ALGEBRAIC REALIZATION OF EQUIVARIANT VECTOR BUNDLES

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ABSTRACT. Let $G$ be a compact Lie group. We consider the problem of algebraically realizing real $G$ vector bundles over a closed smooth $G$ manifold. Theorem B provides assumptions under which this is possible. It also allows us to improve on a previous theorem of ours which tells when a closed smooth $G$ manifold is algebraically realized, i.e., when it is equivariantly diffeomorphic to a non-singular real algebraic $G$ variety.

1. Introduction

A central question in real algebraic geometry is whether a given smooth situation can be realized algebraically. Results of Seifert [Se] and a program established by Nash [N] motivated the

Nash–Tognoli Theorem$^1$ [T]. Every closed smooth manifold (i.e., a compact manifold without boundary) is diffeomorphic to a non-singular real algebraic variety.

To simplify language, we say that a smooth manifold is algebraically realized if it is diffeomorphic to a non-singular real algebraic variety. The variety, or the variety together with the diffeomorphism, is then called an algebraic realization of the manifold.

Subsequent work of Akbulut and King [AK1], [AK2] did not only provide a more conceptual approach to the algebraic realization problem for manifolds, but it also opened the way to study the corresponding problem for manifolds which carry an additional structure. In many situations such an additional structure, like the one of a smooth vector bundle over the manifold, is captured by a map to a classifying space. Assuming that the classifying space is a real algebraic variety $Y$, and $\chi : M \to Y$ is such a map, one would like to find an algebraic realization for $(M, \chi)$, i.e., a non-singular real algebraic variety $X$ and a diffeomorphism $\varphi : X \to M$ such that $\chi \circ \varphi$ is homotopic to an entire rational map. We have additional structures in mind whose isomorphism classes are in $1-1$ correspondence with homotopy classes of classifying maps, which accounts for the use of homotopy in the

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$^1$The question which is answered by this theorem has often been referred to as ‘Nash Conjecture’, e.g., see [T], [K], [DMP]. As Akbulut and King point out, this name is not supported by Nash’s paper, and recently they started to use the word ‘Nash Conjecture’ with a different meaning [AK3], [AK4].
definition. Of particular interest is the algebraic realization problem for vector bundles. This problem was considered and solved by Benedetti and Tognoli [BT].

We like to study the algebraic realization problem in the equivariant category. Let $G$ be a compact Lie group. The expression algebraic realization has its obvious meaning in this category, all spaces are assumed to have an action of $G$, and all maps are assumed to be equivariant. (The reader who is not familiar with all of the terms used in this introduction should consult Section 2 for further definitions and additional background material.) Then we have the

**Algebraic Realization Conjecture.** Let $G$ be a compact Lie group. Every closed smooth $G$ manifold is algebraically realized.

We addressed this conjecture in a previous paper, and as a partial answer we obtained

**Theorem A** (see [DM]). Let $G$ be a compact Lie group. A closed smooth $G$ manifold $M$ is algebraically realized if one of the following assumptions holds.

1. $G$ is of odd order.
2. The action of $G$ on $M$ is semifree.

In the present paper we study the algebraic realization problem for equivariant vector bundles. This problem arose already in the proof of Theorem A. It is also more general than the problem of realizing smooth $G$ manifolds algebraically in so far as the algebraic realization of a bundle includes the realization of its base space. Furthermore, it is essential for improving on Theorem A by verifying the algebraic realization conjecture in more generality. We use maps to Grassmannians to classify $G$ vector bundles and note that Grassmannians are naturally endowed with an algebraic structure (see Section 2c). With this understood we can be specific and state the

**Algebraic Realization Problem for Equivariant Vector Bundles.** Let $G$ be a compact Lie group and $M$ a closed smooth $G$ manifold. Does there exist a non-singular real algebraic $G$ variety $X$ and an equivariant diffeomorphism $\phi : X \to M$ such that for every real $G$ vector bundle $\xi$ over $M$ the classifying map of $\phi^*(\xi)$ is equivariantly homotopic to an equivariant entire rational map?

We use the following terminology. In case of a positive solution to the bundle realization problem we say that $(X, \phi)$ provides an algebraic realization of the set of all real $G$ vector bundles over $M$. Without specifying $(X, \phi)$ we say that the set of all real $G$ vector bundles over $M$ is algebraically realized. The same terminology is used for individual $G$ vector bundles as well as arbitrary sets of $G$ vector bundles. To simplify language, we also allow the case where $M$ consists of several components (which may have different dimensions) and the dimension of the fibre depends on the component. In this case we require that $M$

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2Throughout this paper we follow the convention that a manifold has a dimension, i. e., all of its components have the same dimension. In part, this is motivated by the same convention for non-singular varieties. This conjecture and the results of this paper can be applied to manifolds with components of different dimension by applying them separately to the components of a fixed dimension, and then taking a disjoint union.
decomposes as a union of collections of components \( M_j \), and the set of bundles restricted to each \( M_j \) is realized in the sense of the definition. (Equivalently, we could use unions of Grassmannians as classifying spaces.) As the major result of this paper we obtain

**Theorem B.** The set of all real \( G \) vector bundles over a closed smooth \( G \) manifold is algebraically realized if one of the following assumptions holds:

1. \( G \) is the product of a group of odd order and a 2-torus.
2. The action of \( G \) on the manifold is semifree.

Part (1) of the theorem constitutes an improvement of Theorem A (1). The theorem is proved in Sections 4 and 5.

We explain the strategy of the proof of Theorem B. In a natural way, any solution of such an algebraic realization problem has two steps. In the first step the problem is reduced to a bordism problem. Let \( M_1 \) and \( M_2 \) be closed smooth \( G \) manifolds, \( Y \) a \( G \) space and \( f_i : M_i \to Y \), \( i = 1, 2 \), equivariant maps. We say that \((M_1, f_1)\) and \((M_2, f_2)\) are equivariantly cobordant if there exists a compact smooth \( G \) manifold \( W \) and an equivariant map \( F : W \to Y \) such that the boundary of \( W \) is the disjoint union of \( M_1 \) and \( M_2 \) and \( F \) restricts to \( f_i \) on \( M_i \). With disjoint union as addition, the bordism classes form a group \( \mathcal{N}_G(Y) \), called the equivariant bordism group of \( Y \). We say that \((M, f)\) represents a class of dimension \( n \) if \( M \) is of dimension \( n \). These classes are called homogeneous and form \( \mathcal{N}_n^G(Y) \). Suppose \( Y \) is a real algebraic \( G \) variety. We say that a bordism class \( \alpha \in \mathcal{N}_n^G(Y) \) is algebraically realized if it has a representative \( \phi : X \to Y \) where \( X \) is a non-singular real algebraic \( G \) variety and \( \phi \) is an equivariant entire rational map. The reduction of the algebraic realization problem to a bordism problem is expressed in

**Theorem C.** Let \( G \) be a compact Lie group. An equivariant map from a closed smooth \( G \) manifold to a non-singular real algebraic \( G \) variety is algebraically realized if and only if its equivariant bordism class is algebraically realized.

For \( Y \) a point this theorem has been shown in [DMP]. In an application in a different paper we need more specific information about the way in which a map is algebraically realized. We formulate this as an addendum to Theorem C in Section 7 where we prove both, the theorem and its addendum. This concludes the first step. In the second step we need to address the

**Algebraic Bordism Problem for Equivariant Maps.** Let \( Y \) be a real algebraic \( G \) variety. Is every homogeneous class \( \alpha \in \mathcal{N}_*^G(Y) \) algebraically realized?

Even in the non-equivariant setting the answer to this problem depends in a subtle way on the algebraic structure of \( Y \). Akbulut and King introduce the idea of a variety having *totally algebraic homology* [AK1]. It is a consequence of their paper that for a non-singular variety \( Y \) every homogeneous class \( \alpha \in \mathcal{N}_*^G(Y) \) is algebraically realized if and only if \( Y \) has totally algebraic homology (see [AK1, Lemma 2.5]). Only few varieties, such as Grassmannians with their natural algebraic structure, are known to have totally algebraic homology. Benedetti and Dedò showed that there are closed smooth manifolds (in all dimensions \( \geq 11 \))
such that none of their algebraic realizations has totally algebraic homology \cite[Theorem 2]{BD}. In the absence of a Künneth formula in equivariant bordism, which is used to reduce the bordism problem to a homology problem, it seems most appropriate to say:

**Definition.** A real algebraic $G$ variety $Y$ has totally algebraic bordism if every homogeneous class in $\mathcal{N}_G^G(Y)$ is algebraically realized.

In Section 3 we show

**Theorem D.** Let $G$ be a group of odd order and $Y$ a real algebraic $G$ variety such that $Y^H$ (viewed as a variety without group action) has totally algebraic bordism (or totally algebraic homology) for every subgroup $H$ of $G$. Then $Y$ (viewed as a $G$ variety) has totally algebraic bordism.

The special case of particular interest in this paper is as follows. Let $\Xi$ be an orthogonal representation of $G$ and $G_\mathbb{R}(\Xi, k)$ the Grassmannian of $k$-dimensional subspaces of $\Xi$. It is a finite approximation of the classifying space for $k$-dimensional $G$ vector bundles. An analysis of $G_\mathbb{R}(\Xi, k)$ shows:

**Proposition E.** Let $G$ be a group of odd order, and $\Xi$ an orthogonal representation of $G$. The $G$ variety $G_\mathbb{R}(\Xi, k)$ has totally algebraic bordism, i.e., every homogeneous class in $\mathcal{N}_G^G(G_\mathbb{R}(\Xi, k))$ is algebraically realized.

We prove this proposition in a slightly stronger form in Section 3, see Proposition 3.3. As another tool we need a localization principle. It is used as induction technique in the proof of Theorem B.

**Proposition F.** Let $G$ be a compact Lie group, $\tau \in G$ a central element of order two in $G$ and $M$ a closed smooth $G$ manifold. The set of all real $G$ vector bundles over $M$ is algebraically realized if and only if the set of all $G$ vector bundles over $M^\tau$ is algebraically realized.

A version of this proposition, relating the algebraic realizations of $M$ and of the $G$ normal bundle of $M^\tau$ in $M$, was proved in \cite{DM} and applied in the proof of Theorem A.

Theorem C, Proposition E and Proposition F are essential ingredients in the proof of Theorem B. In return we obtain a stronger form of Proposition E. We prove it in Section 4.

**Corollary E.** Let $G$ be the product of a group of odd order and a 2-torus and $\Xi$ an orthogonal representation of $G$. Then $G_\mathbb{R}(\Xi, k)$ has totally algebraic bordism.

We like to thank Professor Richard Palais for some advice on the wording in this paper to avoid any conflict with the historical and recent use of the word ‘Nash Conjecture’. As he told us, in the 70’s, he and Henry King thought about a theorem similar to our Theorem C in the special case where $Y$ is a point, but they never wrote out a detailed argument.

## 2. Some Background Material

In this section we recall some background material which not all readers may be familiar with. Much of it is discussed in more detail and proved in \cite{DM}. Some parts are new. There are three subsections, basic notation, induction, and strongly algebraic bundles.
2a. Basic Notation.

First of all, we need to define the concept of a real algebraic $G$ variety. Let $G$ be a compact Lie group and $\Omega$ an orthogonal representation of $G$. Here we think of an orthogonal representation as an underlying Euclidean space $\mathbb{R}^n$ together with an action of $G$ via orthogonal maps.

**Definition 2.1.** A real algebraic $G$ variety is the set of common zeros of polynomials $p_1, \ldots, p_m : \Omega \to \mathbb{R}$,

$$V = \{x \in \Omega \mid p_1(x) = \cdots = p_m(x) = 0\},$$

which is invariant under the action of $G$. We also say that $G$ acts real algebraically on $V$.

As a topological space we will consider $V$ with two topologies, the subspace topologies induced by the Euclidean topology and the Zariski topology on $\mathbb{R}^n$. Most of the time we use the Euclidean topology and do not mention this explicitly. Whenever we use the Zariski topology we say so. There is an equivalent, more algebraically minded definition of a real algebraic $G$ action. For details we refer the reader to [DM].

**Definition 2.2.** The variety $V \subset \mathbb{R}^n$ is said to be non-singular at $x \in V$ if there are polynomials $q_1, \ldots, q_s : \mathbb{R}^n \to \mathbb{R}$ which vanish on $V$ and a Zariski open neighbourhood $U$ of $x$ in $\mathbb{R}^n$ such that

1. $V \cap U = U \cap q_1^{-1}(0) \cap \cdots \cap q_s^{-1}(0)$
2. the gradients $(\nabla q_i)_x$ are linearly independent for $i = 1, \ldots, s$.

We say that $V$ is non-singular if $V$ is non-singular at each point $x \in V$, and all connected components of $V$ have the same dimension.

More specifically, $x \in V$ is said to be non-singular of dimension $n - s$ in $V$. Then one defines

1. $\dim V = \max\{d \mid \text{there exists a point } x \in V \text{ which is non-singular of dimension } d\}$
2. $\text{Nonsing } V = \{x \in V \mid x \text{ is non-singular of dimension } = \dim V\}$
3. $\text{Sing } V = V \setminus \text{Nonsing } V$.

In the main body of this paper we only need the idea of non-singularity. Item (3)–(5), and the following proposition are needed only in Section 7 where we prove Theorem C.

**Proposition 2.3.** If $V$ is a real algebraic $G$ variety, then

1. $\text{Sing } V$ is a real algebraic $G$ variety.
2. $\dim \text{Sing } V < \dim V$
3. $\text{Nonsing } V$ is a smooth $G$ manifold.

Next we discuss the morphisms in the algebraic category. Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be real algebraic varieties. A map $f : V \to W$ is said to be regular if it extends to a map $F : \mathbb{R}^n \to \mathbb{R}^m$ such that each component of $F$ is a polynomial. We say that $f$ is entire rational if there are regular maps $p : \mathbb{R}^n \to \mathbb{R}^m$ and $q : \mathbb{R}^n \to \mathbb{R}$ such that $f = p/q$ on

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3Not all authors agree what a rational and an entire rational map should be. In this paper it is essential that maps are defined everywhere on their domain, and this seems to be assumed by all authors when they use the adjective ‘entire rational’.
V and q does not vanish on V. Occasionally, we even call a regular map a polynomial. If 
V and W are G varieties and f is in addition equivariant, then we call f an equivariant
regular map, resp., an equivariant entire rational map.

Occasionally we make maps equivariant by averaging them (compare e. g. [B]). Let Ω
and Ξ be representations of a compact Lie group G, and f : Ω → Ξ. Denote the Haar
measure of G by dg, and let x be a point in Ω. Then

\[ A(f)(x) = \int_G g^{-1} f(gx) \, dg. \]

Lemma 2.4. With the notation from above:

(1) A(f) is equivariant, and A(f) = f if f is equivariant.

(2) If f is a polynomial, then so is A(f).

There are a few simple but important observation. Let V ⊂ Ω and W ⊂ Ξ be real
algebraic G varieties and f : V → W equivariant regular map. Then, by Lemma 2.4, the
regular extension F : Ω → Ξ of f may be assumed to be equivariant. For equivariant entire
rational maps we have

Proposition 2.5. Let V ⊂ Ξ and W ⊂ Ω be real algebraic G varieties, and f : V → W
an equivariant entire rational map. There exist equivariant polynomials P : Ξ → Ω and
Q : Ξ → R such that f = P/Q, restricted to V, and Q does not vanish on Ξ.

Proposition 2.6. Every real algebraic G variety can be expressed as the zero set of a single
equivariant polynomial. If the variety is compact, then it is the set of zeros of a proper G
invariant polynomial (proper means that the inverse images of compacta are compact).

The first part of this proposition is a standard result, the second one is proved in [DMP,
Lemma 4.7] as a generalization of the corresponding non-equivariant result in [AK1, p. 430].

Proposition 2.7.

(1) The disjoint union of two (non-singular) real algebraic G varieties (of the same
dimension) is a (non-singular) real algebraic G variety.

(2) The cartesian product of two (non-singular) real algebraic G varieties is a (non-
singular) real algebraic G variety.

2b. Induction.

