) to an entire rational map \( \Omega \to \text{End}(\Xi) \) by \( \bar{\alpha} \), and \( \bar{\alpha} = \beta / \gamma \) is its description as a quotient of regular maps.

To construct the \( t_i \) we use local coordinates. Let \( x_1, \ldots, x_n \) be an orthonormal basis for \( \Xi \) such that \( \alpha(b) = Q \) and \( L \) correspond to the matrices

\[ \mathcal{M}_Q = \begin{pmatrix} Id(k) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{M}_L = \begin{pmatrix} Id(1) & 0 \\ 0 & 0 \end{pmatrix}. \]

Here \( Id(j) \) is an identity matrix of size \( j \times j \), and \( 0 \) is a zero matrix of the appropriate size. Given any matrix \( T \in \text{End}(\Xi) \), we express it as a matrix \( \mathcal{M}_T = (a_{i,j}(T)) \). We define a map

\[ s : \text{End}(\Xi) \to \Xi \quad \text{where} \quad s(T) = \sum_{i=1}^n a_{i,1}(T) x_i. \]

Expressed with respect to the basis, \( s(T) \) has as coefficients the first column of \( \mathcal{M}_T \). We note that

\[ (3) \quad Ts(T) = s(T) \text{ if } T \in G(\Xi, j) \text{ for any } 0 \leq j \leq n. \]

Let \( \cdot \cdot \cdot \) denote the standard inner product. For \( 1 \leq i \leq n - k \) we define

\[ t_i(x, T) = x_{k+i} \cdot [\gamma(b)\gamma(x)(Id - \alpha(b))(Id - \bar{\alpha}(x))s(T)]. \]

If \( (x, T) \in \mathbb{R} P(E) \), then \( (Id - \bar{\alpha}(x))T = 0 \) and \( t_i(x, T) = 0 \), so \( t_i \) vanishes on \( \mathbb{R} P(E) \). It remains to verify conditions (1) and (2).
We verify (1). Because \( t_i \) vanishes on \( \mathbb{R}P(E) \), we see containment "\( \subseteq \)". We show "\( \supseteq \)". Suppose that \((x,T) \in (B \times G(\Xi,1)) \cap U \) and \( t_1(x,T) = \cdots = t_{n-k}(x,T) = 0 \). By construction, \( x_{k+1}, \ldots, x_n \) span the image of \( Id - \alpha(b) \). Note also that \( \gamma(b) \gamma(x) \neq 0 \). This implies that
\[
(Id - \alpha(b))(Id - \tilde{\alpha}(x))s(T) = 0.
\]
By definition, \( \alpha(b) \) has rank \( k \), hence \( Id - \alpha(b) \) has rank \( n - k \). The rank condition in the definition of \( U \) implies that
\[
\text{rk}(Id - \alpha(b)) = \text{rk}((Id - \alpha(b)) \circ (Id - \tilde{\alpha}(x))) = n - k
\]
and that \( Id - \alpha(b) \) is injective on the image of \( Id - \tilde{\alpha}(x) \). It follows from (4) that
\[
(Id - \tilde{\alpha}(x))s(T) = 0.
\]
As \((x,T) \in U \), hence \( TL \neq 0 \), the first column of \( \mathcal{M}_T \) is non-zero and \( s(T) \neq 0 \). We pointed out that (see (3)) \( Ts(T) = s(T) \), and this implies that \([(Id - \tilde{\alpha}(x))T](Ts(T)) = 0 \).

The rank of \( T \) is one, and \((Id - \tilde{\alpha}(x))T \) vanishes on the image of \( T \), hence \((Id - \tilde{\alpha}(x))T = 0 \).

This implies that \((x,T) \in \mathbb{R}P(E) \), completing our verification of (1).

We verify (2). Observe that
\[
t_i(x,T) = \gamma^2(x)a_{k+i,1}(T).
\]

Differentiation with respect to \( a_{j,1} \), \( k + 1 \leq j \leq n \), i.e., with respect to position \((j,1)\) of the matrix \( T \) results in
\[
\frac{\partial t_i(x,T)}{\partial a_{j,1}}\bigg|_{(b,L)} = \gamma^2(b)\delta_{k+i,j} \quad \text{where} \quad k + 1 \leq j, i + k \leq n.
\]

This verifies that \( \{(\nabla t_i)_{(b,L)}\} \) is a set of linearly independent vectors in \( \tau_{(b,L)} \). We need to refine this to a statement about \( \{\pi((\nabla t_i)_{(b,L)})\} \).

Consider a smooth curve \( C : \mathbb{R} \to \Omega \times \text{End}(\Xi) \) with \( C(0) = (b,L) \) for which we make the additional assumption that \( C(u) = (b,T(u)) \) and \( T(u) \in G(\Xi,1) \). Its derivative \( (dC(u)/du)_{\mid u=0} \) defines an element in \( \tau^0_{(b,L)} \). Specifically, for \( 1 \leq i \leq n - k \) we define curves \( C_i(u) = (b,T_i(u)) \) where \( T_i(u) \) is the orthogonal projection from \( \Xi \) onto the line through \( 0 \) and \( (1 - u)x_1 + ux_{k+i} \). Observe that
\[
a_{k+i,1}(T_j(u)) = \frac{u(1-u)}{(1-u)^2 + u^2}\delta_{i,j} \quad \text{and} \quad \frac{d}{du}(t_i \circ (C_j(u)))_{\mid u=0} = \gamma^2(b)\delta_{i,j}.
\]

