REAL ALGEBRAIC TRANSFORMATION GROUPS

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1. Introduction

The starting point of this paper is the Fixed Point Conjecture. Its formulation, what we know about it at this time, and its relation to problems in (complex) algebraic transformation groups is our first topic. Our work led us naturally into the field which is best described by the expression "*real algebraic transformation groups*". Here we use non-equivariant real algebraic geometry and smooth transformation groups as the two principal areas from which to draw inspiration. So far, the problems we formulate are mostly motivated by theorems in non-equivariant real algebraic geometry, and the methods for solving them are mostly based on techniques in smooth transformation groups. This is the second topic of this paper. We try to provide sufficient background in the real algebraic category even for the less experienced reader by giving the basic definitions and the principal results pertaining to this paper. In Section 2 we define many of the

The authors wish to thank their colleague Mikiya Masuda. Much of the material of this survey is taken from joint papers with him, some of them are only available in preprint form, some are not even written up. In fact, many ideas of this paper were developed in [DM2], much is copied out of that paper, and many ideas for future work are based on this preprint.

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concepts used in this introduction and later in the paper. We assume that the reader is somewhat familiar with the smooth category. For background in smooth transformation groups we refer the reader to Bredon's book [Br], and for more specialized results we provide references. We also formulate questions which we are working on right now, or we think are worth tackling in the future. In this introduction we are as non-technical as possible and rely on the reader's intuition. We develop a broad outline of the theory and in this respect the section is much more detailed than an introduction more commonly. In the following sections we will go into more depth, provide proofs of some of the results stated in this section, and discuss related problems.

The objects considered in real algebraic geometry are real algebraic varieties. We define this concept in the equivariant setting. Throughout this paper G denotes a compact Lie group. Let Ω denote an orthogonal representation of G. Here we think of an orthogonal representation as an underlying Euclidean space $\mathbb{R}^n$ with an action of G via orthogonal maps.

Definition 1.1. *A real algebraic G variety is the set of common zeros of polynomials $p_1, \ldots, p_m : \Omega \to \mathbb{R}$,*

$$V = \{x \in \Omega \mid p_1(x) = \cdots = p_m(x) = 0\},$$

which is invariant under the action of G. We also say that G acts real algebraically on V.

We recall in Section 2 what it means that a variety is non-singular. This concept is important to us when we compare smooth manifolds and real algebraic varieties because every non-singular variety is in a natural way a smooth manifold, though the converse is not true. E. g., in [M2, p. 12, Example C] Milnor describes varieties which are smooth manifolds, but they are singular. An elementary example of a real algebraic G variety is the unit sphere

$$S(\Xi) = \{x \in \Xi \mid x_1^2 + \cdots + x_n^2 - 1 = 0\}$$

in an orthogonal representation Ξ of G with underlying space $\mathbb{R}^n$. More examples will be mentioned later.

More exactly, the varieties which we just defined should be called affine varieties, but as we are not considering any other varieties, such as projective ones, we omit this adjective throughout. Definition 1.1 is made up from the view point of smooth transformation groups. A more algebraically minded reader may prefer a different definition. We give such a definition in Section 7 and show in which sense it is equivalent to the one 1.1.

Fixed Point Conjecture (see [DMP, p. 50]). *A compact Lie group acts real algebraically and without a fixed point on a variety diffeomorphic to Euclidean space if and only if it acts smoothly and without a fixed point on a disk.*

The word *disk* is used for the points of norm ≤ 1 in some Euclidean space. The necessity part ($\Longrightarrow$) has been shown by Petrie and Randall [PR1], [P2]. In fact, this was the topic of Petrie's lectures at the first KIT workshop in 1986. The basic idea is as follows. Given a real algebraic variety (at least it needs to be smooth) with an algebraic action of a compact Lie group G, there exists an equivariant polynomial function $f : X \to \mathbb{R}$ which looks very much like a Morse function. Because it is a polynomial it has only finitely many critical values. For a sufficiently large N, $X_N := f^{-1}(-\infty, N]$ is a compact smooth G manifold, and $X = X_N \cup_{\partial X_N} (\partial X_N \times [0, \infty))$. Assuming X is contractible it follows that X_N is diffeomorphic to a disk. Given that the action on X has no fixed point it follows that the action on X_N has no fixed point. Petrie and Randall's work is in the construction of f and the verification that f has basically all the properties of a Morse function.