In [DM] we considered the induction construction in the algebraic category. We recall
the basic result. Let H be a subgroup of a group G, and let X be an H space. The induced
G space Ind^G_H X is defined as the orbit space G ×_H X. More explicitly, elements in G ×_H X
are equivalence classes of elements in G × X, where (g, x) is equivalent to (gh, h^{-1}x) for
all h ∈ H. If X is a (closed) smooth H manifold and H is a closed subgroup of a compact
Lie group G, then Ind^G_H X is a (closed) smooth G manifold. This procedure is functorial.
Given an H equivariant map f : X → Y of H spaces, the map Id × f : G × X → G × Y
induces a G map Ind^G_H f : Ind^G_H X → Ind^G_H Y of orbit spaces. If X and Y are smooth H
manifolds and f is smooth, then so is Ind^G_H f. In [DM] we proved a similar result for real
algebraic varieties.
Proposition 2.8. Suppose $G$ is a compact Lie group, and $H$ is a closed subgroup of finite index. If $X$ is a (non-singular) real algebraic $H$ variety, then there is a natural procedure to define a (non-singular) real algebraic $G$ variety structure on $\text{Ind}_H^G X$. Given two real algebraic $H$ varieties $X$ and $Y$ and an $H$ equivariant regular (resp. entire rational) map $f : X \to Y$, there is a naturally defined $G$ equivariant regular (resp. entire rational) map $\text{Ind}_H^G f : \text{Ind}_H^G X \to \text{Ind}_H^G Y$.

2c. Strongly Algebraic Vector Bundles.

The appropriate idea of a vector bundle in our setting is the one of a strongly algebraic vector bundle. It is the concept which underlies the one of an algebraically realized bundle. We need this notion with real as well as the complex and quaternion coefficients.

Let $\Lambda$ stand for $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let $\Xi$ be a representation of $G$ over $\Lambda$, in particular, its underlying space is $\Lambda^n$ for some $n$. We assume that the action of $G$ preserves the standard bilinear form on $\Lambda^n$ over $\Lambda$. Let $\text{End}_\Lambda(\Xi)$ denote the set of endomorphisms of $\Xi$ over $\Lambda$. It is a representation of $G$ with the action given by $G \times \text{End}_\Lambda(\Xi) \to \text{End}_\Lambda(\Xi)$ with $(g, L) \mapsto gLg^{-1}$.

Let $k$ be a natural number. We set

$$
G_\Lambda(\Xi, k) = \{ L \in \text{End}_\Lambda(\Xi) \mid L^2 = L, \ L^* = L, \ \text{trace } L = k \}$$

$$
E_\Lambda(\Xi, k) = \{ (L, u) \in \text{End}_\Lambda(\Xi) \times \Xi \mid L \in G_\Lambda(\Xi, k), \ Lu = u \}.
$$

Here $L^*$ denotes the adjoint of $L$. If one chooses an orthonormal (resp. unitary or symplectic) basis of $\Xi$, then $\text{End}_\Lambda(\Xi)$ is canonically identified with the set of $n \times n$ matrices $\Lambda^{n^2}$, and $L^*$ is obtained by transposing $L$ and conjugating its entries. This description specifies $G_\Lambda(\Xi, k)$ and $E_\Lambda(\Xi, k)$ as real algebraic $G$ varieties. Define $p : E_\Lambda(\Xi, k) \to G_\Lambda(\Xi, k)$ as projection on the first factor. This defines a $G$ vector bundle, which is called the universal bundle over $G_\Lambda(\Xi, k)$, and which is denoted by $\gamma_\Lambda(\Xi, k)$.

Definition 2.9. A strongly algebraic $G$ vector bundle over $\Lambda$ is a pair $(X, \mu)$ where $X$ is a real algebraic $G$ variety and $\mu : X \to G_\Lambda(\Xi, k)$ is an equivariant entire rational map.

Convention. Assuming that $\Xi$ is a summand of a representation $\Xi'$ of $G$, we have an embedding $i : G_\Lambda(\Xi, k) \to G_\Lambda(\Xi', k)$. In this sense we identify the strongly algebraic $G$ vector bundles $(X, \mu : X \to G_\Lambda(\Xi, k))$ and $(X, i\mu : X \to G_\Lambda(\Xi', k))$.

In contrast to existing literature ([BT], [I], [BCR]), we defined strongly algebraic vector bundles as an independent concept. Previously, authors considered first another bundle theory, such as algebraic vector bundles. Then they defined strongly algebraic vector bundles as bundles in their theory which are isomorphic to pullbacks of the universal bundle under entire rational maps.

Remark. There is no apparent ‘natural’ way to define the concept of isomorphism for strongly algebraic vector bundles. One might say that two strongly algebraic $G$ vector
bundles \( \mu_i : X \to G_\Lambda(\Xi, k) \) \((i = 0, 1)\) are isomorphic if there exists a continuous equivariant map \( \eta : X \times [0, 1] \to G_\Lambda(\Xi, k) \) such that \( \eta_{|X \times t} = \mu_i \), and \( \eta_{|X \times t} \) is entire rational for each \( t \in [0, 1] \). This defines an equivalence relation. It is tempting to require that \( \eta \) is defined on \( X \times \mathbb{R} \), and that \( \eta \) is entire rational, but then it is not clear that this defines an equivalence relation. Something interesting can still be said at this point. Suppose \((X, \mu_1)\) and \((X, \mu_2)\) are strongly algebraic \( G \) vector bundles, and \( \mu_1 \) and \( \mu_2 \) are equivariantly homotopic. Then \( \mu^*_1(\gamma_\Lambda(\Xi, k)) \) and \( \mu^*_2(\gamma_\Lambda(\Xi, k)) \) are not only isomorphic as real \( G \) vector bundles, but even equivariantly entire rationally isomorphic as algebraic \( G \) vector bundles. This is an equivariant generalization of a result in [BCR, p. 265] proved by Kawakami [Ka].

**Proposition 2.10** [DM]. Let \((B, \alpha : B \to G_\mathbb{R}(\Xi, k))\) be a strongly algebraic \( G \) vector bundle. The total space

\[
\mathbb{R}P(E) = \{(x, T) \in B \times G_\mathbb{R}(\Xi, 1) \mid (Id - \alpha(x))T = 0\}
\]

of the associated projective bundle is a real algebraic \( G \) variety and the natural projection map \( p : \mathbb{R}P(E) \to B \) is an equivariant entire rational map. If \( B \) is non-singular, then \( \mathbb{R}P(E) \) is non-singular.

There are a number of standard constructions (such as the Whitney sum of vector bundles) which may be applied to equivariant topological vector bundles as well as strongly algebraic \( G \) vector bundles. With the obvious meaning we apply notation used for vector bundles to the maps which classify bundles. In the introduction we defined what it means that a real \( G \) vector bundle is algebraically realized. In analogy we say that a \( k \)-dimensional \( G \) vector bundle \( \xi = (E, p, B) \) over \( \Lambda \) is algebraically realized if there exists a non-singular real algebraic \( G \) variety \( X \) and an equivariant diffeomorphism \( \phi : X \to B \) such that the classifying map of \( \phi^*(\xi) \) is equivariantly homotopic to an equivariant entire rational map. Just as we did it for real bundles, we apply this definition with the modifications spelled out in the introduction. In our next proposition we summarize a few basic facts about constructions with strongly algebraic vector bundles (e. g., see [BCR, Chapter 12, Section 1]).

**Proposition 2.11.** Let \( X \) be a real algebraic \( G \) variety and \( \mu : X \to G_\Lambda(\Xi, k) \) and \( \mu' : X \to G_\Lambda(\Xi', k') \) be two strongly algebraic \( G \) vector bundles over \( \Lambda \). Then

1. The Whitney sum \( \mu \oplus \mu' : X \to G_\Lambda(\Xi \oplus \Xi', k + k') \) is a strongly algebraic \( G \) vector bundle over \( \Lambda \).

2. The tensor product \( \mu \otimes \mu' : X \to G_\Lambda(\Xi \otimes \Xi', \epsilon_\Lambda \cdot k \cdot k') \) is a strongly algebraic \( G \) vector bundle over \( \Lambda' \). Here \( \Lambda' = \mathbb{R} \) if \( \Lambda = \mathbb{R} \) or \( \mathbb{H} \), and \( \Lambda' = \mathbb{C} \) if \( \Lambda = \mathbb{C} \), and \( \epsilon_\Lambda = 1 \) if \( \Lambda = \mathbb{R} \) or \( \mathbb{C} \), and \( \epsilon_\Lambda = 4 \) if \( \Lambda = \mathbb{H} \).

3. The dual \( \mu^* \) of \( \mu \) is a strongly algebraic \( G \) vector bundle over \( \mathfrak{t} = \mathbb{R} \) if \( \Lambda = \mathbb{R} \) or \( \mathbb{H} \), and over \( \mathfrak{t} = \mathbb{C} \) if \( \Lambda = \mathbb{C} \). Furthermore, \( \text{Hom}(\mu, \mu') \) (which is isomorphic to \( \mu^* \otimes \mu' \)) is strongly algebraic over \( \mathfrak{t} \), where \( \mathfrak{t} \) is as in the previous sentence.

4. If \( \mu' \) is a strongly algebraic subbundle of \( \mu \), then the orthogonal complement of \( \mu' \) in \( \mu \) is strongly algebraic.
(5) Every strongly algebraic $G$ vector bundle over $\Lambda$ may be viewed as a strongly algebraic $G$ vector bundle over $\mathbb{R}$.

(6) If $\Gamma$ is a $k$-dimensional representation of $G$ over $\Lambda$, then the product bundle over $X$ with fibre $\Gamma$, represented by the constant map to any point in the component of $G_{\Lambda}(\Xi, k)^G$ associated to the representation $\Gamma$, is a strongly algebraic $G$ vector bundle over $\Lambda$.

(7) In the assumption and conclusion for (1)–(6) we may replace strongly algebraic $G$ vector bundles by algebraically realized bundles. But in (1)–(3) we need to assume that $\mu = (E, p, B)$ and $\mu' = (E', p', B)$ are algebraically realized over the same non-singular variety $X$.

Item (4) may need some explanation. Let $\mu : X \rightarrow G_{\Lambda}(\Xi, k)$. We say that $\mu' : X \rightarrow G_{\Lambda}(\Xi, k')$ is a subbundle if $k' \leq k$ and $\mu(x) \circ \mu'(x) = \mu'(x)$. Both, $\mu$ and $\mu'$, are assumed to be equivariant entire rational maps. The orthogonal complement $\mu^\perp : X \rightarrow G_{\Lambda}(\Xi, k - k')$ of $\mu'$ in $\mu$ is then given by $\mu^\perp(x) = (Id - \mu'(x)) \circ \mu(x)$. This is the analog of the ‘orthogonal complement’ construction for bundles in terms of their classifying maps.

We need a description of $G_{\mathbb{R}}(\Xi, k)^H$ for $H \subset G$. The proof of the following proposition together with the precise description of the $F_j$’s has been given in [DM].

**Proposition 2.12.** Let $G$ be a compact Lie group, $\Xi$ an orthogonal representation of $G$, $k$ a non-negative integer and $H$ a closed subgroup of $G$. Then

$$G_{\mathbb{R}}(\Xi, k)^H = F_1 \cup \cdots \cup F_s$$

is a disjoint union of non-singular subvarieties $F_j$. Each $F_j$ is a product, and each factor is a real, complex, or quaternion Grassmannian with its standard structure as a variety.

Our next proposition is a generalization of a non-equivariant result of Benedetti and Tognoli [BT] (see also [I, Section 4]). The idea of proof is similar to theirs.

**Proposition 2.13.** Let $G$ be a compact Lie group and $M$ a closed smooth $G$ manifold. Suppose every finite collection of $G$ vector bundles over $M$ can be algebraically realized, then the set of all $G$ vector bundles over $M$ can be algebraically realized.

The key lemma in the proof of the proposition is as follows.

**Lemma 2.14.** Let $G$ be a compact Lie group, $X$ a compact non-singular real algebraic $G$ variety, and $\xi$, $\xi_0$, and $\xi_1$ a real $G$ vector bundles over $X$. Suppose that the classifying maps of $\xi$ and $\xi_0$ are equivariantly homotopic to entire rational maps, and $\xi_0 \oplus \xi_1 \cong \xi$, then the classifying map of $\xi_1$ is equivariantly homotopic to an entire rational map.

**Proof.** We approximate a bundle monomorphism $\varphi : \xi_0 \rightarrow \xi$ by an algebraic bundle monomorphism $\varphi_a$. By assumption, we have equivariant entire rational maps $\mu : X \rightarrow G_{\mathbb{R}}(\Xi, k)$ and $\mu_0 : X \rightarrow G_{\mathbb{R}}(\Xi_0, k_0)$ which classify $\xi$ and $\xi_0$, respectively. Here $\Xi$ and $\Xi_0$ are appropriate orthogonal representations of $G$, and we assume that the fibre dimensions of $\xi$ and $\xi_0$ are constant (otherwise the argument is carried out over components of $X$.
over which it is constant). Then \( \text{Hom}(\xi_0, \xi) \cong \xi_0^* \otimes \xi \) (see Proposition 2.11) has an entire rational classifying map \( \tilde{\mu} : X \to G_R(\Xi, \tilde{k}) \) induced from \( \mu \) and \( \mu_0 \), where \( \tilde{k} = kk_0 \) and \( \Xi = \text{Hom}(\Xi_0, \Xi) \). To simplify notation, we identify \( \xi \) with \( \mu^*(\gamma_R(\Xi, k)) \), \( \xi_0 \) with \( \mu_0^*(\gamma_R(\Xi_0, k_0)) \), and \( \text{Hom}(\xi_0, \xi) \) with \( \tilde{\mu}(\gamma_R(\Xi, \tilde{k})) \). This in mind, we consider the fibres of the bundles as subrepresentations of the representation specified by the Grassmannian.

We have a bundle epimorphism \( p : X \times \Xi \to \text{Hom}(\xi_0, \xi) \) which induces an equivariant entire rational map of total spaces which we denote by \( p \) as well.

An equivariant embedding \( \varphi : \xi_0 \to \xi \) of bundles defines an equivariant section \( \varphi' \) of \( \text{Hom}(\xi_0, \xi) \). The total space of \( \text{Hom}(\xi_0, \xi) \) is a non-singular real algebraic subvariety of \( X \times \Xi \). Denote the projection of \( X \times \Xi \) onto its second factor by \( p_2 \). We approximate \( p_2 \varphi' \) by an equivariant entire rational map \( \psi_a : X \to \Xi \). This can be done by first taking a non-equivariant polynomial approximation, and then averaging it. Then we get an equivariant polynomial approximation (see Lemma 2.4 and Lemma 6.1). Define an equivariant entire rational section \( \varphi'_a \) of \( \text{Hom}(\xi_0, \xi) \) setting \( \varphi'_a(x) = p(x)(\psi_a(x)) \), and denote the associated bundle map by \( \varphi_a : \xi_0 \to \xi \). Observe that \( \varphi_a \) is a bundle monomorphism if \( \psi_a \) is a close approximation of \( p_2 \varphi \), hence \( \varphi_a \) a close approximation of \( \varphi \). This is the algebraic approximation \( \varphi_a \) of \( \varphi \) which we set out to construct.