By the chain rule
\[
\frac{d}{du}(t_i \circ (C_j(u)))_{\mid u=0} = (\nabla t_i)_{(b,L)} \cdot (C_j(u)/du)_{\mid u=0} = \gamma^2(b)\delta_{i,j}.
\]
Note also that

\[ \pi \left( (\nabla t_1)_{(b,L)} \right) \cdot (C_j(u)/du)|_{u=0} = (\nabla t_1)_{(b,L)} \cdot (C_j(u)/du)|_{u=0}. \]

It follows from (5) and (6) that \( \pi \left( (\nabla t_1)_{(b,L)} \right), \ldots, \pi \left( (\nabla t_{n-k})_{(b,L)} \right) \) are linearly independent. This completes the verification of (2) and the proof of the proposition. \( \square \)

There are a number of standard constructions (such as the Whitney sum of vector bundles) which may be applied to equivariant topological vector bundles as well as strongly algebraic \( G \) vector bundles. With the obvious meaning we apply notation used for vector bundles to the maps which classify bundles. More generally than above, we consider strongly algebraic \( G \) vector bundles over \( \Lambda = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H} \). They are defined as equivariant entire rational maps \( \mu : X \to G_{\Lambda}(\Xi,k) \), where \( X \) is a real algebraic \( G \) variety, and the Grassmannians \( G_\Lambda(\Xi,k) \) are real algebraic \( G \) varieties as defined earlier in this section. Finally, in the introduction we defined what it means that a real \( G \) vector bundle is algebraically realized. In analogy we say that a \( k \)-dimensional vector bundle \( \xi = (E,p,B) \) over \( \Lambda \) is algebraically realized if there exists a non-singular real algebraic \( G \) variety \( X \) and an equivariant diffeomorphism \( \phi : X \to B \) such that the classifying map \( X \to G_{\Lambda}(\Xi,k) \) of \( \phi^*(\xi) \) is equivariantly homotopic to an equivariant entire rational map. In our next proposition we summarize a few basic facts about constructions with strongly algebraic vector bundles (e. g., see [BCR, Chapter 12, Section 1]).

**Proposition 5.3.** Let \( X \) be a real algebraic \( G \) variety and \( \mu : X \to G_{\Lambda}(\Xi,k) \) and \( \mu' : X \to G_{\Lambda}(\Xi',k') \) be two strongly algebraic \( G \) vector bundles over \( \Lambda \). Then

1. The Whitney sum \( \mu \oplus \mu' : X \to G_{\Lambda}(\Xi \oplus \Xi', k + k') \) is a strongly algebraic \( G \) vector bundle over \( \Lambda \).
2. The tensor product \( \mu \otimes \mu' : X \to G_{\Lambda'}(\Xi _{\Lambda} \Xi', k \cdot k') \) is a strongly algebraic \( G \) vector bundle over \( \Lambda ' \). Here \( \Lambda ' = \mathbb{R} \) if \( \Lambda = \mathbb{R} \) or \( \mathbb{H} \), and \( \Lambda ' = \mathbb{C} \) if \( \Lambda = \mathbb{C} \), and \( \epsilon_{\Lambda} = 1 \) if \( \Lambda = \mathbb{R} \) or \( \mathbb{C} \), and \( \epsilon_{\Lambda} = 4 \) if \( \Lambda = \mathbb{H} \).
3. The dual \( \mu^* \) of \( \mu \) is a strongly algebraic \( G \) vector bundle over \( \mathbb{R} \) if \( \Lambda = \mathbb{R} \) or \( \mathbb{H} \), and over \( \mathbb{C} \) if \( \Lambda = \mathbb{C} \).
4. Every strongly algebraic \( G \) vector bundle over \( \Lambda \) may be viewed as a strongly algebraic \( G \) vector bundle over \( \mathbb{R} \).
5. If \( \Gamma \) is a \( k \)-dimensional representation of \( G \) over \( \Lambda \), then the product bundle over \( X \) with fibre \( \Gamma \), represented by the constant map to any point in the component of \( G_{\Lambda}(\Xi,k)^G \) associated to the representation \( \Gamma \), is a strongly algebraic \( G \) vector bundle over \( \Lambda \).
6. In the assumption and conclusion for (1)–(5) we may replace strongly algebraic \( G \) vector bundles by algebraically realized bundles. But in (1) and (2) we need to assume that \( \mu = (E,p,B) \) and \( \mu' = (E',p',B) \) are algebraically realized over the same non-singular \( X \).

**Proof.** All claims in this proposition are elementary. E. g., for (1) one notes that the map

\[ G_{\Lambda}(\Xi,k) \times G_{\Lambda}(\Xi',k') \to G_{\Lambda}(\Xi \oplus \Xi', k + k') \]
which produces the Whitney sum is an equivariant entire rational map. For (4) one notes that the inclusion map
\[ G_{\Lambda}(\Xi, k) \rightarrow G_{\mathbb{R}}(\Xi, \epsilon k) \]
is entire rational. Here we understand \( \Xi \) as a representation of \( G \) over \( \Lambda \) as well as over \( \mathbb{R} \), and \( \epsilon = 1, 2, \) or \( 4 \) depending on whether \( \Lambda = \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \). \( \square \)

We need a more detailed description of \( G_{\mathbb{R}}(\Xi, k)^H \) as a variety \( (H \subset G) \). We use the following notation.