In the context of this paper we are more concerned with the sufficiency part of the Fixed Point Conjecture. First of all, it is important to know which groups can act smoothly on a disk without a fixed point. Oliver showed

Theorem 1.2 (see [O]). *A finite group G acts smoothly and without* *a fixed point on a disk if and only if there does not exist a normal series* $P \lhd H \lhd G$ *such that* P *and* G/H *are of prime power order and* H/P *is cyclic. (In particular, a finite abelian group acts without a fixed point on a disk if and only if it has at least three non-cyclic Sylow subgroups.)*

Later in this paper we are concerned with the more elementary necessity ($\Longrightarrow$) part of this theorem. Suppose that G acts on the disk D without a fixed point and that G has a normal series as described in the theorem. We will derive a contradiction. One may assume that the action is simplicial, i. e., the disk has a simplicial decomposition, and any element in the group either leaves any simplex pointwise fixed, or it maps it to a different simplex. It follows from Smith theory (e. g., see [Br, Chapter III]) that the P fixed point set $D^P = \{x \in D \mid gx = x \text{ for all } g \in P\}$ is a homology disk with coefficients in $\mathbb{Z}_p$ if P is of p-power order, i. e., the reduced homology groups $\widetilde{H}_*(D^P, \mathbb{Z}_p)$ vanish. In particular, the Euler characteristic of D^P is one. The Lefschetz fixed point theorem implies that the Euler characteristic of $D^H = (D^P)^{H/P}$ is also one [Sp]. Here one uses that P/H is cyclic. A simple argument shows that $\chi(D^H) \equiv \chi(D^G)$ (mod q) where χ denotes the Euler characteristic and G/H is assumed to be of q-power order. Here one uses that $(D^H)^{G/H} = D^G$. In particular, $\chi(D^G) \neq 0$ and $D^G \neq \emptyset$. $\square$

If we want to prove the Fixed Point Conjecture for a given group, or a class of groups, then we have to construct fixed point free actions on real algebraic varieties diffeomorphic to some Euclidean space. We state three results. We discuss the first two of them in some detail later.

Theorem 1.3 (see [DMP], [DM1]). *The alternating group* A_5 *acts real algebraically and without a fixed point on a variety diffeomorphic to a Euclidean space.*

Theorem 1.4 (see [DKS], [DM2]). *Abelian groups of odd order with at least three non-cyclic Sylow subgroups act real algebraically and* *without a fixed point on varieties diffeomorphic to a Euclidean space.*

Proposition 1.5 (see [DMP]). *Let $\mathcal{G}$ denote the set of all compact Lie groups which act real algebraically, effectively, and without fixed point on a variety diffeomorphic to Euclidean space, and let G be a compact Lie group.*

(1) *If $H \subset G$ is a subgroup of finite index and $H \in \mathcal{G}$, then $G \in \mathcal{G}$.*
(2) *If G surjects onto K and $K \in \mathcal{G}$, then $G \in \mathcal{G}$.*

Let us draw some conclusions for the Fixed Point Conjecture. It follows from Oliver's work (see Theorem 1.2) that the symmetric group S_k and the alternating group A_k act smoothly on a disk without a fixed point if and only if $k \geq 5$. Earlier, Floyd and Richardson (see [FR] or [Br, p. 58]) had shown that A_5 acts smoothly on a disk without a fixed point, and it follows from an elementary construction that A_k and S_k do as well if $k \geq 5$. Based on this argument and (1.3)–(1.5) we conclude

Corollary 1.6. *The Fixed Point Conjecture holds for symmetric groups, alternating groups, and odd order abelian groups.*

As a trivial extension of the above we could write down infinitely many more groups for which the Fixed Point Conjecture holds. A more interesting extension would be the proof of the Fixed Point Conjecture for odd order groups. We propose two ways to do this (see the end of this section and Section 3). Obviously we are interested in a proof of the Fixed Point Conjecture in general, but at this time we do not have a promising strategy.