We now define a map \( \mu_0^a : X \to G_R(\Xi, k_0) \), it maps \( x \) to the orthogonal projection from \( \Xi \) onto \( \varphi_a(\xi_0)_x \). By construction, \( \mu_0^a \) classifies \( \xi_0 \), \( \mu_0^a \) is entire rational, \( \mu(x)\mu_0^a(x) = \mu_0^a(x)\mu(x) = \mu_0^a(x) \), and \( \text{trace}(\mu_0^a(x)) = k_0 \). This describes \( \mu_0^a \) as a strongly algebraic subbundle of \( \mu \). The orthogonal complement of \( \varphi_a(\xi_0) \) in \( \xi \) is isomorphic to \( \xi_1 \). It is classified by the entire rational map \( \mu_0^a : X \to G_R(\Xi, k_1) \) where \( \mu_0^a(x) = (Id - \mu_0^a(x)\mu(x))\mu(x) \) (see 2.11 (4)). This completes the proof. □

Proof of Proposition 2.13. By definition, elements in \( KO_G(M) \) are formal differences \( \xi_+ - \xi_- \) of real \( G \) vector bundles over \( M \) modulo the equivalence relation \( \xi_+ - \xi_- \cong \xi'_+ - \xi'_- \) if and only if \( \xi_+ \oplus \xi_- \leq \eta \cong \xi'_+ \oplus \xi_- \leq \eta \) for some real \( G \) vector bundle \( \eta \). It is known that \( KO_G(M) \) is finitely generated as a module over the real representation ring \( RO(G) \), see [Sg, p. 147]. Let \( \mathfrak{f} = \{ \xi_1, \ldots, \xi_m \} \) be a collection of \( G \) vector bundles which generate \( KO_G(M) \) over \( RO(G) \), and suppose \( \mathfrak{f} \) is algebraically realized by \((X, \phi)\), i.e., \( X \) is a non-singular real algebraic \( G \) variety, \( \phi : X \to M \) is an equivariant diffeomorphism, and the classifying maps of \( \phi^*(\xi) \) are equivariantly homotopic to entire rational maps for all \( \xi \) in \( \mathfrak{f} \).

Let \( \zeta \) be a real \( G \) vector bundle over \( M \). We may write \( \zeta \) as a formal difference \( \xi_+ - \xi_- \) where \( \xi_{\pm} = \sum_i \alpha_i^\pm \xi_i \) are real vector bundles over \( M \). The \( \alpha_i \) are real representations of \( G \), the \( \xi_i \) are in \( \mathfrak{f} \), product means tensor product, and addition means Whitney sum. In terms of actual bundles we have the equation

\[
\zeta \oplus \xi_- \oplus \eta \cong \xi_+ \oplus \eta.
\]

Adding to both sides of the equation the orthogonal complement of \( \eta \) in some product bundle, we may assume that \( \eta \) is a product bundle. It follows from Proposition 2.11 that \( \xi_- \oplus \eta \) and \( \xi_+ \oplus \eta \) are algebraically realized over \( X \), i.e., the classifying maps of \( \phi^*(\xi_- \oplus \eta) \) and \( \phi^*(\xi_+ \oplus \eta) \) are equivariantly homotopic to entire rational maps. Lemma 2.14 implies
that the classifying map of $\phi^*(\zeta)$ is equivariantly homotopic to an entire rational map, and this means that $\zeta$ is algebraically realized over $X$. As $\zeta$ was any real $G$ vector bundle over $M$, this means that $(X, \phi)$ provides an algebraic realization for the set of all real $G$ vector bundles over $M$. This proves the proposition. $\square$

Finally, here is an example which will be of interest later.

**Proposition 2.15.** The tangent bundle and the normal bundle of a non-singular real algebraic $G$ variety are strongly algebraic.

More precisely, let $X \subset \Xi$ be a non-singular real algebraic $G$ variety of dimension $m$ where $\Xi$ is an orthogonal representation of $G$ of dimension $n$. Earlier in this section we described the real Grassmannian $G_\mathbb{R}(\Xi, k)$ and the total space $E_\mathbb{R}(\Xi, k)$ of the universal bundle $\gamma_\mathbb{R}(\Xi, k)$ as solutions of polynomial equations on $\text{End}_\mathbb{R}(\Xi)$ and $\text{End}_\mathbb{R}(\Xi) \times \Xi$, respectively. Define $\chi_T : X \to G_\mathbb{R}(\Xi, m)$ and $\chi_\nu : X \to G_\mathbb{R}(\Xi, n - m)$ by letting $\chi_T(x)$ be the orthogonal projection from $\Xi$ onto $T_xX$ and $\chi_\nu(x)$ the orthogonal projection from $\Xi$ onto the normal fibre of $X$ in $\Xi$ at $x$, respectively. Here $x$ is any point in $X$. Note that $\chi_T$ and $\chi_\nu$ are well defined, and not only up to homotopy. It has been shown that $\chi_T$ and $\chi_\nu$ are entire rational [AK1, Lemma 2.3], [BCR, p. 260]. By definition they are also equivariant, hence they are entire rational maps.

3. The Odd Order Group Case (Theorem D and Proposition E)

In this section we prove Theorem D and Proposition E. We need to recall some bordism theoretic notation. Cartesian product of representatives defines a product on $\mathcal{R}_\ast^G(\text{point}) = \mathcal{R}_\ast^G$, so $\mathcal{R}_\ast^G(Y)$ is a module over $\mathcal{R}_\ast^G$. Furthermore, $\mathcal{R}_0^G$ is a ring and $\mathcal{R}_+^G$ is an algebra over $\mathcal{R}_0^G$. Consider the ring $\mathcal{R}_\ast$ of bordism classes of closed manifolds. We may think of manifolds without group action as manifolds with trivial action. In this sense, $\mathcal{R}_\ast^G(Y)$ is also a module over $\mathcal{R}_\ast$. We indicate bordism classes by $[\ ].$

For any subgroup $H$ of $G$ we define a ring homomorphism $\phi_H : \mathcal{R}_0^G \to \mathbb{Z}_2$. Let $A$ represent a class in $\mathcal{R}_0^G$, so $A$ is a finite $G$ set. We set $\phi_H([A]) = |A^H| \mod 2$. Let $(H)$ denote the conjugacy class of $H$ in $G$. It is elementary to show (e. g., see [C, Section 1])

**Proposition 3.1.** Let $G$ be a group of odd order. There exist idempotents $e_H \in \mathcal{R}_0^G$, one for each subgroup $H$ of $G$, such that for all conjugacy classes $(K)$ of subgroups of $G$

1. $\phi_K(e_H) = 1$ if $(H) = (K)$.
2. $\phi_K(e_H) = 0$ if $(H) \neq (K)$.
3. $\sum_H e_H = 1$ where the summation ranges over subgroups of $G$.

We set up some notation. Let $H$ be a subgroup of $G$, $V$ a non-trivial irreducible representation of $H$, and $i$ a positive integer. Set

$$R_{H,V,i} = \mathbb{R}P((\eta_i \otimes_\mathbb{C} V) \oplus \mathbb{R}).$$

Here $\eta_{i-1}$ is the canonical line bundle over $\mathbb{C}P^{i-1}$ with trivial action of $H$. This bundle is tensored with the product bundle $V$ over $\mathbb{C}P^{i-1}$ with fibre $V$. We give $V$ any of its complex
structures. Then we add the product bundle $\mathbb{R}$ with fibre $\mathbb{R}$ and take the total space of the associated projective bundle. The result is $R_{H,V,i}$.

Consider formal variables $\gamma_{H,V,i}$. Let $\Psi_0(\gamma_{H,V,i}) = R_{H,V,i}$. To any polynomial $P$ in the $\gamma_{H,V,i}$ with coefficients in $\mathfrak{N}_*$, represented by closed manifolds, we associate the corresponding expression $\Psi_0(P)$ in the $R_{H,V,i}$ where products translate into cartesian products and sums into disjoint unions. Thus we defined $\Psi_0(P)$ for any element $P$ in $\mathfrak{N}_*[\gamma_{H,V,i}]$ where $V$ ranges over the non-trivial irreducible representations of $H$ and $i = 1, 2, \ldots$.

Consider a $G$ space $Y$ and a generator $P \otimes \alpha \in \mathfrak{N}_*[\gamma_{H,V,i}] \otimes \mathfrak{N}_*(Y^H)$ where $\alpha$ is represented by a map $f : M \to Y^H$. Let $p_2$ denote projection on the second factor. We define

$$\Psi_1(P \otimes \alpha) = (\tilde{f} = f \circ p_2 : \Psi_0(P) \times M \to Y^H).$$

It represents a class in $\mathfrak{N}_H(Y^H)$. There exists a unique $G$ equivariant extension $\bar{f} : G \times_H (\Psi_0(P) \times M) \to Y$ of $\tilde{f}$. It represents a class $\beta(P \otimes \alpha)$ in $\mathfrak{N}_G^G(Y)$. We use the module structure of $\mathfrak{N}_G^G(Y)$ over $\mathfrak{N}_G^G$ and the $e_H$ from Proposition 3.1 to define

$$\Psi_H(P \otimes \alpha) = e_H \cdot \beta(P \otimes \alpha).$$

Our next result is basically taken from a paper of Costenoble [C], but with some work a similar (and equally useful) result could be obtained from older work of Stong [S1].

**Theorem 3.2.** Let $G$ be a group of odd order. There exists a surjective graded $\mathfrak{N}_*$ module homomorphism

$$\Psi : \prod_{(H)} \mathfrak{N}_*[\gamma_{H,V,i}] \otimes \mathfrak{N}_*(Y^H) \to \mathfrak{N}_G^G(Y)$$

where the product is taken over all conjugacy classes of subgroups of $G$, $V$ ranges over the non-trivial irreducible representations of $H$, and $i = 1, 2, \ldots$. The (abstract) polynomial generators $\gamma_{H,V,i}$ live in dimension $|\gamma_{H,V,i}| = 2(i - 1) + \dim \mathbb{R} V$. On each of the factors $\Psi$ is given by the map $\Psi_H$ defined above.

**Remarks on the Proof.** The statement in this theorem can be extracted from [C]. Costenoble studies a map

$$\Phi : \mathfrak{N}_G^G(Y) \to \prod_{(H)} (\mathfrak{N}_*[\gamma_{H,V,i}] \otimes \mathfrak{N}_*(Y^H))^{WH}.$$ 

Here $WH = NH/H$ acts on the set of irreducible representations of $H$ by conjugating them (i.e., the action permutes these representations), and it acts on $\mathfrak{N}_*(Y^H)$ via its action on $Y^H$. Our map $\Psi$ is the explicit description of the converse of $\Phi$. Specifically, $\Psi \circ \Phi$ is the identity on $\mathfrak{N}_G^G(Y)$. But $\Phi \circ \Psi$ is the identity only if one restricts oneself to the $WH$ invariant part of the factor corresponding to $H$. The need to pass to $WH$ fixed sets arises
in Proposition 4.3 of [C]. This is the strategy to deduce the claim in the first sentence of our theorem from Costenoble’s paper.

We discuss the formula which describes $\Psi$. Observe that we consider the $\gamma_{H,V,i}$ as abstract elements, and Costenoble’s $\gamma_{H,V,i}$ correspond to our $\Psi_H(\gamma_{H,V,i} \otimes 1)$. He considers the $\gamma_{H,V,i}$ only when $Y$ is a point so that 1 stands for the map point $\mapsto$ point. In this case we identify $\mathfrak{H}_G$ with $\mathfrak{H}_G(\text{pt})$ and write $\Psi_H(\gamma_{H,V,i} \otimes 1) = e_H[G \times_H R_{H,V,i}]$, which is the formula given on page 284 of [C] for abelian groups. (In the abelian group case the explicit expression for $\Psi_H(\gamma_{H,V,i})$ is of particular interest because then $WH$ acts trivially on the set of $H$ representations, hence on $\mathfrak{M}_*[\gamma_{H,V,i}]$. This leads to a perfectly explicit calculation of $\mathfrak{M}_G^*$ in Theorem 3.2.) Actually, the same analysis as the one Costenoble went through provides the expression for $\Psi_H(\gamma_{H,V,i} \otimes 1)$ also in the non-abelian case. Here we still use $Y = \text{point}$. The description of $\Psi_H(P \otimes \alpha)$ for $P \otimes \alpha \in \mathfrak{M}_*[\gamma_{H,V,i}] \otimes \mathfrak{M}_*(Y^H)$ is obtained from Costenoble’s description of $\Phi$, reversed by our $\Psi$, in connection with the Künneth formula for bordism theory. This is the strategy to verify the second sentence of the theorem. □

Obviously, we only sketched the strategy to prove Theorem 3.2, but the details would occupy several pages where we would only add an occasional technical remark to Costenoble’s proof.

Proof of Theorem D. Consider an element $P \otimes \alpha \in \mathfrak{M}_*[\gamma_{H,V,i}] \otimes \mathfrak{M}_*(Y^H)$ and represent $\alpha$ by a map $f : M \to Y^H$ where $M$ is a non-singular real algebraic variety and $f$ is an entire rational map. We may represent $\alpha$ in this way as $Y^H$ is assumed to have totally algebraic bordism. We need to show that $\Psi_H(P \otimes \alpha)$ is algebraic.

Obviously the canonical line bundle $\eta_{i-1}$ is a strongly algebraic vector bundle over $\mathbb{C}P^{i-1}$ (over $\Lambda = \mathbb{C}$ in Definition 2.9). Setting $\mathbb{C}P^{i-1} = G_\mathbb{C}(\mathbb{C}^i,1)$, the classifying map of $\eta_{i-1}$ is the identity map. A combination of the statements in 2.11 implies that $(\eta_{i-1} \otimes_\mathbb{C} V) \oplus \mathbb{R}$ is a strongly algebraic $H$ vector bundle over $\mathbb{C}P^{i-1}$ (over $\Lambda = \mathbb{R}$). Proposition 2.10 implies that $R_{H,V,i} = \mathbb{R}P((\eta_{i-1} \otimes_\mathbb{C} V) \oplus \mathbb{R})$ is a non-singular real algebraic $H$ variety. Consider a polynomial $P$ in the $\gamma_{H,V,i}$ with coefficients in $\mathfrak{M}_*$. Represent the coefficients by non-singular real algebraic varieties, which is possible by the Nash–Tognoli theorem. It follows from Proposition 2.7 that $\Psi_0(P)$, the corresponding polynomial expression in the $R_{H,V,i}$, is a non-singular real algebraic $H$ variety. Another application of Propositions 2.7 and 2.8 implies that $e_H \cdot (G \times_H (\Psi_0(P) \times M))$ is a non-singular real algebraic $G$ variety. This is the domain of $\Psi_H(P \otimes \alpha)$.

We have to show that the map

$$F : e_H \cdot (G \times_H (\Psi_0(P) \times M)) \to Y$$

obtained in our construction is equivariant and entire rational. Because $p_2$ in the definition of $\Psi_1$ is entire rational, it suffices to show that

\begin{equation}
(*) \quad G \times_H M \to G \times_H Y^H \to Y
\end{equation}

is equivariant and entire rational. The first map in $(*)$ is obtained from $f : M \to Y^H$ by applying induction, and according to Proposition 2.8, $\text{Ind}_H^G f$ is equivariant and entire
rational. The second map is the equivariant extension of the inclusion \( \iota : Y^H \to Y \) defined by mapping \([g, x] \in G \times_H Y^H \) to \( gx \in Y \). Because \( \iota \) is the embedding of a subvariety, hence entire rational, and \( H \) equivariant, and because \( G \) acts algebraically on \( Y \), it follows that the second map in (\( \ast \)) is equivariant and entire rational. As their composition, the map in (\( \ast \)) is equivariant and entire rational, and so is \( F \). Thus \( \Psi_H(P \otimes \alpha) \) is algebraically represented.

It follows from Theorem 3.2 and Proposition 2.7 that every class in \( \mathcal{N}^G_n(Y) \) has an algebraic representative. This is what we needed to show. \( \square \)

**Proposition 3.3.** Let \( G \) be a group of odd order, \( \Xi_i \) orthogonal representations of \( G \) and \( k_i \) natural numbers, \( i = 1, \ldots, m \). The \( G \) variety \( \prod_{i=1}^m G_\mathbb{R}(\Xi_i, k_i) \) has totally algebraic bordism, i.e., every homogeneous class in \( \mathcal{N}^G_\ast(\prod_{i=1}^m G_\mathbb{R}(\Xi_i, k_i)) \) is algebraically realized.

For \( m = 1 \) this is Proposition E which we stated in the introduction.

**Remark on totally algebraic homology.** Akbulut and King [AK1] showed that real Grassmannians have totally algebraic homology, hence totally algebraic bordism. The same proof shows that complex and quaternion Grassmannians also have totally algebraic bordism. Akbulut and King also show that the disjoint union and the cartesian product of varieties with totally algebraic bordism again have totally algebraic bordism.