Let \( \chi \) be an irreducible representation of a compact Lie group \( H \) over \( \mathbb{R} \), and let \( D(\chi) \) the division ring \( \text{Hom}_H(\chi, \chi) \) consisting of all \( H \)-equivariant endomorphisms of \( \chi \) (i.e., \( D(\chi) = \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \)). We note that \( \chi \) is the realification of a representation of \( H \) over \( D(\chi) \). Let \( \Xi \) be a representation of a compact Lie group \( G \) over \( \mathbb{R} \), and let \( H \) be a closed subgroup of \( G \). Restrict \( \Xi \) to a representation of \( H \) which is then denoted by \( \text{Res}_H \Xi \). Write \( \text{Res}_H \Xi \) as the sum of multiples of the irreducible representations of \( H \), so \( \text{Res}_H \Xi = \sum m(\chi, \Xi) \chi \).

The integers \( m(\chi, \Xi) \) are defined by this equation.

**Proposition 5.4.** Let \( G \) be a compact Lie group, \( \Xi \) an orthogonal representation of \( G \), \( k \) a non-negative integer and \( H \) a closed subgroup of \( G \). Then

\[ G_{\mathbb{R}}(\Xi, k)^H = F(V_1) \cup \cdots \cup F(V_s) \]
is a disjoint union of non-singular subvarieties, where \( V_j \) ranges over the \( k \)-dimensional representations of \( H \) which are isomorphic to summands of \( \text{Res}_H \Xi \). As a variety

\[ F(V_i) = \prod_{\chi} G_{D(\chi)}(D(\chi)^{m(\chi, \Xi)}, m(\chi, V_i)) \]

Here \( \chi \) ranges over the irreducible representations of \( H \) and the integers \( m(\cdot, \cdot) \) are as defined above.

A version of this proposition for classifying spaces of more general \( G \) bundles has been given by Lashof [L]. In comparison, our result is in the algebraic category, and for the finite approximation of the classifying space under consideration. We prove this proposition in Section 10.

It is worthwhile to point out

**Proposition 5.5 ([DS, Section 5]).** Let \( X \) be a non-singular real algebraic \( G \) variety and \( H \) a closed subgroup of \( G \). The collection \( F \) of components of \( X^H \) of a given dimension (or slice type) is a non-singular subvariety of \( X \), and the tangent bundle \( TX \) of \( X \) as well as the normal bundle \( \nu(F) \) of \( F \) in \( X \) are strongly algebraic \( G \), resp. \( NH \), vector bundles.

**Remark.** The statements in this proposition are proved in [DS]. We like to point out that the classifying maps of the bundles under consideration are actually well defined, and not only up to homotopy, because \( X \) is a submanifold of a Euclidean space, and not only an abstract manifold. For the proof that the tangent bundle of a non-singular variety (without group action) is strongly algebraic see [AK1, Lemma 2.3] or [BCR, p. 260]. The construction is natural enough to apply also in the equivariant setting. Suppose \( X \) is of
dimension $k$ and realized in $\Xi$, and $\mu : X \to G_\mathbb{R}(\Xi, k)$ is the classifying map of $TX$. Since $\mu$ is entire rational, $X^H \cap \mu^{-1}(F(V))$ is a real algebraic variety, and so is $F$. It is the disjoint union of $X^H \cap \mu^{-1}(F(V_i))$, where $V_i$ varies over the representations of $H$ which occur as $T_x X$ for $x \in F$. The proof that $F$ is non-singular is based on work of Whitney [W].

6. Odd Order Groups – Proof of Theorem A (1)

In this section we prove Theorem A (1). As explained in the introduction, we need to show that equivariant bordism classes have algebraic representatives.

**Theorem 6.1.** Let $G$ be a group of odd order. Every closed smooth $G$ manifold is equivariantly cobordant to a non-singular real algebraic $G$ variety.

**Proof of Theorem A (1).** This is an immediate consequence of Theorem 6.1 and Theorem 1.2. □

To prove Theorem 6.1 we need some preparation. We recall some bordism theoretic notation. Disjoint union and cartesian product of representatives define an addition and a product for equivariant bordism classes, so $\mathcal{N}_G$ and $\mathcal{N}_G^0$ are rings and $\mathcal{N}_G^0$ is an algebra over $\mathcal{N}_G^*$. Consider the ring $\mathcal{N}_*$ of bordism classes of closed manifolds. We may think of manifolds without group action as manifolds with trivial action. In this sense, $\mathcal{N}_G^*$ is also an algebra over $\mathcal{N}_*$. We indicate bordism classes by $[\cdot]$.

For any subgroup $H$ of $G$ we define a ring homomorphism $\phi_H : \mathcal{N}_G^0 \to \mathbb{Z}_2$. Let $A$ represent a class in $\mathcal{N}_G^0$, so $A$ is a finite $G$ set. We set $\phi_H([A]) = |A^H|$ mod 2. It is elementary to show (e. g., see [C, Section 1])

**Proposition 6.2.** Let $G$ be a group of odd order. There exist idempotents $e_H \in \mathcal{N}_G^0$, one for each subgroup $H$ of $G$, such that

1. $\phi_H(e_K) = 1$ if $(H) = (K)$.
2. $\phi_H(e_K) = 0$ if $(H) \neq (K)$.
3. $\sum_{(H)} e_H = 1$ where the summation ranges over subgroups of $G$, taking one representative in each conjugacy class of subgroups.