Let us explain the approach used to prove Theorem 1.3. We need a few definitions and results to set this up. Two closed smooth G manifolds M_1 and M_2 are said to be *equivariantly cobordant* if there exists a compact smooth G manifold whose boundary is the disjoint union of M_1 and M_2.

A result which was obtained in the non-equivariant setting by Tognoli [T], and which was his break through in the solution of the Nash

Conjecture (to be discussed below) is of great importance throughout this paper. It is a convenient abbreviation to say that a smooth G manifold is *algebraically realized* if it is equivariantly diffeomorphic to a non-singular real algebraic G variety.

Theorem 1.7 (see [DMP, Theorem 1.3]). *Let G be a compact Lie group. A closed smooth G manifold is algebraically realized if it is equivariantly cobordant to a non-singular real algebraic G variety. In particular, the boundary of a compact smooth G manifold is algebraically realized.*

Identify A_5 with the icosahedral group I. Its elements are the rotations of the icosahedron, and as such we consider it as a subgroup of the special orthogonal group $SO(3)$. Left translation defines a smooth orientation preserving action of I on the left coset space

$$\Sigma = SO(3)/I = \{gI \mid g \in SO(3)\}.$$

The manifold Σ is the well-known Poincaré homology sphere.

Lemma 1.8. *With the just described action of I*

(1) *Σ is algebraically realized.*
(2) *The action on Σ has exactly one fixed point.*
(3) *The $2k$-fold cartesian product $\Sigma \times \cdots \times \Sigma$ is equivariantly cobordant to a homotopy sphere S for $k = 1$ and for k even and ≥ 4, and the cobordism may be chosen to be relative to the fixed point set. In particular, the actions on $\Sigma \times \cdots \times \Sigma$ and S have exactly one fixed point.*

The first claim follows from arguments given in a paper by Schwarz [Sw1] as Σ is the quotient of a compact Lie group by a compact subgroup, see [DMP, Proposition 2.2]. The second claim follows because $(SO(3)/I)^I = N_{SO(3)}I/I = \{1\}$. The third part of the lemma is quite substantial. It is an application of equivariant surgery theory. The case when $k \geq 4$ is proved in [DMP]. The case when $k = 1$ is more explicit and requires less background. Its proof is given in [DM1].

Lemma 1.9 (compare [M2, p. 105] or [DMP, p. 50]). *Let V be a non-singular real algebraic G variety and $x \in V$ a fixed point. Then $V \setminus x$ is equivariantly diffeomorphic to a real algebraic G variety. If V is a homotopy sphere and the action has exactly one fixed point, then $V \setminus x$ is diffeomorphic to a Euclidean space on which the action has no fixed point.*

Proof. Let Ξ denote the representation of G in which V is the zero set. There exists a polynomial $p : \Xi \to \mathbb{R}$ such that $p^{-1}(0) = V$ (see 2.4). Find an invariant polynomial q such that $q^{-1}(0) = x$. Then

$$V \setminus x = \{y \in \Xi \mid p(y) = 0 \text{ and } q(y) \neq 0\}.$$

The assignment which maps y to $(y, 1/q(y))$ defines an equivariant diffeomorphism

$$V \setminus x \to A = \{(y,t) \in \Xi \oplus \mathbb{R} \mid p(y) = 0 \text{ and } tq(y) - 1 = 0\}$$

Obviously, A is a real algebraic variety, and the action is real algebraic. If we remove a point from a homotopy sphere (only dimensions ≥ 6 matter here) then the resulting space is diffeomorphic to Euclidean space. □