**Proof of Proposition 3.3.** In Proposition 2.12 we described the \( H \) fixed point set \( G_\mathbb{R}(\Xi_i, k_i)^H \) as a disjoint union of components, where each component is a cartesian product and each factor a Grassmannians over \( \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \). Hence \( G_\mathbb{R}(\Xi_i, k_i)^H \) has totally algebraic bordism for each \( i \). But \( \left( \prod_{i=1}^m G_\mathbb{R}(\Xi_i, k_i) \right)^H = \prod G_\mathbb{R}(\Xi_i, k_i)^H \), hence the \( H \) fixed point set of this product of Grassmannians also has totally algebraic homology. Theorem D implies that \( \prod_{i=1}^m G_\mathbb{R}(\Xi_i, k_i) \) has totally algebraic bordism as a \( G \) variety. \( \square \)

**Corollary 3.4.** Let \( G \) be a group of odd order and \( M \) a closed smooth \( G \) manifold. The set of all real \( G \) vector bundles over \( M \) is algebraically realized.

**Proof.** First, consider a finite collection \( \Phi = \{ \xi_1, \ldots, \xi_m \} \) of real \( G \) vector bundles over \( M \), and let \( \mu_i : M \to G_\mathbb{R}(\Xi_i, k_i) \) be their classifying maps, \( i = 1, \ldots, m \). Set \( Y = \prod G_\mathbb{R}(\Xi_i, k_i) \). The product \( \mu = \prod \mu_i : M \to Y \) represents an element of \( \mathcal{R}^G_\ast(Y) \). By Proposition 3.3 it has an algebraic representative. It follows from Theorem C that \((M, \mu)\) can be algebraically realized by a pair \((X, \chi)\), i.e., \( X \) is a non-singular real algebraic variety, \( \chi \) is an equivariant entire rational map, and there is an an equivariant diffeomorphism \( \phi : X \to M \) so that \( \chi \) is equivariantly homotopic to \( \mu \circ \phi \). Let \( \pi_i : Y \to G_\mathbb{R}(\Xi_i, k_i) \) be the projection. Then \((X, \pi_i \circ \chi)\) realizes \((M, \mu_i)\) algebraically. This means that \( \xi_i \) is algebraically realized over \( X \). It follows that \( \Phi \) is algebraically realized over \( X \) as \((X, \phi)\) is the same for all \( \xi_i \in \Phi \).

If \( M \) is not connected and the \( \xi_i \) do not have a constant fibre dimension, we decompose \( M \) as a disjoint union of \( M_j \)’s, where each \( M_j \) is a \( G \) invariant union of components of \( M \), and all \( \xi_i \) have constant fibre dimension restricted to each \( M_j \). Restricted to each \( M_j \) the \( \xi_i \) are algebraically realized by the argument just given, so our proof is also complete in case of the finite set of \( G \) vector bundles \( \Phi \) in this more general setting.
It follows from Proposition 2.13 that the set of all real $G$ vector bundles over $M$ is algebraically realized. □

4. Proof of Proposition F and Theorem B (1)

In this section we prove Proposition F and Theorem B (1). As an important ingredient we prove a bordism result for bundles, Lemma 4.2. We begin this section with a construction of a bundle needed later in this section.

Let $\rho = (E_\rho \to B)$ be a real $G$ vector bundle with classifying map $\alpha : B \to G_\mathbb{R}(\Xi, k)$. In Section 2 we defined the total space of the associated projective bundle:

$$\mathbb{R}P(E_\rho) = \{(x, T) \in B \times G_\mathbb{R}(\Xi, 1) \mid (Id - \alpha(x))T = 0\}$$

Projection on the second factor defines a map $\tilde{\alpha} : \mathbb{R}P(E_\rho) \to G_\mathbb{R}(\Xi, 1)$. We use $\tilde{\alpha}$ to pull back the canonical line bundle $\gamma_\mathbb{R}(\Xi, 1)$ over $G_\mathbb{R}(\Xi, 1)$. The total space of the bundle we obtain is

$$Q(E_\rho) = \{(b, T, v) \in B \times G_\mathbb{R}(\Xi, 1) \times \Xi \mid (Id - \alpha(x))T = 0, \; Tv = v\}.$$ 

Thus we defined a $G$ line bundle $L(\rho) = (Q(E_\rho) \to \mathbb{R}P(E_\rho))$. As an immediate consequence of the construction we have

**Lemma 4.1.** If $(B, \alpha)$ is a strongly algebraic $G$ vector bundle, then so is $(\mathbb{R}P(E_\rho), \tilde{\alpha})$. Furthermore, if $\rho$ is an algebraically realized bundle, then so is $L(\rho)$.

The setting for Lemma 4.2 is as follows. Let $G$ be a compact Lie group, $\tau$ an element of order two in its center, and $M$ a closed smooth $G$ manifold. Let $\xi = (E \to M)$ be a $G$ vector bundle, and $\xi_\mid$ its restriction over $M^\tau$. Let $\xi_\mid^\pm$ be the $\pm 1$ eigen bundles of $\xi_\mid$ with respect to $\tau$. Since $\tau$ is central, $\xi_\mid^\pm$ are again $G$ vector bundles. Finally, let $\nu = (\nu(M^\tau) \to M^\tau)$ be the $G$ normal bundle of $M^\tau$ in $M$, and $L = (Q(\nu(M^\tau) \oplus \mathbb{R}) \to \mathbb{R}P(\nu(M^\tau) \oplus \mathbb{R}))$, the $G$ line bundle obtained from $\nu$ in the way described above after adding a one dimensional trivial bundle. We abbreviate $Q(\nu(M^\tau) \oplus \mathbb{R})$ as $Q$ and $\mathbb{R}P(\nu(M^\tau) \oplus \mathbb{R})$ as $P$. The projection from $P$ to $M^\tau$ is denoted by $p$.

We need the concept of a bordism between bundles. In the introduction we defined the concept of a bordism between maps. We apply it to bundles by saying that two bundles over closed manifolds are cobordant if their classifying maps are cobordant. Equivalently, two bundles are cobordant if there is a cobordism between their base spaces and a bundle over this cobordism which extends the given bundles.

**Lemma 4.2.** There exists a $G$ cobordism $W$ between $M$ and $P$ such that for every $G$ vector bundle $\xi$ over $M$ there exists a $G$ cobordism of bundles over $W$ between $\xi$ and $p^*\xi_\mid^+ \oplus (p^*(\xi_\mid^-) \otimes L)$.

We now prove Proposition F and Theorem B (1), and then we return to the proof of Lemma 4.2.
Proof of Proposition F. We use the notation from Lemma 4.2. Let $\mathfrak{F}$ be a collection of $G$ vector bundles over $M$. For $\xi \in \mathfrak{F}$ we define $\xi' = p^*\xi^+ \oplus (p^*(\xi^-) \otimes \mathcal{L})$, and we set $\mathfrak{F}' = \{\xi' \mid \xi \in \mathfrak{F}\}$. Let $W$ be the cobordism between $M$ and $P$, and let $\xi$ be the cobordism between $\xi$ and $\xi'$ over $W$ provided by Lemma 4.2. In the first step we show that $\mathfrak{F}'$ is algebraically realized. In the second step we apply Theorem C to obtain an algebraic realization for $\mathfrak{F}$ from the one of $\mathfrak{F}'$.

Let $\xi$ be a $G$ vector bundle over $M$, $\xi|_\tau$ its restriction over $M^\tau$, and $\xi^\pm$ the $\pm 1$ eigenbundle with respect to $\tau$. Denote their classifying maps by $\mu$, $\mu_1$ and $\mu^\pm$ respectively. Let $\nu(M^\tau)$ be the normal bundle of $M^\tau$ and $\chi$ its classifying map. By assumption, there exists a non-singular real algebraic $G$ variety $X_0$ and a $G$ equivariant diffeomorphism $\varphi : X_0 \to M^\tau$ which realizes the set of all $G$ vector bundles over $M^\tau$. In particular, $\varphi \circ \varphi$ is $G$ equivariantly homotopic to an entire rational map if $\vartheta = \mu^\pm$ and if $\vartheta = \chi$.

Form the total space $P \nu$ of the projective bundle associated with the strongly algebraic $G$ vector bundle $(\chi \oplus 1) \circ \varphi$, the summand 1 indicates that we added a one dimensional product bundle with trivial action on the fibre. Then $P \nu$ is a non-singular real algebraic $G$ variety, and the projection $p \nu : P \nu \to X_0$ is equivariant and entire rational (see Proposition 2.10). By construction, there exists a $G$ equivariant diffeomorphism $\Phi : P \nu \to P$ such that the diagram

$$
\begin{array}{ccc}
P \nu & \xrightarrow{\Phi} & P \\
p \nu \downarrow & & \downarrow p \\
X_0 & \xrightarrow{\varphi} & M^\tau
\end{array}
$$

commutes. The pair $(P \nu, \Phi)$ provides an algebraic realization of the bundles $p^*(\xi^\pm)$ over $P$. It also provides an algebraic realization for $\nu(M^\tau) \oplus \mathbb{R}$, hence by Lemma 4.1 an algebraic realization for the bundle $\mathcal{L}$ introduced above. It follows from Proposition 2.11 that $(P \nu, \Phi)$ provides an algebraic realization of $\xi' = p^*(\xi^+) \oplus (p^*(\xi^-) \otimes \mathcal{L})$.

To summarize this first step, we constructed a collection $\mathfrak{F}' = \{\xi' \mid \xi \in \mathfrak{F}\}$ of $G$ vector bundles over $P$, this collection is algebraically realized by $(P \nu, \Phi)$, and by Lemma 4.2 there exists a $G$ cobordism $W$ between $M$ and $P$ which is the base space for $G$ cobordisms of bundles $\xi'$ between $\xi$ over $M$ and $\xi'$ over $P$ for all $\xi \in \mathfrak{F}$. To avoid unnecessary notation we now assume (w. l. o. g.) that $P$ is a non-singular real algebraic $G$ variety, and the classifying maps $\mu'$ of the bundles $\xi' \in \mathfrak{F}'$ are entire rational.

Suppose now that $\mathfrak{F}$ is any finite set of $G$ vector bundles over $M$. Let $\xi$ and $\xi'$ be classified by maps to a Grassmannian $G(\xi)$, and let $G(\mathfrak{F}) = \prod G(\xi)$, where the product is taken over all $\xi \in \mathfrak{F}$. Let $\eta = \prod \mu$ and $\eta' = \prod \mu'$ be the products of classifying maps of $\xi$ and $\xi'$ for $\xi \in \mathfrak{F}$, so $\eta : M \to G(\mathfrak{F})$ and $\eta' : P \to G(\mathfrak{F})$. By construction, $\eta'$ is entire rational. Let $\mu : W \to G(\xi)$ be a classifying map of $\tilde{\xi}$ which extends $\mu$ and $\mu'$, and $\tilde{\eta} : W \to G(\mathfrak{F})$ the product of all $\tilde{\mu}$ as $\xi$ varies over the elements in $\mathfrak{F}$. Observe that $(W, \tilde{\eta})$ provides a cobordism between $(M, \eta)$ and $(P, \eta')$.

Theorem C implies that there exists a non-singular real algebraic $G$ variety $X$ and an equivariant diffeomorphism $\phi : X \to M$ such that $\eta \circ \phi$ is equivariantly homotopic to an
equivariant entire rational map. Denote the projection from \( G(\mathcal{F}) \) onto \( G(\xi) \) by \( \pi_\xi \). Then \( \pi_\xi \circ \eta \circ \phi : X \to G(\xi) \) realizes \( \mu : M \to G(\xi) \) algebraically for each \( \xi \in \mathcal{F} \), and \((X, \phi)\) provides an algebraic realization of \( \mathcal{F} \). As \( \mathcal{F} \) was any finite set of \( G \) vector bundles over \( M \), it follows from Proposition 2.13 that the set of all \( G \) vector bundles over \( M \) can be algebraically realized. This is what we set out to show. \( \square \)

We state Proposition F in the form in which it is applied in the proof of Theorem B (1).

**Corollary F.** Let \( G = H \times \mathbb{Z}_2 \) be a compact Lie group, \( \tau \) a generator of the \( \mathbb{Z}_2 \)-factor of \( G \) and \( M \) be a closed smooth \( G \) manifold. The set of all real \( G \) vector bundles over \( M \) is algebraically realized if the set of all real \( H \) vector bundles over \( M^\tau \) is algebraically realized.

**Proof.** By Proposition F it suffices to show that the set of all real \( G \) vector bundles over \( M^\tau \) is algebraically realized. We show this. Let \( \xi \) be a \( G \) vector bundle over \( M^\tau \), and \( \xi^\pm \) the \( \pm 1 \) eigen bundle of \( \xi \) with respect to \( \tau \). These bundles are \( G \) vector bundles, but \( \xi^\pm \) are uniquely determined by their structure as \( H \) vector bundles and the decision whether they are the \( +1 \) or \( -1 \) eigen bundle. By assumption these \( H \) vector bundles are algebraically realized. We realize them as \( G \) vector bundles. On the base space the action of \( H \) extends to an action of \( G \) by letting \( \tau \) act trivially, and on the fibres of these bundles the action of \( \tau \) is either trivial or multiplication with \(-1\). If the \( H \) bundles \( \xi^+ \) and \( \xi^- \) are algebraically realized over a non-singular real algebraic \( H \) variety \( X_1 \), then the \( G \) bundles \( \xi^+ \) and \( \xi^- \), as well as their sum \( \xi = \xi^+ \oplus \xi^- \) (see Proposition 2.11), are algebraically realized over a non-singular real algebraic \( G \) variety \( X_0 \). The varieties are the same, only the action of \( H \) is extended to an action of \( G \) by letting \( \tau \) act trivially. The reader is invited to express these ideas in terms of classifying maps to make them more explicit. This argument applies to arbitrary sets of bundles, and this implies our claim. \( \square \)

**Proof of Theorem B (1).** Let \( K \) be a group of odd order and \( G = K \times T(n) \) where \( T(n) = (\mathbb{Z}_2)^n \). It follows from Corollary 3.4 that the set of all \( K \) vector bundles over \( M^{T(n)} \), which is a \( K \) manifold, is algebraically realized. Factor \( T(n) \) as \( T(1) \times T(n-1) \). Corollary F implies that the set of all \( K \times T(1) \) vector bundles over \( M^{T(n-1)} \) is algebraically realized. Continued application of Corollary F implies that the set of all \( G \) vector bundles over \( M \) is algebraically realized. \( \square \)

**Proof of Corollary E.** Let \( \alpha \) be a class in \( \mathfrak{N}_\mathbb{R}^G(\varXi, k) \). Represent \( \alpha \) by an equivariant map \( \chi : M \to G_{\mathbb{R}}(\varXi, k) \), where \( M \) is a closed smooth \( G \) manifold. Let \( \xi = \chi^*(\gamma_{\mathbb{R}}(\varXi, k)) \) be the bundle obtained by pulling back the universal \( G \) vector bundle over \( G_{\mathbb{R}}(\varXi, k) \). The algebraic realization of \( \xi \), obtained from Theorem B (1), provides the algebraic representative of \( \alpha \). \( \square \)

We now turn our attention to Lemma 4.2. Let \( G, \tau, M, \nu, \) and \( P \) be as in the setup for this lemma. One way to obtain \( P \) is as follows. Let \( \pi : D(\nu) \to M^\tau \) be the unit disk bundle of \( \nu \), and \( S(\nu) \subset D(\nu) \) the total space of the unit sphere bundle. We obtain \( P \) as a quotient of \( D(\nu) \), identifying \( x \) with \( \tau(x) \) if \( x \in S(\nu) \). We denote the quotient map by \( q : D(\nu) \to P \). Let \( \eta \) be a \( G \) vector bundle over \( M^\tau \) with total space \( E_\eta \) and \( \pi^*(\eta) \) the pull back over \( D(\nu) \) with total space \( \pi^*(E_\eta) \). Furthermore, let \( \tilde{\pi}^*(\eta) \) be the bundle over
P obtained by identifying \((x, v) \in \pi^*(E_\eta) \subset D(\nu) \times E_\eta\) with \((\tau(x), \tau(v))\) if \(x \in S(\nu)\). We observe that \(\tilde{\pi}^*(\eta)\) is a \(G\) vector bundle since \(\tau\) is in the center of \(G\). As before, we denote the projection \(P \to M^\tau\) by \(p\). The \(\tau\) fixed point set of \(P\) consists of a copy of \(M^\tau\), which corresponds to the zero section of \(D(\nu)\), and a space \(C\) which is the quotient space of \(S(\nu)\) obtained by identifying \(x\) with \(-x\).