To state our bordism result we need some more notation. Let $H$ be a subgroup of $G$, $V$ a non-trivial irreducible representation of $H$, and $i$ a positive integer. Set

$$R_{H,V,i} = \mathbb{R}P((\eta_{i-1} \otimes \mathcal{V}) \oplus \mathbb{R}).$$

Here $\eta_{i-1}$ is the canonical line bundle over $\mathbb{C}P^{i-1}$ with trivial action of $H$. This bundle is tensored with the product bundle $\mathcal{V}$ over $\mathbb{C}P^{i-1}$ with fibre $V$. We give $V$ any of its complex structures. Then we add the product bundle $\mathbb{R}$ with fibre $\mathbb{R}$ and take the total space of the associated projective bundle. The result is $R_{H,V,i}$.

Consider formal variables $\gamma_{H,V,i}$. Let $\Psi_0(\gamma_{H,V,i}) = R_{H,V,i}$. To any polynomial $P$ in the $\gamma_{H,V,i}$ with coefficients in $\mathcal{N}_*$, represented by closed manifolds, we associate the corresponding expression $\Psi_0(P)$ in the $R_{H,V,i}$ where products translate into cartesian products and sums into disjoint unions. Thus we defined $\Psi_0(P)$ for any element $P$ in $\mathcal{N}[*_{\gamma_{H,V,i}}]$
where $V$ ranges over the non-trivial irreducible representations of $H$ and $i = 1, 2, \ldots$. We use the module structure of $\mathcal{N}_*^G$ over $\mathcal{N}_0^G$ and the $e_H$ from Proposition 6.2 to define

$$\Psi_H(P) = e_H \cdot [G \times H \Psi_0(P)]$$

Our next result is basically taken from a paper of Costenoble [C], but with some work a similar (and equally useful) result could be obtained from older work of Stong [S1].

**Theorem 6.3.** Let $G$ be a group of odd order. There exists a surjective graded $\mathcal{N}_*$ module homomorphism

$$\Psi : \prod_{(H)} \mathcal{N}_*[\gamma_{H,V,i}] \to \mathcal{N}_*^G$$

where the product is taken over all conjugacy classes of subgroups of $G$, $V$ ranges over the non-trivial irreducible representations of $H$, and $i = 1, 2, \ldots$. The (abstract) polynomial generators live in dimension $|\gamma_{H,V,i}| = 2(i-1) + \dim \mathbb{R} V$. On each of the factors $\Psi$ is given by the map $\Psi_H$ defined above.

**Remarks on the Proof.** The statement in this theorem can be extracted from [C]. Costenoble studies a map $\Phi : \mathcal{N}_*^G \to \prod_{(H)} (\mathcal{N}_*[\gamma_{H,V,i}])^{WH}$. Here $WH = NH/H$ acts on the set of irreducible representations of $H$ by conjugating them, i.e., the action permutes these representations. Our map $\Psi$ is the explicit description of the converse of $\Phi$. Specifically, $\Psi \circ \Phi$ is the identity on $\mathcal{N}_*^G$. But $\Phi \circ \Psi$ is the identity only if one restricts oneself to the $WH$ invariant part of the factor corresponding to $H$. The need to pass to $WH$ fixed sets arises in Proposition 4.3 of [C]. This is the strategy to deduce the claim in the first sentence of our theorem from Costenoble’s paper.

We discuss the formula which describes $\Psi$. Observe that we consider the $\gamma_{H,V,i}$ as abstract elements, and Costenoble’s $\gamma_{H,V,i}$ correspond to our $\Psi_H(\gamma_{H,V,i})$. This is the formula given on page 284 of [C] for abelian groups. (In the abelian group case the explicit expression for $\Psi_H(\gamma_{H,V,i})$ is of particular interest because then $WH$ acts trivially on the set of $H$ representations, hence on $\mathcal{N}_*[\gamma_{H,V,i}]$. This leads to a perfectly explicit calculation of $\mathcal{N}_*^G$ in Theorem 6.3.) Actually, the same analysis as the one Costenoble went through provides the expression for $\Psi_H(\gamma_{H,V,i})$ also in the non-abelian case. This is the strategy to verify the second sentence of the theorem. \(\Box\)

Obviously, we only sketched the strategy to prove Theorem 6.3, but the details would occupy several pages where we would only add an occasional technical remark to Costenoble’s proof.

**Proof of Theorem 6.1.** Obviously the canonical line bundle $\eta_{i-1}$ is a strongly algebraic vector bundle over $\mathbb{C}P^{i-1}$ (over $\Lambda = \mathbb{C}$ in Definition 5.1). A combination of the statements in 5.3 implies that $(\eta_{i-1} \otimes \mathbb{C} V) \oplus \mathbb{R}$ is a strongly algebraic $H$ vector bundle over $\mathbb{C}P^{i-1}$ (over $\Lambda = \mathbb{R}$). Proposition 5.2 implies that $R_{H,V,i} = \mathbb{R}P((\eta_{i-1} \otimes \mathbb{C} V) \oplus \mathbb{R})$ is a non-singular real algebraic $H$ variety. Consider a polynomial $P$ in the $\gamma_{H,V,i}$ with coefficients in $\mathcal{N}_*$. Represent the coefficients by non-singular real algebraic varieties, which is possible by the
Nash Tognoli Theorem. It follows from Proposition 2.6 that $\Psi_0(P)$, the corresponding polynomial expression in the $R_{H,V,i}$, is a non-singular real algebraic $H$ variety. Another application of Propositions 3.1 and 2.6 implies that $e_H \cdot (G \times_H \Psi_0(P))$ is a non-singular real algebraic $G$ variety. Thus $\Psi_H(P)$ is represented by a non-singular real algebraic $G$ variety. It follows from Theorem 6.3 and Proposition 2.6 that every class in $\mathfrak{H}_G^*$ has an algebraic representative, and this is what we wanted to show.