Proof of Theorem 1.3. Let $k = 1$ or k even and ≥ 4. The $2k$-fold cartesian product $X = \Sigma \times \cdots \times \Sigma$ is a non-singular real algebraic I variety (see 2.6) on which I acts with exactly one fixed point (see 1.8 (2)). It follows from 1.8 (3) that X is equivariantly cobordant to a homotopy sphere S on which I acts smoothly with exactly one fixed point. Theorem 1.7 implies that we may consider S as a non-singular real algebraic I variety. Applying Lemma 1.9 to S we obtain a fixed point free real algebraic action on a variety diffeomorphic to Euclidean space. Its existence was our claim. □

At this point we leave the discussion of the Fixed Point Conjecture. Further aspects are discussed in Section 3 and at the end of this section.

We now turn to the second aspect of this survey. It is concerned with one of the important aspects of real algebraic geometry, the question whether a given smooth situation can be realized algebraically. We begin our discussion with the non-equivariant background material. Initial results of Seifert [Se] and Nash [N] led to what is now called the

Nash Conjecture (see [N]). *Every closed smooth manifold (i. e., a compact manifold without boundary) is diffeomorphic to a nonsingular real algebraic variety.*

Tognoli proved this conjecture in 1973 [T]. His break through was the proof of the non-equivariant version of Theorem 1.7. The only other fact needed was a previously known result of Milnor [M1]. It shows that every closed smooth manifold is cobordant to a nonsingular real algebraic variety. Subsequent work of Akbulut and King [AK1], [AK2] did not only provide a more conceptual proof of the Nash Conjecture, but it also opened the way to study the corresponding problem for manifolds which carry an additional structure. A very natural question of this kind is whether a vector bundle over a closed smooth manifold can be realized algebraically. The exact formulation and solution of this problem are due to Benedetti and Tognoli [BT]. (We formulate and address this problem in the equivariant setting below.) Survey articles by Ivanov [I] and King [K] and a book by Bochnak, Coste, and Roy [BCR] describe many of the results and questions in this program. It is important to observe that there are non-compact manifolds which are not diffeomorphic to real algebraic varieties (as an elementary example one may use $\mathbb{R}\setminus\mathbb{Z}$, as more interesting ones on may use the manifolds constructed by Siebenmann in his thesis [Si]), and that the algebraic structure on a closed smooth manifold is not unique. On any manifold of dimension ≥ 1 there are uncountably many such structures [BK1], [BK2].

In real algebraic transformation groups we impose as additional structure an action of a compact Lie group G. This may be viewed as a special case of the program described so far, as well as a generalization of the program in its entirety. Substantial new problems arise. Earlier in this introduction we said that a smooth G manifold is *algebraically realized* if it is equivariantly diffeomorphic to a non-singular real algebraic G variety. The focus of our work on this subject is the

Equivariant Nash Conjecture. *Let G be a compact Lie group. Every closed smooth G manifold is algebraically realized.*

The following theorem confirms this conjecture for groups of odd order and for actions which satisfy some assumptions on the isotropy groups. An action of a group G on a space M is said to be *semifree* if the isotropy group $G_x = \{g \in G \mid gx = x\}$ of x is $\{1\}$ or G for all $x \in M$.

Theorem 1.10. *Let G be a compact Lie group. A closed smooth G manifold M is algebraically realized if*

(1) *G is of odd order.*
(2) *the action of G on M is semifree.*
(3) *the center of G contains a non-trivial 2-torus $H = (\mathbb{Z}_2)^k$ and $M^H = \emptyset$.*

There are two principles which, taken together, provide the proof of Theorem 1.10. One of them is an approximation technique which reduces the algebraic realization problem to an algebraic bordism realization problem. This is our Theorem 1.7. The other one shows that, up to bordism, the given situation has an algebraic representative.