**Proposition 4.3.** With the notation from above, and \(\mathcal{L}\) as in Lemma 4.2 we have

1. \(\mathcal{L}|_{M^\tau}\) is a product bundle with the trivial representation \(\mathbb{R}\) as fibre.
2. \(\tau\) acts by multiplication with \(-1\) on the fibres of \(\mathcal{L}\) restricted over \(C\).
3. \(\mathcal{L}\) is obtained from \(D(\nu) \times \mathbb{R}\) by identifying \((x, t)\) with \((-x, -t)\) if \(x \in S(\nu)\).
4. \(\tilde{\pi}^*(\eta) \cong p^*(\eta)\) if \(\tau\) acts trivially on \(E_\eta\).
5. \(\tilde{\pi}^*(\eta) \cong p^*(\eta) \otimes \mathcal{L}\) if \(\tau\) acts by multiplication with \(-1\) on \(E_\eta\).
6. The operation \(\tilde{\pi}^*\) is compatible with direct sums and tensor products.

**Proof.** To see (1) we observe that there is a canonical section \(s : M^\tau \to P\) in the bundle \(\mathcal{L}\) which, in the notation from the beginning of the section, maps \(b \in M^\tau\) to \(T_c \in G_\mathbb{R}(\Xi, 1)\), where \(T_c\) stands for the orthogonal projection onto the summand \(\mathbb{R}\) which we added to \(\nu(M^\tau)\) when we formed \(\mathcal{L}\). This corresponds to the embedding \(M^\tau \to P\) which we defined in the paragraph before the proposition. Then \(\alpha(x) = T_c\) for all \(x \in M^\tau\), where \(\alpha\) is the classifying map of \(\mathcal{L}\) defined in the beginning of this section, hence \(\mathcal{L}|_{M^\tau}\) is a product bundle. The action on the fibre is trivial because \(T_c\) is the orthogonal projection onto \(\mathbb{R}\) with trivial action.

To see (2) we observe that the points in \(C\) correspond to lines (or orthogonal projection onto lines) in the summand \(\nu\) of \(\nu \oplus \mathbb{R}\) of which \(P\) is the total space of the associated projective bundle. As \(\tau\) acts by multiplication with \(-1\) on \(\nu\), it acts in the same way on each of the lines which correspond to the points in \(C\), hence on the fibres of \(\mathcal{L}|_C\).

(3) is obtained as an easy reformulation of the construction of \(\mathcal{L}\) in the language developed in the previous two arguments.

To see (4) we observe that \(p^*(\eta)\) is obtained from \(\pi^*(\eta)\) by identifying \((x, v) \in \pi^*(E_\eta) \subset D(\nu) \times E_\eta\) with \((-x, v)\) if \(x \in S(\nu)\). In this case \((-x, v) = (\tau(x), \tau(v))\). So, in the construction of \(p^*(\eta)\) we perform exactly the same identification as the one used in the construction of \(\tilde{\pi}^*(\eta)\) and \(p^*(\eta) \cong \tilde{\pi}^*(\eta)\) as claimed.

We show (5). Consider \(\tilde{\pi}^*(E_\eta)\) as a quotient space of \(\pi^*(E_\eta)\) as done before in this proof, and consider \(\pi^*(E_\eta)\) as a subset of \(D(\nu) \times E_\eta\). In the construction of \(\tilde{\pi}^*(E_\eta)\) we identify \((x, v) \in \pi^*(E_\eta)\) with \((\tau(x), \tau(v)) = (-x, v)\) if \(x \in S(\nu)\). Similarly, in the construction of the total space of \(p^*(\eta) \otimes \mathcal{L}\) as a quotient of \(\pi^*(E_\eta) \otimes (D(\nu) \times \mathbb{R}) \subset (D(\nu) \times E_\eta) \otimes (D(\nu) \times \mathbb{R})\) we identify \((x, v) \otimes (x, t)\) with \((-x, v) \otimes (-x, -t)\) if \(x \in S(\nu)\). We have an isomorphism \(\beta : \pi^*(\eta) \to \pi^*(\eta) \otimes \pi^*(\mathcal{L}|_{M^\tau})\) of \(G\) vector bundles over \(D(\nu)\) defined by \(\beta(x, v) = (x, v) \otimes (x, 1)\). Then \(\beta(-x, -v) = (-x, v) \otimes (-x, 1) = (-x, v) \otimes (-x, -1)\). This means that \(\beta\) is compatible with the identifications described above, and that \(\beta\) induces a \(G\) equivariant isomorphism \(\tilde{\pi}^*(\eta) \to p^*(\eta) \otimes \mathcal{L}\).

Our last claim (6) is an immediate consequence of the constructions involved in the definition of \(\tilde{\pi}^*\). \(\square\)
Proof of Lemma 4.2. Set $M_0 = M − \text{Int}(D(\nu(M^\tau)))$, where $D$ stands for the disk bundle. It is a compact smooth $G$ manifold on which $\tau$ acts freely. Let $Z$ be the total space of the $D^1$ bundle associated with the principal $\mathbb{Z}_2$ bundle $M_0 \to M_0/\tau$. Since $\tau$ is central in $G$, $Z$ is a smooth $G$ manifold. We note that $\partial Z = M_0 \cup \partial M_0 \times_\tau D^1$. The action of $\tau$ on $D^1$ is antipodal, and the union is taken over $\partial M_0 = \partial M_0 \times_\tau S^0$. Consider $M \times [0, 1]$ and regard $M_0$ as a submanifold of $M \times \{1\}$. We glue $Z$ and $M \times [0, 1]$ together along $M_0$. The resulting $G$ manifold $W = M \times [0, 1] \cup M_0$ is an equivariant cobordism between $M$ and $P = \mathbb{R}P(\nu(M^\tau) \oplus \mathbb{R})$.

We extend $\xi$ to a $G$ vector bundle $\tilde{\xi}$ over $W$. The construction is analogous to the one of $W$, but carried out with bundles. Let $\xi_0 = (E_0 \to M_0)$ be the restriction of $\xi$ over $M_0$. The total space of $\tilde{\xi}$ is $\tilde{E} = E \times [0, 1] \cup \mathcal{E}_0$ and the projection to $W$ is the obvious one. We denote the restriction of $\tilde{\xi}$ by $\xi' = (E' \to P)$.

We observe that, by construction, $\xi'$ is isomorphic to $\tilde{\pi}^*(\xi)$. It follows from 4.3 (6) that $\tilde{\pi}^*(\xi)$ is isomorphic to $\tilde{\pi}^*(\xi^+) \oplus \tilde{\pi}^*(\xi^-)$, and from 4.3 (4) and (5) that it is isomorphic to $p^*(\xi^+) \oplus (p^*(\xi^-) \otimes \mathcal{L})$. Thus $\xi'$ is isomorphic to $p^*(\xi^+) \oplus (p^*(\xi^-) \otimes \mathcal{L})$ as claimed. As required, all bundles $\tilde{\xi}$ for $\xi \in \mathfrak{g}$ have the same base space $W$. This concludes the proof of the lemma.

5. Proof of Theorem B (2) – The Semifree Case

As basic tool in the proof of Theorem B (2) we need a bordism theoretic result. To state this result we need the notion of free bordism. A group $G$ is said to act freely on a space $X$ if $G_x = \{g \in G \mid gx = x\} = \{1\}$ for all $x \in X$. Let $Y$ be a $G$ space. We define the free bordism group $\mathfrak{N}_G^G[\text{free}] (Y)$. Its elements are represented by equivariant maps $f : M \to Y$, where $M$ is a closed smooth $G$ manifold on which $G$ acts freely. Two maps $f_0 : M_0 \to Y$ and $f_1 : M_1 \to Y$ are said to be freely cobordant (or in the same class in $\mathfrak{N}_G^G[\text{free}] (Y)$) if there exists a compact smooth $G$ manifold $W$ with free action of $G$ and an equivariant map $F : W \to Y$ such that $\partial W$ is the disjoint union of $M_1$ and $M_2$ and $F$ extends $f_0$ and $f_1$.

Let $G$ be a compact Lie group. We define a subgroup $N_2T$. Consider the exact sequence

$$1 \to T \to NT \to W \to 1.$$ 

Here $T$ is a maximal torus of $G$, $NT$ its normalizer in $G$, and $W$ the Weyl group. Let $W_2$ be the 2-Sylow subgroup of $W$ and $N_2T$ the subgroup of $NT$ given by the exact sequence

$$1 \to T \to N_2T \to W_2 \to 1.$$ 

If $G$ is finite, then $N_2T$ is the 2-Sylow subgroup of $G$.

**Lemma 5.1.** Let $G$ and $N_2T$ be as above. If $G$ is not of odd order, then there exists a non-trivial element $\tau \in N_2T$ which is central in $N_2T$.

**Proof.** There is an action of $W_2$ on $T$ given by

$$W_2 \times T \to T \quad \text{with} \quad (g, t) \mapsto gtg^{-1}.$$
As elements of order 2 in $T$ are again mapped to elements of order 2, there is an induced action of $W_2$ on the 2-torus $H = (\mathbb{Z}_2)^k$ in $T$. Here $k = \dim T$. As this action has the unit element $e \in H$ as a fixed point, there must be another, non-trivial $W_2$ fixed point in $H$. We call it $\tau$ and note that $\tau$ lies in the center of $N_2 T$. This concludes the proof. \(\square\)

Let $G$ and $N_2 T$ be as above. Given an $H$ equivariant map $f : M \to Y$ we obtain a map $F : G \times_H M \to Y$ by setting $F[g, x] = gf(x)$, where $[g, x]$ denotes the class of $(g, x) \in G \times M$ in $G \times_H M$. We use this induction process in our next proposition. It is different from the one considered in Section 2. There is no possibility of conflict of notation, so we denote this construction also by $\text{Ind}$. Our next result is a generalization of a well known result for finite groups.

**Proposition 5.2.** Let $G$ be a compact Lie group, $N_2 T$ as above, and $Y$ a $G$ CW complex of finite type. Set $d = \dim G$ and $d_1 = \dim N_2 T$. Then $\text{Ind} : \mathcal{N}_{*+d}^{N_2 T}[\text{free}](Y) \to \mathcal{N}_{*+d}^G[\text{free}](Y)$ is surjective.

**Proof.** Let $Y_G$ denote the Borel construction on $Y$, i.e., $Y_G = EG \times_G Y$ where $EG$ is the universal contractible $G$ space. We also use $Y_{N_2 T} = EG \times_{N_2 T} Y$. In the proof we use the following diagram.

\[
\begin{array}{ccc}
\mathcal{N}_* \otimes_{\mathbb{Z}_2} H_*(Y_G, \mathbb{Z}_2) & \xrightarrow{\cong} & \mathcal{N}_*(Y_G) \\
\uparrow i_* & & \uparrow i_* \\
\mathcal{N}_* \otimes_{\mathbb{Z}_2} H_*(Y_{N_2 T}, \mathbb{Z}_2) & \xrightarrow{\cong} & \mathcal{N}_*(Y_{N_2 T}) \\
\end{array}
\]

We describe the maps in the diagram. The definition of the induction maps $\text{Ind}$ has been given above. There are two maps denoted by $i_*$. They are induced by the inclusion map $N_2 T \to G$. It defines a quotient map $Y_{N_2 T} \to Y_G$, which in turn defines the maps $i_*$ in the diagram. The first pair of horizontal isomorphisms is obtained from the Künneth formula in bordism [CF, Section 17]. This map is not natural, and the first square may not commute. For the second pair of isomorphisms we refer the reader to [S1, Proposition 5.2]. Still, we recall its definition to show that the second square in the diagram commutes. Consider a class in $\mathcal{N}_{*+d}^G[\text{free}](Y)$ represented by a map $f : M \to Y$. Consider the fibration $M \to M/G$. Its classifying map to $BG$ is covered by a map $\chi : M \to EG$. The product $\chi \times f : M \to EG \times Y$ is an equivariant map between free $G$ spaces. The induced map of quotient spaces represents a class in $\mathcal{N}_*(Y_G)$. This defines the isomorphism $\mathcal{N}_{*+d}^G[\text{free}](Y) \to \mathcal{N}_*(Y_G)$. The map $\mathcal{N}_{*+d_1}^{N_2 T}[\text{free}](Y) \to \mathcal{N}_*(Y_{N_2 T})$ is defined in the corresponding way.

Consider the fibration

\[G/N_2 T \to Y_{N_2 T} \xrightarrow{i} Y_G\]

and the composition

\[H_*(Y_G, \mathbb{Z}_2) \xrightarrow{tr} H_*(Y_{N_2 T}, \mathbb{Z}_2) \xrightarrow{i_*} H_*(Y_G, \mathbb{Z}_2)\]

Here $tr$ denotes the Becker-Gottlieb transfer [G, Theorem C], [BG]. As $i_* \circ tr$ is multiplication with the Euler characteristic of $G/N_2 T$, and $\chi(G/NT) = 1$ (see [HS], [We], or
Recall some details from [CF, Section 17]. Choose \( \kappa \geq n \) depends on some choices which we like to make compatibly for \( Z \).

Second factor. That \( Y \) those \( \{ \mu \} \) map to the basis elements \( \{ j,n \} \) the first square in the diagram given above may not commute. Nevertheless, if \( B \) denotes the subspace of \( H_n(Y_{N_2T}, \mathbb{Z}_2) \) spanned by those \( b_{j,n} \) which map to the basis elements \( c_{j,n} \), then the square commutes restricted to \( \mathfrak{N}_* \otimes_{\mathbb{Z}_2} B \). By construction, \( \mathfrak{N}_* \otimes_{\mathbb{Z}_2} B \) surjects onto \( \mathfrak{N}_* \otimes_{\mathbb{Z}_2} H_n(Y_G, \mathbb{Z}_2) \), and it follows that \( i_* : \mathfrak{N}_*(Y_{N_2T}) \to \mathfrak{N}_*(Y_G) \) is onto. Commutativity of the second square implies that \( \text{Ind} : \mathfrak{N}_{*+d_1}^{N_2T} [\text{free}](Y) \to \mathfrak{N}_{*+d}^{G} [\text{free}](Y) \) is surjective.

**Proof of Theorem B (2), Case 1.** First, consider the case where \( G \) acts freely. The case where \( G \) is of odd order is covered by the first part of the theorem. So, suppose that \( G \) is not of odd order, \( M \) is a closed smooth \( G \)-manifold on which \( G \) acts freely, and \( \mathfrak{g} \) is an arbitrary finite collection of \( G \)-vector bundles over \( M \). Suppose that \( \xi \in \mathfrak{g} \) is classified by a map \( \mu(\xi) : M \to G(\xi) \), where \( G(\xi) \) is an appropriate Grassmannian. Set \( G(\mathfrak{g}) = \prod_{\xi \in \mathfrak{g}} G(\xi) \) and \( \mu(\mathfrak{g}) = \prod_{\xi \in \mathfrak{g}} \mu(\xi) \). Consider the class of \( \mu(\mathfrak{g}) : M \to G(\mathfrak{g}) \in \mathfrak{N}_*(G(\mathfrak{g})) \). We show that it is zero.