7. A Simple Bordism Construction – Proof of Theorem A (2)

The first result of this section is based on a simple and well known bordism construction. It is the essential tool in the proof of our localization principle stated in the introduction (see Proposition 1.4).

**Proposition 7.1.** Let $M$ be a closed smooth $G$ manifold. Suppose $G$ contains a central element $\tau$ of order two. Then $M$ is equivariantly cobordant to $\mathbb{R}P(\nu(M^\tau) \oplus \mathbb{R})$.

Here $M^\tau = \{x \in M \mid \tau x = x\}$ is the $\tau$ fixed set and $\nu(M^\tau)$ denotes the normal bundle of $M^\tau$ in $M$, to which we add the product bundle with fibre $\mathbb{R}$.

**Proof.** Set $M_0 = M - \text{Int}(D(\nu(M^\tau)))$, where $D$ stands for the disk bundle. It is a compact $G$ manifold on which $\tau$ acts freely. Let $Z$ be the total space of the $D^1$ bundle associated with the principal $\mathbb{Z}_2$ bundle $M_0 \to M_0/\tau$. Since $\tau$ is central in $G$, $Z$ is a smooth $G$ manifold. We note that

$$\partial Z = M_0 \cup \partial M_0 \times_\tau D^1.$$  

The action of $\tau$ on $D^1$ is antipodal, and the union is taken over $\partial M_0 = \partial M_0 \times_\tau S^0$. Consider $M \times [0,1]$ and regard $M_0$ as a submanifold of $M \times \{1\}$. We glue $Z$ and $M \times [0,1]$ together along $M_0$. The resulting $G$ manifold is an equivariant cobordism between $M$ and $\mathbb{R}P(\nu(M^\tau) \oplus \mathbb{R})$, as claimed. □

**Proof of Proposition 1.4.** Suppose $M$ is algebraically realized, so there exists a non-singular real algebraic $G$ variety $X$ and an equivariant diffeomorphism $\phi : X \to M$. It follows from Proposition 5.5 that $\nu(X^\tau)$ is, in a natural way, a strongly algebraic $G$ vector bundle if we restrict it over the components of $X^\tau$ of a given dimension. By definition, this means that $\nu(M^\tau)$ is algebraically realized.

Suppose now that $\nu(M^\tau)$ is algebraically realized. It follows from Corollary 1.3 that $\mathbb{R}P(\nu(M^\tau) \oplus \mathbb{R})$ is algebraically realized. Proposition 7.1 states that $M$ is equivariantly cobordant to $\mathbb{R}P(\nu(M^\tau) \oplus \mathbb{R})$, and it follows from Theorem 1.2 that $M$ is algebraically realized. □

We need one simple result concerning the algebraic realization of equivariant vector bundles. This concept was introduced in the introduction. In the following proposition we assume that the total space of the bundle has a well defined dimension, even if its base space consists of several components of possibly different dimension. In the proof we need the concept of a variety having totally algebraic homology. It was defined by Akbulut and King [AK1]. In particular, if $X$ has totally algebraic homology, then every class in $\mathfrak{N}_*(X)$ has an algebraic representative, i.e., it can be represented by an entire rational map from a non-singular real algebraic variety to $X$. Akbulut and King showed that the
real Grassmannians have totally algebraic homology. An analogous argument shows that
the complex and quaternion Grassmannians have totally algebraic homology. In addition,
the product of varieties with totally algebraic homology as well as their disjoint union
again have totally algebraic homology.

**Proposition 7.2.** Let $G$ be a compact Lie group and $\xi$ a real $G$ vector bundle over a
closed smooth manifold on which $G$ acts trivially. Then $\xi$ is algebraically realized.

**Proof.** According to the definition of an algebraically realized bundle, it suffices to prove
the statement when the base space is connected. By a result of Segal [Sg, Proposition 2.2]
and the remark on the real case [Sg, p. 133–134], any $G$ vector bundle $\xi$ over a base space $B$
with trivial action can be written as a Whitney sum

$$\xi = \bigoplus \xi_\chi \otimes D(\chi) \chi.$$

Here $\chi$ ranges over the irreducible representations of $G$ and $D(\chi) = \text{Hom}_G(\chi, \chi)$ is the
set of equivariant endomorphisms of $\chi$. The division ring $D(\chi)$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.
In this decomposition $\xi_\chi = \text{Hom}_G(\chi, \xi)$ is a $D(\chi)$ vector bundle, which is then tensored with
the product bundle over $B$ with fibre $\chi$. Let

$$\mu_\chi : B \to G_{D(\chi)} (D(\chi)^{n_\chi}, k_\chi)$$

be the classifying map for $\xi_\chi$, and

$$\mu : B \to \prod_\chi G_{D(\chi)} (D(\chi)^{n_\chi}, k_\chi)$$

their product. The Grassmannians in this product have totally algebraic homology, and so
does their product. It follows from [AK1, Proposition 2.8] that there exists a non-singular
real algebraic variety $X$, an entire rational map

$$\eta : X \to \prod_\chi G_{D(\chi)} (D(\chi)^{n_\chi}, k_\chi)$$

and a diffeomorphism $\phi : X \to B$ such that $\eta$ is homotopic to $\mu \circ \phi$. (This is the approach
used by Benedetti and Tognoli in the realization of vector bundles over closed manifolds
by strongly algebraic vector bundles.) Compose $\eta$ with the entire rational map

$$\rho : \prod_\chi G_{D(\chi)} (D(\chi)^{n_\chi}, k_\chi) \to G_{\mathbb{R}}(\Xi, k).$$