Surjectivity of the induction map \( \text{Ind} : \mathfrak{N}_{*+d_1}^{N_2T} [\text{free}](Y) \to \mathfrak{N}_{*+d}^{G} [\text{free}](Y) \) implies that there exists a closed smooth \( G \)-manifold \( M' \) and a \( G \)-equivariant map \( \eta : M' \to G(\mathfrak{g}) \) such that \( (M', \mu(\mathfrak{g})) \) is equivariantly cobordant to \( (M', \eta) \), and \( (M', \eta) = \text{Ind}(M_0, \eta_0) \) for some \( N_2T \) map \( \eta_0 : M_0 \to G(\mathfrak{g}) \). Let \( p(\xi) : G(\mathfrak{g}) \to G(\xi) \) be the projection. The map \( \mu'_0(\xi) = p(\xi) \circ \eta_0 \) classifies an \( N_2T \) bundle \( \xi'_0 \) over \( M_0 \). Let \( \tau \) be a non-trivial central element of order two in \( N_2T \) (see Lemma 5.1). Then \( M_0 \) is the boundary of the mapping cylinder \( W_0 \) of the quotient map \( M_0 \to M_0/\tau \), and \( W_0 \) is a smooth \( N_2T \) manifold because \( \tau \) is central in \( N_2T \). The same mapping cylinder construction applied to \( \xi'_0 \) provides an extension of \( \xi'_0 \) to an \( N_2T \) bundle over \( W_0 \). Hence \( \mu'_0(\xi) \) extends to an \( N_2T \) equivariant map \( W_0 \to G(\xi) \). Let \( W = \text{Ind} W_0 \). Then \( M' = \partial W \), and induction provides us with a \( G \)-equivariant map \( \mu'(\xi) : M' \to G(\xi) \) which extends \( G \) equivariantly over \( W \). The product of these classifying maps, \( \prod_{\xi \in \mathfrak{g}} \mu'(\xi) : M' \to G(\mathfrak{g}) \), is a boundary representing zero in \( \mathfrak{N}_*(G(\mathfrak{g})) \). It is also \( G \) equivariantly cobordant to \( (M, \mu(\mathfrak{g})) \), so \( (M, \mu(\mathfrak{g})) \) bounds as well.
Theorem C implies the existence of a non-singular real algebraic $G$ variety $X$ and an equivariant diffeomorphism $f : X \to M$ such that $\mu(\mathfrak{F}) \circ f$ is equivariantly homotopic to an entire rational map. The projection maps $p(\xi) : G(\mathfrak{F}) \to G(\xi)$ are equivariant and entire rational, and so are the classifying maps $p(\xi) \circ \mu(\mathfrak{F}) \circ f = \mu(\xi) \circ f$ for the bundles $f^*(\xi)$ for all $\xi \in \mathfrak{F}$. This means that we found an algebraic realization of the finite set of $G$ vector bundles $\mathfrak{F}$. Proposition 2.13 now implies that the set of all $G$ vector bundles over $M$ is algebraically realized, and this is what we wanted to show.

**Case 2.** We consider the case where $G$ does not act freely. Again, the odd order group case has been taken care of in the first part of the theorem. So, suppose that $G$ is not of odd order. Let $x \in M$ be a fixed point of the action of $G$. Then $G$ acts freely on the unit sphere in the fibre of the normal bundle of $M^G$ in $M$ at $x$. Milnor showed that every element of order two in $G$ must be central [M]. Hence $G$ has a central element $\tau$ of order two. Obviously, $M^\tau = M^G$. Theorem 7.2 of [DM] states that every $G$ vector bundle over $M^G$ can be algebraically realized. In the proof of this theorem we actually realize a finite number of $G$ vector bundles over a closed manifold with trivial action of $G$, so the theorem has as an immediate corollary of its proof the conclusion that any finite collection of $G$ vector bundles over $M^G$ can be algebraically realized. Proposition 2.13 implies that the set of all $G$ vector bundles over $M^G$ is algebraically realized, and Proposition F implies that the set of all $G$ vector bundles over $M$ is algebraically realized. This was our claim. \(\Box\)

6. Review of some Differential Topology

In preparation of the proof of Theorem C we review some differential topology. The proof involves some delicate approximations of maps, so we explain in which sense these are to be understood. Next we relate this to the notion of an isotopy. We also apply a ‘transversality argument’, and what this means is the final topic of this section.

**$C^1$ Approximations.** Let $G$ be a compact Lie group, and $\Xi$ and $\Omega$ orthogonal representations of $G$. Then $C^r(\Xi, \Omega)$ denotes the set of all $r$ times continuously differentiable functions from $\Xi$ to $\Omega$. On this space we consider the $C^\rho$ topology for $\rho \leq r$ [H]. More specifically, let $p_{K,q}$ be the semi-norm on the set $C^r(\Xi, \Omega)$ defined by

$$p_{K,q}(f) = \sup_{|s| \leq q, x \in K} |D^s f(x)|$$

where $K$ is a compact subset of $\Xi$, $q \leq r$ and $|y| = \sqrt{y_1^2 + \cdots + y_m^2}$ for $y = (y_1, \ldots, y_m) \in \Omega$. For $q \leq \rho \leq r$ these semi-norms induce the $C^\rho$ topology on $C^r(\Xi, \Omega)$.

Let $A$ be the averaging operator considered in Section 2. In [DMP, Lemma 4.1] we showed

**Lemma 6.1.**  
(1) If $0 \leq r \leq \infty$ and $f \in C^r(\Xi, \Omega)$, then $A(f) \in C^r(\Xi, \Omega)$.

(2) The selfmap of $C^r(\Xi, \Omega)$ induced by $A$ is continuous with respect to the $C^\rho$ topology for any $\rho \leq r$ and $\rho < \infty$.

(3) If $f \in C^1(\Xi, \Omega)$, then $D^s A(f)(x) \leq |D^s f(x)|$ for all partial derivatives $D^s$ of order at most one.
Isotopy. Let $X$ be a smooth submanifold of $M$, and let $j$ be the inclusion. An isotopy of $X$ is a smooth map $h : X \times I \to M$ such that $h(x, 0) = j(x)$ and $h(\cdot, t) : X \times \{t\} \to M$ is an embedding for each $t \in I$. With the obvious identifications this last map is also denoted by $h_t : X \to M$. The isotopy is said to be small if $h_t$ is close to $h_0$ for each $t \in I$. Two submanifolds $X$ and $X_1$ of $M$ are called isotopic if there is an isotopy $h : X \times I \to M$ of $X$ such that $h(X, 1) = X_1$, and $h$ is called an isotopy between $X$ and $X_1$. The isotopy fixes $L \subset X$ if $h(x, t) = j(x)$ for all $x \in L$ and for all $t \in I$. Some authors prefer to extend isotopies to parametrized families of diffeomorphisms of $M$, which is always possible by the isotopy extension theorem, and call this family of diffeomorphisms the isotopy. In our convention, this family of diffeomorphism would be called an ambient isotopy. To define equivariant isotopies we assume that all spaces have a $G$ action and that all maps are equivariant.

Transversality. Let $f : M \to N$ be a smooth map between smooth manifolds. Let $Y$ be a smooth submanifold of $N$. Then $f$ is said to be transverse to $Y$ if for all $y \in Y$ and for all $x \in f^{-1}(y)$ the composition of the derivative and the quotient map

$$T_xM \xrightarrow{Df_x} T_yN \to \nu_y(Y, N) \xrightarrow{\text{Def}} T_yN/T_yY$$

is surjective. With the obvious meaning we also use the phrase ‘$f$ is transversal to $N$ at $x’$. With the help of a Riemannian metric on $Y$ one can find a direct sum decomposition $T_yN = T_yY \oplus \nu_y(Y, N)$ and consider the second map as the projection on the second factor. Observe that $f^{-1}(Y)$ is a smooth submanifold of $M$ if $f$ is transverse to $Y$. Transversality is an open condition. That means, if $f : M \to N$ is transverse to $Y \subset N$, then there is a neighbourhood $U$ of $f$ in $C^\infty(M, N)$ with the $C^1$ topology such that every function $f' \in U$ is also transverse to $Y$. This applies if $M$ is compact. In case of a non-compact manifold $M$ we will have to apply some additional arguments to control the situation outside of a compact set.

Isotopy and Transversality. Finally, we relate isotopies and transversality. Let $H : M \times I \to N$ be a smooth map. It restricts to maps $H_t : M \times \{t\} = M \to N$, $t \in I$. Suppose $H_t$ is transverse to $Y \subset N$ for each $t \in I$. It is a theorem that $H_0^{-1}(Y)$ and $H_1^{-1}(Y)$ are isotopic. This theorem follows from the Morse Lemma. The homotopy $H$ between $H_0$ and $H_1$ is called small if $H_t$ and $H_0$ are close to each other for all $t \in I$. If the homotopy is small, then the isotopy between $H_0^{-1}(Y)$ and $H_1^{-1}(Y)$ may be chosen small as well. This one may check easily. If the homotopy is relative to $L \subset M$ (that is, $H(x, t) = H(x, 0)$ for all $x \in L$ and $t \in I$) then the isotopy may be chosen to fix $L$. The connection between transversality and isotopy explained in this paragraph will be used repeatedly in this section. We refer to it as a transversality argument.

7. Proof of Theorem C

In this section we prove Theorem C. This theorem generalizes, in part, a non-equivariant result of Akbulut and King [AK1] in so far as we are working in the equivariant category. It also generalizes an algebraic approximation result for closed smooth $G$ manifolds of our
own [DMP] in so far as we are now approximating a closed smooth $G$ manifold together with an equivariant map to a non-singular $G$ variety by a non-singular real algebraic $G$ variety and an equivariant entire rational map. In the previous section we clarified the meaning of the word approximation as it is used in this section.

**Convention.** Throughout this section we use $C^1$ approximations of $C^\infty$ maps. Using the notation from Lemma 6.1 we set $r = \infty$ and $\rho = 1$. Furthermore, $G$ denotes a compact Lie group. We repeat this in some central statements for emphasis.

We do not attempt to state or prove the results in this section in greater generality than needed for the proof of Theorem C. On the other hand, we proceed along the lines provided by Akbulut and King so that we can take advantage of their work. First of all, we recall a procedure to remove a subvariety from a variety. It was used by Tognoli [T]. The basic ingredient in the proof is the complexification/realification process from [W].

**Lemma 7.1.** Suppose $V_a \subset \mathbb{R}^n$ is a non-singular real algebraic variety and $V_b \subset V_a$ is a non-singular real algebraic subvariety with $\dim V_a = \dim V_b$. Then $V_a \setminus V_b \subset \mathbb{R}^n$ is a non-singular real algebraic variety.

**Proof.** We show that $V_b$ is the union of irreducible components of $V_a$. Here we use the word components with respect to the Zariski topology. Then $V_c = V_a \setminus V_b$ is the union of those irreducible components of $V_a$ which are not contained in $V_b$, hence it is a real algebraic variety. Let $U$ be an irreducible component of $V_a$ such that $U \cap V_b$ is non-empty. We show that $U \subset V_b$. Denote the complexification of a variety by a superscript $\mathbb{C}$. Then $U^\mathbb{C}$ is irreducible by [W, Lemma 7]. Non-singularity of $V_d$ implies that $V_d^\mathbb{C}$ is a 2dim$V_d$ dimensional manifold near $V_d$ for $d = a, b$. So there is a neighbourhood of $V_b$ in $V_b^\mathbb{C}$ which is contained in $V_a^\mathbb{C}$. The non-singular points of a complex variety are dense, so if $U \cap V_b$ is non-empty, then we must have $\dim(U^\mathbb{C} \cap V_b^\mathbb{C}) = \dim(U^\mathbb{C} \cap V_a^\mathbb{C}) = \dim(U^\mathbb{C})$. The last equation follows as $U^\mathbb{C} \subset V_a^\mathbb{C}$. But a proper subvariety of an irreducible variety has strictly smaller dimension [W, Lemma 2]. It follows that $U^\mathbb{C} \cap V_b^\mathbb{C} = U^\mathbb{C}$ and $U \cap V_b = U$. Hence $U \subset V_b$ and, as claimed above, $V_b$ is a union of components of $V$ and so is $V_c$. It is immediate from the definition of non-singularity that points of $V_c$ which are non-singular points of $V_a$ are non-singular points of $V_c$, hence $V_c$ is non-singular. $\Box$

**Definition.** Let $Z$ be a smooth $G$ manifold and $D$ a closed $G$ invariant submanifold of codimension 1. We say that $D$ separates $Z$ compactly and regularly if there exists an equivariant smooth function $\sigma : Z \to \mathbb{R}$ such that

1. $\sigma^{-1}(0) = D$,
2. $0$ is a regular value of $\sigma$,
3. $\sigma^{-1}([0, \infty])$ is compact.

Next we state two results from [AK1] which will be used in the proofs of our main results of this section. The notation is adjusted to fit into our paper. The first one is an equivariant version of Lemma 2.2 in [AK1]. See also [DMP, Lemma 5.2].
Lemma 7.2. Let $Z$ be a real algebraic $G$ variety, $D$ a closed $G$ invariant submanifold of Nonsing $Z$ of codimension 1 which separates $Z$ compactly and regularly, and $C$ a non-singular real algebraic $G$ variety which is contained in $D$. Then there are arbitrarily small $G$ isotopies of Nonsing $Z$ which fix $C$ and carry $D$ to a non-singular real algebraic $G$ variety.

We note that the proofs of Lemma 2.2 in [AK1] and Lemma 5.2 in [DMP] are false. In the proof of either of these lemmata it is assumed that $D$ separates $Z$ compactly and asserted that there is a function with the properties listed in the definition of compact, regular separation. A counter example to this assertion is as follows. Let $Z = S^1$ and $D$ a point in $Z$. According to the definition in [AK], $D$ separates $Z$ compactly. It is an exercise in calculus to show that there is no smooth function $\sigma : Z \to \mathbb{R}$ with $\sigma^{-1}(0) = D$ which has 0 as regular value. We do not see an easy way to correct the proofs of the lemmata mentioned above. On the other hand, one can make the stronger assumption of regular separation and verify it later in the paper when the lemma is applied. With this stronger assumption the proof of Lemma 2.2 in [AK] and Lemma 5.2 in [DMP] proceed as carried out in these references.

The final result which we need in preparation of the proof of Theorem C is an equivariant version of Proposition 2.8 of [AK1], but we only need the weak version stated within Akbulut and King’s proof. Compare also [DMP, Proposition 5.5]. To formulate it we need the idea of a germ. Let $M$ and $M'$ be submanifolds of $\mathbb{R}^n$, and let $L$ be a submanifold of $M$ and $M'$. The germ of $M$ at $L$ is the family of all tubular neighbourhoods of $L$ in $M$. We say that the germ of $M$ at $L$ is the germ of $M'$ at $L$ if these two germs have a common element, i. e., there exists a subset $N$ of $M \cap M'$ which is a tubular neighbourhood of $L$ in $M$ and in $M'$. We say that $f : M \to W$ and $f' : M' \to W$ have the same germ at $L$ if for some $N$ as above $f |_N = f' |_N$. In the equivariant category we assume that all of the spaces have a $G$ action, and all maps are equivariant. We prove our next proposition later in this section. We assume that $L$ is non-singular so that we need not discuss further technical aspects of germs.

Proposition 7.3. Let $G$ be a compact Lie group, $M$ a closed smooth $G$ submanifold of an orthogonal representation $\Xi$ of $G$, $W$ a non-singular real algebraic $G$ variety realized in an orthogonal representation $\Gamma$ of $G$, and $f : M \to W$ an equivariant smooth map. Suppose $L \subset M$ is a non-singular real algebraic $G$ variety, the germ of $M$ at $L$ is the germ of a non-singular real algebraic $G$ variety $V$ at $L$, and the germ of $f$ at $L$ is the germ of an equivariant entire rational map $v : V \to W$ at $L$.

Then there exist an orthogonal representation $\Omega$ of $G$, an equivariant isotopy $h_t$ of $\Xi \times \Omega$, a real algebraic $G$ variety $Z \subset \Xi \times \Omega$ and an equivariant entire rational map $r : Z \to W$ so that

1. $h_1(M \times 0)$ is the union of non-singular components of $Z$,
2. $h_t(x, 0) = (x, 0)$ for all $x \in L, t \in [0, 1],$
3. $r(x, 0) = f(x)$ for all $x \in L$,
4. $h_t$ can be as small as we want and $rh_1 |_{M \times 0}$ can be as close an approximation of $f$ as we want (for some choice of $Z$ and $r$). Here we identify $M$ with $M \times 0$. 
Proof of Theorem C. The ‘only if’ part of the theorem is trivial. We show the ‘if’ part. Let \( \beta_0 : Q_0 \to W \) be an equivariant smooth map from a closed smooth \( G \) manifold \( Q_0 \) to a non-singular real algebraic \( G \) variety \( W \) for which there is an equivariant cobordism \((\mathcal{M}, F)\) between \((Q_0, \beta_0)\) and \((Q, \beta)\), where \( Q \) is a non-singular real algebraic \( G \) variety and \( \beta \) is an equivariant entire rational map. This means, \( \mathcal{M} \) is a compact smooth \( G \) manifold with \( \partial \mathcal{M} = Q_0 \sqcup Q \) and \( F \) restricts to \( \beta_0 \) on \( Q_0 \) and to \( \beta \) on \( Q \). We like to realize \((Q_0, \beta_0)\) algebraically. First of all, we construct data to which we can apply Proposition 7.3.