This map is such that the factor which belongs to $\chi$ is first tensored with $\chi$, then each factor
is realified, and finally the direct factors are added using Whitney sum (see Proposition
5.3). The values for $\Xi$ and $k$ are obtained from the construction. Observe that $\rho \circ \eta$ is
entire rational, hence it represents a strongly algebraic $G$ vector bundle. This shows that
$\xi$ is algebraically realized by the triple $(X, \phi, \rho \circ \eta)$. □
Proof of Theorem A (2). We want to show that every smooth manifold with a semifree action of a compact Lie group $G$ is algebraically realized. We showed this in case that $G$ is of odd order (this is Theorem A (1) proved in Section 6), and if $G$ acts freely (see Corollary 4.2). Thus we may assume that $G$ is not of odd order and that the action has a fixed point.

Let $x \in M$ be a fixed point of the action of $G$. Then $G$ acts freely on the unit sphere in the normal slice at $x$. Milnor showed that every element of order two in $G$ must be central. Hence $G$ has a non-trivial central element $\tau$ of order two. Obviously, $M\tau = M^G$. It follows from Proposition 7.2 that $\nu(M^\tau)$ is algebraically realized, and from Proposition 1.4 that $M$ is algebraically realized. This is what we wanted to show. □

8. Applications

Our first application is an alternative proof of a result of Conner and Floyd.

**Theorem** (see [CF, 31.3]). There is no smooth action of $(\mathbb{Z}_2)^k$ on a closed connected smooth manifold of positive dimension with exactly one fixed point.

**Proof.** First of all we note that it suffices to prove the theorem for a $k = 1$. If $k > 1$ and $x$ is a fixed point, then there is a proper subgroup of $(\mathbb{Z}_2)^k$ which has a closed fixed point component of positive dimension which contains $x$. This creates the same problem, but for a smaller value of $k$.

Denote the manifold by $M$, the fixed point by $x$, and denote $\mathbb{Z}_2$ by $G$. By Theorem A we may assume that $M$ is a non-singular real algebraic variety and that $G$ acts real algebraically on it. Let $O(M)$ be the coordinate ring of $M$ and $m_x$ be the maximal ideal which vanishes at $x$. There is an exact sequence of $G$ modules

$$0 \to m_x^2 \to m_x \xrightarrow{d} T_xM^* \to 0$$

where $T_xM^* = \text{Hom}_\mathbb{R}(T_xM, \mathbb{R})$ and $d$ is given by

$$df(v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$ 

Here $v \in T_xM$, $f \in m_x$. (Since $M \subset \mathbb{R}^m$ for some $m$, the expression $x + tv$ makes sense.)

Choose a homomorphism $s : T_xM^* \to m_x$ such that the composition $d \circ s$ is the identity map and consider

$$\phi(w) = (s(w) + gs(gw))/2$$

where $g \in G$ is the generator. Then $\phi$ is an equivariant homomorphism and the composition $d \circ \phi$ is the identity map. Note that $O(T_xM)$ is the symmetric algebra of $T_xM^*$, so $\phi$ extends to a unique ring homomorphism $A : O(T_xM) \to m_x \subset O(M)$. This induces an equivariant polynomial map $f_A : M \to T_xM$ with $f_A(x) = 0$. As is easily seen

$$df_A(x) : T_xM \to T_0(T_xM) = T_xM$$

is the identity map (cf. [PR2]).

Let $\delta$ be the non-trivial 1-dimensional $G$ module. Then $T_xM = n\delta$ where $n = \dim M$. Take a projection map $p : n\delta \to (n-1)\delta$. Then $(p \circ f_A)^{-1}(0)$ is a (possibly singular) real algebraic $G$ variety of dimension one which contains $x$ and is non-singular at $x$ because
$df_A(x)$ is the identity. We resolve its singularities if they exist. The connected component of the resulting (non-singular) real algebraic $G$ variety containing $x$ is diffeomorphic to a circle and the $G$ action on it has exactly one fixed point. But it is easy to see that this is impossible. □

The second application which we stated in the introduction is the proof of the Fixed Point Conjecture for odd order abelian groups. Here is our argument.

**Proof.** An abelian group of odd order acts smoothly and without a fixed point on a disk if and only if it acts smoothly on a (homotopy) sphere with exactly one fixed point. This is a consequence of results of Oliver [O] and Petrie [P1]. According to Theorem A (1) the action on the sphere is algebraically realized. Call the sphere $S$ and the fixed point $x$. It is easy to show that $S \setminus x$ is equivariantly diffeomorphic to a real algebraic $G$ variety (see [M2, p. 105] or [DMP, p. 50–51]), and $S \setminus x$ is diffeomorphic to Euclidean space. Obviously, the action on $S \setminus x$ has no fixed point. □

9. **Algebraic Groups and Compact Lie Groups (Appendix)**

The use of algebraic groups and $G$ modules may seem to be more appropriate to algebraic geometers, while the use of compact Lie groups and representations may seem more natural to topologists. We want to reconcile these two concepts. For groups the result is as follows:

**Theorem** (see [OV, p. 247]). On any compact Lie group there is a unique real algebraic structure. If $L$ is a compact subgroup of $K$ then, viewed as real algebraic groups, $L$ is an algebraic subgroup of $K$.

For representations and modules we have the following easy consequence.

**Proposition.** Let $G$ be a compact Lie group and $\Omega$ an orthogonal representation of $G$ with underlying space $\mathbb{R}^n$. Then, viewed as an algebraic group, $G$ acts algebraically on $\mathbb{R}^n$, and in this sense $\Omega$ can be considered as a real algebraic $G$ module.

In algebraic transformation groups a different definition of a real algebraic $G$ variety may seem more appropriate.

**Definition.** Let $V$ be a real algebraic variety, and let $G$ be a real algebraic group. An action $\theta : G \times V \to V$ is said to be algebraic if $\theta$ is regular.

This definition seems to be weaker than the one given in the introduction. Suppose $V$ is realized as the zero set of a polynomial $p : \mathbb{R}^n \to \mathbb{R}$. According to the definition in the introduction, one would require that $\theta$ extends to a linear action $\overline{\theta} : G \times \mathbb{R}^n \to \mathbb{R}^n$. Actually, in the following sense, these definitions are the same.

**Theorem.** Let $V$ be a real algebraic variety with an algebraic action of the group $G$. There exists a real algebraic $G$ variety $V'$ (in the sense of the introduction) and a regular isomorphism $\phi : V \to V'$ which is equivariant.

For a proof of this theorem see [Kr, II.2.4] and a remark on the real case in [Sc, (1.5)].
10. THE ALGEBRAIC STRUCTURE OF $G_{\mathbb{R}}(\Xi, k)$ (APPENDIX)

We restate and prove Proposition 5.4.

**Proposition 5.4.** Let $G$ be a compact Lie group, $\Xi$ an orthogonal representation of $G$, $k$ a non-negative integer and $H$ a closed subgroup of $G$. Then

$$G_{\mathbb{R}}(\Xi, k)^H = F(V_1) \cup \cdots \cup F(V_s)$$

is a disjoint union of non-singular subvarieties, where $V_j$ ranges over the $k$-dimensional representations of $H$ which are isomorphic to summands of $\text{Res}_H \Xi$. As a variety

$$F(V_i) = \prod_{\chi} G_{D(\chi)}(D(\chi)^{m(\chi, \Xi)}, m(\chi, V_i)),$$

Here $\chi$ ranges over the irreducible representations of $H$, the integers $m(\chi, U)$ denote the multiplicity of $\chi$ in $U$, and $D(\chi)$ is the division ring $\text{Hom}_H(\chi, \chi)$.

In the proof of the proposition we will make use of

**Lemma 10.1.** Let $H$ be a compact Lie group and $\chi$ an irreducible representation of $H$ of dimension $\delta$. Set $D = D(\chi)$. There is an algebraic isomorphism

$$G_{\mathbb{R}}(a\chi, b\delta)^H \to G_D(D^a, b)$$

where $a, b$ are any natural numbers, $a \geq b$.

**Proof.** In this set-up we suppose an equivariant inner product on $\chi$, which then defines one on $a\chi$. Observe that

$$G_{\mathbb{R}}(a\chi, b\delta)^H = \{ L \in G_{\mathbb{R}}(a\chi, b\delta) \mid hLh^{-1} = L \text{ for all } h \in H \},$$

which means that $G_{\mathbb{R}}(a\chi, b\delta)^H$ consists of exactly those $L \in G_{\mathbb{R}}(a\chi, b\delta)$ such that

1. $L^2 = L$ and $L^* = L$. (As before, $^*$ indicates an adjoint transformation, or, if we think in terms of matrices, that the matrix has been transposed.)
2. $L$ is $H$ equivariant
3. trace $L = b\delta$ (which, using (1) and (2), is equivalent to $L(a\chi) \cong b\chi$).

Composition with $L$ defines a homomorphism

$$L_0 : \text{Hom}_H(\chi, a\chi) \to \text{Hom}_H(\chi, a\chi).$$

Identifying $\text{Hom}_H(\chi, \chi)$ with $D$ through an algebra homomorphism, hence $\text{Hom}_H(\chi, a\chi)$ with $D^a$, induces a homomorphism

$$L_1 : D^a \to D^a.$$ 

Properties (1)–(3) translate into

1. $L_1^2 = L_1$ and $L_1^* = L_1$. (Now the adjoint is taken with respect to the canonical inner product on $D^a$, and if we represent endomorphisms through matrices, then $^*$ indicates that the matrix is transposed and its entries are conjugated in $D$.)
2. $L_1$ is a homomorphism over $D$.
3. trace $L_1 = b$. 