Let \( \Xi_0 \) and \( \Gamma \) be orthogonal representations of \( G \) such that \( Q \) is realized as a variety in \( \Xi_0 \), \( W \) is realized as a variety in \( \Gamma \), and \( \beta : Q \to W \) extends to an equivariant rational map, realizing \( \beta \) as an equivariant entire rational map. Possibly choosing \( \Xi_0 \) bigger, we suppose that the embedding of \( Q \) into \( \Xi_0 \) extends to an embedding of \( \mathcal{M} \) into \( \Xi_0 \). We slightly round the corners of \( \partial (\mathcal{M} \times [-1, 1]) \) so that it becomes a smooth \( G \) manifold which we then denote by \( Y \). It is embedd in \( \Xi_0 \times \mathbb{R} \). Composing the projection onto the first factor with \( F \), we obtain an equivariant map from \( \mathcal{M} \times [-1, 1] \) to \( W \), which restricts to a smooth equivariant map \( \eta : Y \to W \). We observe that \( Q \times 0 \subset Y \cap (\Xi_0 \times 0) \), and the germ of \( Y \) at \( Q \times 0 \) is the germ of \( Q \times \mathbb{R} \) at \( Q \times 0 \). Let \( \beta' : Q \times \mathbb{R} \to W \) be defined by \( \beta'(y, t) = \beta(y) \). Then the germ of \( \eta \) at \( Q \times 0 \) is the germ of \( \beta' \) at \( Q \times 0 \).

It is easy to verify the assumptions of Proposition 7.3 if we let

(a) \( M = \{(y, t, z) \in \Xi_0 \times \mathbb{R} \times \Gamma \mid (y, t) \in Y, z = \eta(y, t)\} \) be the graph of \( \eta \) which is contained in \( \Xi_0 \times \mathbb{R} \times \Gamma \),

(b) \( W \) and \( \Gamma \) have the same meaning here as in Proposition 7.3,

(c) \( f : \mathcal{M} \to W \) be the smooth equivariant map defined by \( f(y, t, z) = z \),

(d) \( L = \{(y, 0, \beta(y)) \in \Xi_0 \times \mathbb{R} \times \Gamma \mid y \in Q\} \subset M \) be the graph of \( \eta\{Q \times 0\} \) (as the graph of an equivariant entire rational map between non-singular real algebraic \( G \) varieties \( L \) is a non-singular real algebraic \( G \) variety which is algebraically isomorphic to \( Q \), and \( f|_{L} \) is equivariant and entire rational),

(e) \( V = \{(y, t, \beta(y)) \in \Xi_0 \times \mathbb{R} \times \Gamma \mid y \in Q\} \) be the graph of the equivariant entire rational map \( \beta' \) defined above, hence \( V \) is a non-singular real algebraic \( G \) variety, and

(f) \( v : V \to W \) be the equivariant entire rational map defined by \( v(y, t, z) = \beta(y) \).

Because the germ of \( Y \) at \( Q \times 0 \) is the germ of \( Q \times \mathbb{R} \) at \( Q \times 0 \) and the germ of \( \eta \) at \( Q \times 0 \) is the germ of \( \beta' \) at \( Q \times 0 \), the germ of \( M \) at \( L \) is the germ of \( V \) at \( L \) and the germ of \( f \) at \( L \) is the germ of \( v \) at \( L \). There is an obvious equivariant diffeomorphism \( \alpha_{Q_0} : Q_0 \to Q'_0 \) def \( \{(y, 0, \beta_0(y)) \mid y \in Q_0\} \subset M \) such that \( \beta_0 = f \circ \alpha_{Q_0} \), and an equivariant algebraic isomorphism \( \alpha_{Q} : Q \to L \) such that \( \beta = f \circ \alpha_{Q} \). In this sense, we replaced \((Q_0, \beta_0)\) and \((Q, \beta)\) by isomorphic objects \((Q'_0, f|_{Q'_0})\) and \((L, f|_{L})\).

Proposition 7.3 provides us with an orthogonal representation \( \Omega \) of \( G \), an equivariant isotopy \( h_t \) of \( \Xi \times \Omega \), a real algebraic \( G \) variety \( Z \subset \Xi \times \Omega \) and an equivariant entire rational map \( r : Z \to W \) so that

1. \( Z' = h_1(M \times 0) \) is the union of non-singular components of \( Z \),
2. \( h_t(x, 0) = (x, 0) \) for all \( x \in L, t \in [0, 1] \),
3. \( r(x, 0) = f(x) \) for all \( x \in L \),
4. \( h_t \) can be as small as we want and \( rh_{1\mid M \times 0} \) can be as close an approximation of \( f \)
as we want (for some choice of $Z$ and $r$). Here we identify $M$ with $M \times 0$.

Decompose $M$ as $M_+ \cup M_-$ where

$$M_\pm = \{(y, t, z) \in \mathbb{Z}_0 \times \mathbb{R} \times \Gamma \mid (y, t, z) \in M \text{ and } \pm t \geq 0\}$$

Set $M_\cap = M_+ \cap M_- = Q_0' \cup L$. Projection on the second factor defines a compact regular separation of $M$ where $M_\cap$ plays the role of $D$ in the definition. Accordingly, $Z'$ decomposes into $Z'_\pm = h_1(M_\pm \times 0)$. Composition of the separating function for $M$ with the diffeomorphism $h_1$ defines a compact regular separation of $Z'$ by $Z_\cap := h_1(M_\cap \times 0) = h_1(Q_0' \times 0) \cup L \times 0$. Defining the separating function to be $-1$ on $Z \setminus Z'$ defines a separating function $\sigma : Z \to \mathbb{R}$ such that $\sigma^{-1}([-\infty, 0]) = Z'_- \cup (Z - Z')$, $\sigma^{-1}([0, \infty]) = Z_+$, and $\sigma^{-1}(0) = Z_\cap$. By construction, $h_1(M_\cap \times 0)$ is a $G$ invariant submanifold of Nonsing $Z$ of codimension 1. The assumptions of Lemma 7.2 are satisfied if we set $D = Z_\cap$, $C = L \times 0$, and $Z = Z$. We conclude that there are arbitrarily small equivariant isotopies $H_1$ of Nonsing $Z$ which fix $L \times 0$ and carry $Z_\cap$ to a non-singular real algebraic $G$ variety. So $H_1(Z_\cap)$ is a non-singular real algebraic $G$ variety and $L \times 0$ is a subvariety. Let

$$V_c = H_1(h_1(Q_0' \times 0)) = H_1(Z_\cap) \setminus (L \times 0).$$

The real algebraic $G$ varieties $V_a = H_1(Z_\cap)$ and $V_b = L \times 0$ have the same dimension, they are both non-singular, and $V_b$ is a non-singular subvariety in $V_a$. It follows from Lemma 7.1 that their difference $V_c$ is a non-singular real algebraic $G$ variety.

We show that $r : V_c \to W$ realizes $\beta_0 : Q_0 \to W$ algebraically. The composition $h'_1 := H_t \circ h_t$ provides a small equivariant isotopy between $Q_0' \times 0$ and $V_c$, hence $h'_1 : Q_0' \times 0 \to V_c$ is an equivariant diffeomorphism. By definition

$$Q_0' \times 0 \overset{r h'_1}{\longrightarrow} W$$

$$\downarrow h'_1 \quad \downarrow \text{Id}$$

$$V_c \quad \overset{r}{\longrightarrow} \quad W$$

commutes. So $(V_c, r)$ is an algebraic realization of $(Q_0' \times 0, r h'_1)$. By construction, $H_t$ is a small equivariant isotopy and $r h_1$ is close to $f$ (see (4)). Identifying $Q_0' \times 0$ with $Q_0'$ we see that $r h'_1|_{Q_0' \times 0}$ is equivariantly homotopic to $f|_{Q_0'}$. Hence $(V_c, r)$ is an algebraic realization of $(Q_0', f|_{Q_0'})$. Above we defined an equivariant diffeomorphism $\alpha_{Q_0} : Q_0 \to Q_0'$ such that $\beta_0 = f \circ \alpha_{Q_0}$. This implies that $(V_c, r)$ is an algebraic realization of $(Q_0, \beta_0)$, and this is what we wanted to prove. □

As we have just seen, the following more technical statement of Theorem C holds. We remind the reader, that ‘close’ is to be understood in the $C^1$ topology.

**Addendum to Theorem C.** Let $\beta_0 : Q_0 \to W$ be an equivariant map from a closed smooth $G$ manifold to a non-singular real algebraic $G$ variety whose bordism class is algebraically represented. Then there exist non-singular real algebraic $G$ varieties $V_c$, equivariant entire rational maps $r : V_c \to W$, and equivariant diffeomorphisms $\vartheta : Q_0 \to V_c$ such that $r \circ \vartheta$ is arbitrarily close to $\beta_0$. 

Proof. To see this addendum we recall the major steps of the proof of Theorem C. We found an equivariant diffeomorphism \( \alpha_{Q_0} \) which identifies \( Q_0 \) with a smooth \( G \) submanifold \( Q'_0 \) in a (sufficiently large) representation \( \Omega \) of \( G \). We defined \( f : Q'_0 \to W \) such that \( f \circ \alpha_{Q_0} = \beta_0 \). Then we found arbitrarily small equivariant isotopies \( H'_i \) of \( \Omega \) (these extend \( h'_i \) from the proof and exist by the equivariant isotopy extension theorem) such that \( V_i = H'_1(Q'_0) \) is a non-singular real algebraic \( G \) variety in \( \Omega \), so \( h'_i \) defines an equivariant diffeomorphism from \( Q'_0 \) to \( V_i \). Furthermore, we have equivariant entire rational maps \( r : V_c \to W \) such that \( r \circ h'_i \) is arbitrarily close to \( f \). The smaller \( H'_i \) is chosen, the closer \( V_i \) will be to \( Q'_0 \) and \( r \circ h'_i \) to \( f \). This defines the equivariant diffeomorphisms \( \vartheta = h'_1 \circ \alpha_{Q_0} : Q_0 \to V_c \) and the entire rational equivariant maps \( r : V_c \to W \) for which \( r \circ \vartheta \) can be arbitrarily close to \( \beta_0 \). \( \square \)

It remains to prove Proposition 7.3. First we recall one more result from the literature. It is a special case of Lemma 1.4 in [AK1]. The notation is adjusted to fit into our paper.

**Lemma 7.4.** Let \( U \), \( W_0 \) and \( W_1 \) be real algebraic varieties, and \( W_1 \subset W_0 \). Suppose \( w : U \to W_0 \) is an entire rational map. Assume \( z \in w^{-1}(W_1) \) is non-singular of dimension \( r \) in \( U \), \( w(z) \) is non-singular of dimension \( s \) in \( W_1 \), and non-singular of dimension \( t \) in \( W_0 \), and \( w \) is transversal to \( W_1 \) at \( z \). Then \( z \) is non-singular of dimension \( r + s - t \) in \( w^{-1}(W_1) \).

Our next result is the equivariant generalization of a special case of Lemma 2.1 in [AK1].

**Lemma 7.5.** Let \( G \) be a compact Lie group, \( \Xi \) and \( \Omega_0 \) orthogonal representations of \( G \), \( L \) a non-singular real algebraic \( G \) variety contained in \( \Xi \), and \( T \) a compact subset of \( \Xi \) such that \( L \subset \text{Int}T \). Let \( \delta : \Xi \to \Omega_0 \) be a smooth equivariant map, \( V \) a non-singular real algebraic \( G \) variety, and \( N \subset T \) a \( G \) invariant tubular neighbourhood of \( L \) in \( V \). Suppose \( \delta_{|N} = 0 \). Then there are equivariant polynomials \( s : (\Xi, L) \to (\Omega_0, 0) \) which approximate \( \delta \) arbitrarily closely near \( T \).

**Proof.** Forgetting the group action and setting \( \Omega_0 = \mathbb{R} \), our lemma is a special case of Lemma 2.1 of [AK1]. Suppose the dimension of \( \Omega_0 \) is \( m \). Write \( \delta = (\delta_1, \ldots, \delta_m) \) in terms of its coordinates. The result of Akbulut and King provides polynomials \( s_i, i = 1, \ldots, m \), such that the \( s_i \) approximate the \( \delta_i \) arbitrarily closely near \( T \). Hence the \( \tilde{s} = (s_1, \ldots, s_m) \) approximate \( \delta \) arbitrarily closely near \( T \). The averages \( s = A(\tilde{s}) \) are equivariant polynomials (see Lemma 2.4) which approximate \( \delta \) arbitrarily closely near \( T \) (see Lemma 6.1). As each \( \delta_i \) is assumed to vanish on \( L \), it follows from the result of Akbulut and King that each \( s_i \) vanishes on \( L \), and so does \( \tilde{s} \). It follows from Lemma 2.4 (1) that \( s_{|L} = \tilde{s}_{|L} \), so \( s \) vanishes on \( L \). \( \square \)

Our next lemma is related to Lemma 2.4 in [AK1] and to Lemma 5.4 in [DMP]. The notation is chosen such that it can be applied easily in the following proposition. The statement from [AK1] is not only made equivariant, but also slightly modified to fit better in the intended application. In addition, it incorporates a remark which Akbulut and King make after their proof.

**Lemma 7.6.** Let \( G \) be a compact Lie group, \( L, V, W_0 \) be non-singular real algebraic \( G \) varieties with \( L \subset V \subset \Xi \) and \( W_0 \subset \Omega_0 \), where \( \Xi \) and \( \Omega_0 \) are orthogonal representations of
G. Suppose $T$ is a $G$ invariant compact submanifold of $\Xi$ of codimension zero such that $L \subset \text{Int } T$, $N \subset T$ is a $G$ invariant tubular neighbourhood of $L$ in $V$, $f_0 : T \to W_0$ is equivariant and smooth, and there is an equivariant entire rational map $u_0 : (\Xi, L) \to (\Omega_0, W_0)$ such that $f_{0|N} = u_{0|N}$.

Then there are an algebraic $G$ variety $U \subset \Xi \times \Omega_0$, an equivariant entire rational map $w : U \to W_0$ and a smooth $G$ map $\phi : (T, L) \to (\Omega_0, 0)$, which may be chosen arbitrarily small, such that

1. $J \overset{\text{Def}}{=} \{(x, \phi(x)) \mid x \in \text{Int } T\}$ is an open $G$ invariant subset of $\text{Nonsing } U$,
2. The $G$ map $\mu : T \to W_0$ defined by $\mu(x) = w(x, \phi(x))$ can be as close to $f_0$ as we wish (for some choice of $U$, $w$ and $\phi$),
3. $\mu(x) = w(x, 0) = f_0(x)$ for $x \in L$.

**Proof.** We define the data called for in the conclusion of the lemma. Let $k$ be the codimension of $W_0$ in $\Omega_0$ and $\chi : W_0 \to G_\mathbb{R}(\Omega_0, k)$ the classifying map of the normal bundle of $W_0$ in $\Omega_0$. So $\chi$ is equivariant and entire rational (see 2.15) and can be written as a quotient $\chi = P/Q$ where $P : \Omega_0 \to \text{End}_\mathbb{R}(\Omega_0)$ (for the definition of the space of endomorphisms see Section 2c), $Q : \Omega_0 \to \mathbb{R}$ and $Q$ does not vanish anywhere on $\Omega_0$ (see 2.5). Extend $f_0$ to an equivariant smooth map $\Xi \to \Omega_0$ and denote it again by $f_0$. We apply Lemma 7.5 setting $\delta = f_0 - u_0$, and with otherwise matching notation. We find equivariant polynomial maps $s : (\Xi, L) \to (\Omega_0, 0)$ which approximate $f_0 - u_0$ arbitrarily closely near $T$. We define:

1. $v = s + u_0 : (\Xi, L) \to (\Omega_0, W_0)$
2. $w(x, y) = v(x) + y$ \hspace{1em} $(x \in \Xi, y \in \Omega_0)$
3. $\phi(x) = \text{ the vector from } v(x) \text{ to the point closest to } v(x) \text{ on } W_0$ \hspace{1em} $(x \in T)$.