Setting $L_1 = \Phi(L)$ we obtain a regular map

$$\Phi : G_\mathbb{R}(a\chi, b\delta)^H \rightarrow G_D(D^a, b).$$

We construct a regular inverse for $\Phi$. For this purpose, remember that there is a representation $\chi^c$ of $H$ over $D$ which becomes $\chi$ if one reduces the coefficients to $\mathbb{R}$. For the underlying real representations we have natural isomorphisms

$$\chi \cong D \otimes_D \chi^c \quad \text{and} \quad a\chi \cong D^a \otimes_D \chi^c$$

An element $T \in G_D(D^a, b)$, i.e., an endomorphism of $D^a$ which satisfies (1')–(3'), induces a map $\Psi(T) = T \otimes I_d$ which satisfies (1)–(3). This defines a regular map

$$\Psi : G_D(D^a, b) \rightarrow G_\mathbb{R}(a\chi, b\delta)^H$$

which is the inverse of $\Phi$. This shows that $\Phi$ is an algebraic isomorphism. □

Proof of Proposition 5.4. By definition

$$G_\mathbb{R}(\Xi, k)^H = \{ L \in G_\mathbb{R}(\Xi, k) \mid hLh^{-1} = L \text{ for all } h \in H \}.$$ 

The image of any $L \in G_\mathbb{R}(\Xi, k)^H$ is a representation of $H$ and a subrepresentation of $\text{Res}_H\Xi$. Given any real $k$-dimensional summand $V$ of $\text{Res}_H\Xi$ we set

$$F(V) = \{ L \in G_\mathbb{R}(\Xi, k)^H \mid \text{im}(L) \cong V \}.$$ 

Apparently, $F(V) \cap F(V') = \emptyset$ if $V$ and $V'$ are not isomorphic as representations of $H$, and the union of the $F(V)$ is $G_\mathbb{R}(\Xi, k)^H$. The union is taken over the $k$-dimensional representations $V$ of $H$ which are isomorphic to summands of $\text{Res}_H\Xi$. From the following discussion it will be clear that each $F(V)$ is connected. Together this implies the decomposition of $G_\mathbb{R}(\Xi, k)^H$ into components as claimed.

In our definition of the Grassmannian we assumed that the underlying space of $\Xi$ has an inner product which is invariant under the action of $G$, so $\Xi$ is an orthogonal representation. This allows us to talk unambiguously about orthogonal projections onto subspaces.

We show that each $F(V)$ is a subvariety of $G_\mathbb{R}(\Xi, k)$. Obviously, $G_\mathbb{R}(\Xi, k)^H$ is a subvariety of $G_\mathbb{R}(\Xi, k)$. We provide polynomials which distinguish the subsets $F(V_j)$. For this purpose, decompose $\text{Res}_H\Xi$ as a sum of representations $\Gamma_\chi = m(\chi, \Xi)\chi$ where $\chi$ is an irreducible representation of $H$. So $\Gamma_\chi$ contains all of the summands $\chi$ of $\text{Res}_H\Xi$. Let $p_{\Gamma_\chi} : \Xi \rightarrow \Xi$ be the orthogonal projection onto $\Gamma_\chi$. Define regular maps

$$t_\chi : \text{End}(\Xi) \rightarrow \mathbb{R} \quad \text{by} \quad t_\chi(L) = \text{trace}(L \circ p_{\Gamma_\chi})$$

Expressed differently, if $L \in G_\mathbb{R}(\Xi, k)^H$, then $L$ restricts to an endomorphism of $\Gamma_\chi$, and $t_\chi(L)$ is its trace. In this case the traces $t_\chi$ determine the multiplicity of $\chi$ in the image of $L$. Hence the $t_\chi$ determine $\text{im}(L)$ as a representation of $H$, and distinguish the subsets $F(V_i)$ of $G_\mathbb{R}(\Xi, k)^H$. As the $t_\chi$ are regular maps, each $F(V)$ is a subvariety of $G_\mathbb{R}(\Xi, k)$. 
We discuss the non-singularity of the $F(V)$. Based on the explicit nature of the example, one may check this explicitly. We give a more general argument. It follows from Whitney’s work [W] and other principles in algebraic geometry that $X^H$ is non-singular at each point $x \in X^H$ if $X$ is a non-singular real algebraic $G$ variety and $H$ is a subgroup of $G$ (compare [DS]). Because $F(V)$ is connected, as we will see below, $F(V)$ is non-singular.

Finally, we study the decomposition of $F(V)$ into factors. Let $L \in F(V)$, then $L$ decomposes as a direct sum $L = \bigoplus \chi L_\chi$. Each $L_\chi$ is defined as the restriction of $L$ to the summand $\Gamma_\chi$ of $\text{Res}_H \Xi$, which it maps to itself. Observe that $L_\chi(\Gamma_\chi)$ is a representation of $H$, namely $m(\chi, V)_\chi$. Then

1. $L_\chi^2 = L_\chi$ and $L_\chi^t = L_\chi$
2. $L_\chi$ is $H$ equivariant
3. trace $L_\chi = m(\chi, V) \dim_\mathbb{R} \chi$.

The correspondence between $L$ and the collection of the $L_\chi$ defines a regular isomorphism of varieties

$$F(V) \underset{\cong}{\rightarrow} \prod \chi G_\mathbb{R}(\Gamma_\chi, m(\chi, V) \dim_\mathbb{R} \chi)^H.$$ 

It follows from Lemma 10.1 that the factors in this product are algebraically isomorphic to $G_{D(\chi)}(D(\chi)^{m(\chi, \Xi)}, m(\chi, V))$. This provides the factorization of $F(V_i)$ claimed in the proposition. We note that the Grassmannians which occur as factors in this product are connected, hence each $F(V)$ is connected. This completes the proof. □

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