The better $s$ approximates $f_0 - u_0$, the smaller $\phi$ will be. This verifies the assertion that $\phi$ may be chosen arbitrarily small. Furthermore, let

$$U_0 = \{(x, y) \in \Xi \times \Omega_0 \mid w(x, y) \in W_0, \ P(w(x, y))(y) = Q(w(x, y)) \cdot y\}$$

Let the dimension of $\Xi$ be $n$. Below we show

(*) If $x \in \text{Int } T$, then $(x, \phi(x))$ is non-singular of dimension $n$ in $U_0$.

Taking the singular part of $U_0$ several times we construct a variety

$$U = \text{Sing}(\text{Sing}(\ldots (\text{Sing } U_0)\ldots))$$

so that $\dim U = n$. That this is possible follows from (*). The function $v : \Xi \to \Omega_0$ in (i) is an equivariant entire rational map which approximates $f_0$ on $T$. The sets $U_0$ and $J$ are $G$ invariant, and $w$ and $\phi : (T, L) \to (\Omega_0, 0)$ are equivariant because all of the data involved is equivariant. To make sure that $\phi$ is well defined we need to assume that there is a unique closest point to $v(x)$ in $W_0$ for all $x \in T$. This is assured if $v(x)$ is close to $W_0$, or in other words, $\phi$ is small. As $f_0(x)$ is in $W_0$, we just need to make sure that $s$ is a
close approximation of \( f_0 - u_0 \) near \( T \), then \( v(x) \) is close to \( W_0 \). Because \( v \) is smooth, \( \phi \) is smooth as well. As we required that \( w(x, y) \in W_0 \) for \((x, y)\) to be in \( U_0 \), it follows that \( w \) maps \( U \) to \( W_0 \) and that \( J \) is a subset of \( U_0 \). Now we have all of the data to work with.

It remains to check \((1)-(3)\). We start with \((2)\). As \( v \) is close to \( f_0 \) and \( \phi \) is small, \( \mu \) is close to \( f_0 \). Next, observe if \( x \in L \), then \( s(x) = 0 \) and \( f_0(x) = u_0(x) \in W_0 \), so \( \phi(x) = 0 \). This implies \((3)\). We assume \((*)\) and show \((1)\). As pointed out above, \( J \subset \text{Nonsing} U \). Observe that \( J \subset \text{Nonsing} U \), \( \dim J = \dim U \), and \( J \) is the image of the open set \( \text{Int} T \) under the embedding \( x \mapsto (x, \phi(x)) \). This implies that \( J \) is an open \( G \) invariant subset of \( \text{Nonsing} U \) as claimed in \((1)\).

It remains to verify \((*)\). We derive it as a consequence of Lemma 7.4. Consider the map

\[
\psi : \Xi \times \Omega_0 \to \Omega_0 \times \Omega_0 \text{ defined by } \psi(x, y) = (v(x) + y, y) = (w(x, y), y)
\]

and in \( \Omega_0 \times \Omega_0 \) consider the submanifold

\[
Z = \{(x, y) \in W_0 \times \Omega_0 \mid P(z)(y) = Q(z)y\}
\]

We show that \( \psi \) is transversal at \((x, \phi(x))\) if \( x \in \text{Int} T \). A different way of describing \( Z \) is to say that \((x, y) \in W_0 \times \Omega_0 \) is in \( Z \) if \( \chi(z) \in G_\mathbb{R}(\Omega_0, k) \) and \( y \) is in the normal fibre of \( W_0 \) at \( z \). Write

\[
\psi(x, y) = ((v(x) + \phi(x)) + (y - \phi(x)), y) = (\mu(x) + (y - \phi(x)), y)
\]

The fibre of the normal bundle of \( Z \) at \( \psi(x, \phi(x)) \) consists of vectors of the form \((\mu(x) + a, b)\) where \( a \) is an arbitrary vector normal to \( W_0 \) at \( \psi(x, \phi(x)) \), and \( b \) is an arbitrary vector tangent to \( W_0 \) at \( \psi(x, \phi(x)) \). Note also that \((y - \phi(x))\) is normal to \( W_0 \) at \((x, \phi(x))\). Varying \( y \) over elements of \( \Omega \), all the required values for \( a \) and \( b \) are assumed. This shows that \( \psi \) is transversal at \((x, \phi(x))\) if \( x \in \text{Int} T \).

Let \( m \) be the dimension of \( \Omega_0 \). Then

- \((\alpha)\) \((x, \phi(x))\) is non-singular of dimension equal to \( n + m \) in \( \Xi \times \Omega_0 \).
- \((\beta)\) \((x, \phi(x))\) is non-singular of dimension equal to \( m + \dim W_0 \) in \( \Omega_0 \times \Omega_0 \), and non-singular of dimension equal to \( 2m \) in \( W_0 \times \Omega_0 \).

Now we apply Lemma 7.4. Observe that \( \psi^{-1}(Z) = U_0 \). Transversality of \( \psi \) at \((x, \phi(x))\) and the dimension of non-singularity statements in \((\alpha)\) and \((\beta)\) imply that \((x, \phi(x))\) is non-singular of dimension equal to \( n \) in \( U_0 \). By construction, \( U \) contains the points of \( U_0 \) which are non-singular of dimension equal to \( n \), and they are nonsingular in \( U \). This concludes the verification of \((1)\) and the proof of the proposition. \( \square \)

**Proof of Proposition 7.3.** We construct certain data whose properties are listed in \((a)-(d)\) and \((A)-(C)\), which will then be used in the proof of the proposition. As in Section 2c, let \( G_\mathbb{R}(\Xi, k) \) be the real Grassmannian, \( E_\mathbb{R}(\Xi, k) \) the total space of the universal bundle \( \gamma_\mathbb{R}(\Xi, k) \), and \( \text{End}_\mathbb{R}(\Xi) \) the vector space of endomorphisms of \( \Xi \) with the given orthogonal action of \( G \). We use the embeddings

\[
G_\mathbb{R}(\Xi, k) \subset E_\mathbb{R}(\Xi, k) \subset G_\mathbb{R}(\Xi, k) \times \Xi \subset \text{End}_\mathbb{R}(\Xi) \times \Xi.
\]
As a first step we choose a compact $G$ invariant tubular neighbourhood $T$ of $M$ in $\Xi$, and a $G$ invariant tubular neighborhood $N$ of $L$ in $M$ and in $V$. The latter choice is possible because the germ of $M$ at $L$ is the germ of $V$ at $L$. Furthermore we make this choice of $N$ such that $f$ and $v$ agree on $N$, which is again possible by the assumption that the germ of $f$ at $L$ is the germ of $v$ at $L$. We define $u : V \to G_R(\Xi, k)$ as the classifying map of the normal bundle of $V$ in $\Xi$. Here $k$ denotes the codimension of $V$ in $\Xi$. This is an equivariant entire rational map, see Proposition 2.15. Then we construct a smooth $G$ map $\gamma : T \to E_R(\Xi, k)$ such that

(a) $\gamma$ is transverse to $G_R(\Xi, k)$,
(b) $\gamma^{-1}(G_R(\Xi, k)) = M$,
(c) $\gamma|_N = u|_N$.

Above we also made sure that

(d) $f|_N = v|_N$.

The construction of $\gamma$ is as follows. To $x \in T$ we assign the point $y \in M$ which is closest to $x$. We may choose $T$ such that this map is well defined. Let $A \in G_R(\Xi, k)$ be orthogonal projection onto the normal plane of $M$ in $\Xi$ at $y$. Then we set

$$\gamma(x) = (A, A(y - x)) \in E_R(\Xi, k).$$

Statements (a) and (b) are obvious from the construction of $\gamma$. We show (c). For a point $x \in M$, $\gamma(x) = (\chi_\nu(x), 0)$, where $\chi_\nu$ is the classifying map of the normal bundle of $M$ in $\Xi$. As $N$ is also a tubular neighbourhood of $L$ in $V$, we have $\chi_\nu(x) = u(x)$ for all $x \in N$. We identify $G_R(\Xi, k)$ with the zero section of $\gamma_R(\Xi, k)$, and in this sense $\gamma(x) = u(x)$ for all $x \in N$.

Abusing language slightly, we denote by $f : T \to W$ the extension of $f$, obtained as the composition of the projection map $T \to M$ and $f : M \to W$. We now apply the previous lemma. Manifolds, varieties, representations, and maps with the same name play the same role here and in the setup and conclusion of Lemma 7.6. In addition we have the correspondence

<table>
<thead>
<tr>
<th>In 7.6</th>
<th>In 7.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_0$</td>
<td>$W \times E_R(\Xi, k)$</td>
</tr>
<tr>
<td>$f_0$</td>
<td>$f \times \gamma$</td>
</tr>
<tr>
<td>$u_0$</td>
<td>$v \times u$</td>
</tr>
<tr>
<td>$\Omega_0$</td>
<td>$\Omega \overset{\text{Def}}{=} \Gamma \times \text{End}_R(\Xi) \times \Xi.$</td>
</tr>
</tbody>
</table>

Then by Lemma 7.6 there are an algebraic $G$ variety $U \subset \Xi \times \Omega$ and a smooth $G$ map $\phi : (T, L) \to (\Omega, 0)$ and an equivariant entire rational map $w : U \to W \times E_R(\Xi, k)$ so that

(A) $J = \{(x, \phi(x)) \in \Xi \times \Omega \mid x \in \text{Int} T\}$ is an open $G$ invariant subset of Nonsing $U$,

(B) The $G$ map $\mu : T \to W \times E_R(\Xi, k)$ defined by $\mu(x) = w(x, \phi(x))$ is as close as we like to $f \times \gamma$ (for some choice of $U$, $w$, and $\phi$),

(C) $w(x, 0) = (v(x), u(x)) = (f(x), \gamma(x))$ for all $x \in L$, in other words $\mu|_L = (f \times \gamma)|_L$. 


We are now ready to prove the proposition. Consider the smooth $G$ manifolds

(i) \[ M \times 0 = \{(x, 0) \in T \times \Omega \mid (f(x), \gamma(x)) \in W \times G_R(\Xi, k)\} \]

(ii) \[ D_1 \overset{\text{Def}}{=} \{(x, 0) \in T \times \Omega \mid \mu(x) \in W \times G_R(\Xi, k)\} \]

(iii) \[ D_2 \overset{\text{Def}}{=} \{(x, \phi(x)) \in T \times \Omega \mid \mu(x) \in W \times G_R(\Xi, k)\} \]

(iv) \[ D_3 \overset{\text{Def}}{=} \{(x, \phi(x)) \in \text{Int } T \times \Omega \mid \mu(x) \in W \times G_R(\Xi, k)\} \]

(v) \[ Z' \overset{\text{Def}}{=} \{(x, \phi(x)) \in J \mid \mu(x) \in W \times G_R(\Xi, k)\} \]

We construct an equivariant isotopy between (the embeddings of $M$ into $T \times \Omega$ as) $M \times 0$ and $Z'$ which can be as small as we like. The equation in (i) follows because $x \in M$ if and only if $\gamma(x) \in G_R(\Xi, k)$ (see (b)), and because $f(x) \in W$ imposes no restriction, it holds for all $x \in T$.

We construct a small equivariant isotopy between $M \times 0$ and $D_1$. First, consider the homotopy

\[ H_t = (1 - t)(f \times \gamma) + t \mu : T \to \Omega \quad \text{where } t \in [0, 1]. \]

Because this homotopy is as small as we like (see (B)), its image is contained in a tubular neighbourhood of $W \times E_R(\Xi, k)$. Compose $H_t$ with the projection of the tubular neighbourhood to $W \times E_R(\Xi, k)$. This provides an equivariant homotopy $H : T \times [0, 1] \to W \times E_R(\Xi, k)$ between $H_0 = f \times \gamma$ and $H_1 = \mu$. This homotopy can also be assumed to be as small as we wish. Because $\gamma$ is transversal to $G_R(\Xi, k)$ (see (a)), hence $f \times \gamma$ transversal to $W \times G_R(\Xi, k)$, it follows that each $H_t$ may be assumed to be transversal to $W \times G_R(\Xi, k)$ because $T$ is compact and transversality is an open condition. The transversality argument from the previous section implies that the equivariant homotopy $H$ provides a small equivariant isotopy $H_t$ between $M \times 0$ and $D_1$. Condition (C) implies that this isotopy may be chosen such that it fixes $L$.

Because $\phi$ is small, see Lemma 7.6, there is an apparent small equivariant isotopy between $D_1$ and $D_2$. The isotopy fixes $L$ because $\phi|_L = 0$. Remember that $(f \times \gamma)(x) \in W \times G_R(\Xi, k)$ only if $x \in \text{Int } T$. So, if $\mu$ is sufficiently close to $f \times \gamma$, then $\mu(x) \in W \times G_R(\Xi, k)$ only if $x \in \text{Int } T$. Assuming this, $D_2 = D_3$. The definition of $J$ implies that $D_3 = Z'$. Taken together, the above provides the arbitrarily small equivariant isotopy $h_t$ between $M \times 0$ and $Z'$. Specifically, $h_1(x, 0) = (x, \phi(x))$ for $x \in M$. This is the isotopy called for in the claim of the proposition. It leaves $L$ fixed as required in (2) because $\phi|_L = 0$.

We show that $Z'$ is the union of non-singular components of a real algebraic $G$ variety $Z$. We constructed an equivariant entire rational map $w : U \to W \times E_R(\Xi, k)$. Let

\[ Z'' = U \cap w^{-1}(W \times G_R(\Xi, k)). \]

Note that $Z' \subset Z''$. We show that $z \in Z'$ is non-singular of dimension equal to $\dim M$ in $Z''$. Observe that

(α) $z$ is non-singular of dimension equal to $r = \dim \Xi$ in $U$ (see 7.6 (1) and use (A)).
(β) $w(z)$ is non-singular of dimension equal to $s = \dim W + \dim G_R(\Xi, k)$ in $W \times G_R(\Xi, k)$ and non-singular of dimension equal to $t = \dim W + \dim E_R(\Xi, k)$ in $W \times E_R(\Xi, k)$ because both of the varieties are non-singular.

(γ) $w$ is transversal to $W \times G_R(\Xi, k)$ at $z$.

By Lemma 7.4, $z$ is non-singular of dimension $r + s - t = \dim \Xi - k = \dim M$ in $Z''$. Taking the singular part of $Z''$ several times we construct a variety

$$Z = \text{Sing}(\text{Sing}(\ldots(\text{Sing} Z'') \ldots))$$

so that $\dim Z = \dim M$. As the points $z \in Z'$ are non-singular of dimension equal to $\dim M$ in $Z''$, it follows that $Z' \subset Z$, and $Z' = rh_1(M \times 0)$ is a union of components of $Z$. So $Z$ is the variety called for in the proposition and (1) holds.

We define $r = p_1w : M \times 0 \to W$ where $p_1 : W \times G_R(\Xi, k) \to W$ is the projection on the first factor. If $x \in L$, then $r(x) = f(x)$ by (C) as required in (3). Identify $M$ with $M \times 0$, and abusing language, define $f : M \times 0 \to W$ by $f(x, 0) = f(x)$. Then, restricted to $M \times 0$, we have $rh_1(x, 0) = r(x, \phi(x)) = p_1w(x, \phi(x)) = p_1\mu(x)$ and $f(x) = p_1(f(x), \gamma(x))$. As $\mu$ is close to $f \times \gamma$ (see (B)) we conclude that $rh_1|_{M \times 0}$ is close to $f$. This concludes the proof of the proposition. □

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