Some of the cases of Theorem 1.10 (case (2) if G is not of odd order and G acts freely, and case (3)) only require the first principle. Generalizations of bordism theoretic results of Stong [S1], [S2] show that the actions bound equivariantly (see Propositions 4.1 and 4.3). Then one applies Theorem 1.7. To demonstrate this approach we provide the

Proof of Theorem 1.10 for Free Involutions. Consider a free action of $\mathbb{Z}_2$ on a closed smooth manifold M. Let τ denote the generator of

$\mathbb{Z}_2$. The orbit map $M \to M/\tau$ is a smooth S^0 fibre bundle. Consider the associated $D^1 = [-1, 1]$ bundle whose total space we denote by W. The boundary of W is M, and the action of $\mathbb{Z}_2$ on M extends to a smooth action on W. So M is the boundary of a compact smooth $\mathbb{Z}_2$ manifold, and by Theorem 1.7 it is algebraically realized. This concludes the proof of Theorem 1.10 in the considered special case. $\square$

As an example, the reader may consider the case where M is a circle, and τ acts antipodally. In this case M/τ is again a circle, and W is the Möbius band. Considering the Möbius band as a D^1 fibre bundle, τ acts on W by multiplication with -1 on the fibres.

To make any additional progress we need to consider the theory of equivariant vector bundles (see [At] and [Sg]). Let Ξ be an orthogonal representation of G (with underlying space $\mathbb{R}^n$) and $\mathrm{End}_{\mathbb{R}}(\Xi)$ the set of real endomorphisms of Ξ. We choose an ordered basis for $\mathbb{R}^n$ and represent the endomorphisms by $n \times n$ matrices. An action $G \times \mathrm{End}_{\mathbb{R}}(\Xi) \to \mathrm{End}_{\mathbb{R}}(\Xi)$ of G is defined as follows. Let $g \in G$ and $T \in \mathrm{End}_{\mathbb{R}}(\Xi)$. Then we map (g, L) to L^g where $L^g(x) = gL(g^{-1}x)$ for $x \in \Xi$. Let k be a natural number. We set

$$\begin{aligned} G_{\mathbb{R}}(\Xi, k) &= \{L \in \mathrm{End}_{\mathbb{R}}(\Xi) \mid L^2 = L,\ L^t = L,\ \mathrm{trace}\, L = k\} \\ E_{\mathbb{R}}(\Xi, k) &= \{(L, u) \in \mathrm{End}_{\mathbb{R}}(\Xi) \times \Xi \mid L \in G_{\mathbb{R}}(\Xi, k),\ Lu = u\}. \end{aligned}$$

This description specifies $G_{\mathbb{R}}(\Xi, k)$ and $E_{\mathbb{R}}(\Xi, k)$ as real algebraic G varieties. Define $p : E_{\mathbb{R}}(\Xi, k) \to G_{\mathbb{R}}(\Xi, k)$ as projection on the first factor. This defines a G vector bundle, which is called the *universal bundle* over the Grassmann manifold $G_{\mathbb{R}}(\Xi, k)$, and which is denoted by $\gamma_{\mathbb{R}}(\Xi, k)$.

The word universal is justified by the following fact. Let M be a smooth G manifold, or more generally a G CW-complex. If Ξ is sufficiently large (i. e., Ξ contains each irreducible representation of G

sufficiently often) then there is a $1-1$ correspondence between equivariant isomorphism classes of k-dimensional G vector bundles over M and equivariant homotopy classes of maps from M to $G_{\mathbb{R}}(\Xi, k)$:

$$\mathrm{Vect}_G(M) \leftrightarrow [M, G_{\mathbb{R}}(\Xi, k)]_G \tag{1.11}$$

The map from right to left is given by the pullback construction. To $\mu : M \to G_{\mathbb{R}}(\Xi, k)$ one associates $\mu^*(\gamma_{\mathbb{R}}(\Xi, k))$. The total space of this bundle is $\{(x, (L, u)) \in M \times E_{\mathbb{R}}(\Xi, k) \mid f(x) = L\}$, and the projection is the projection on the first factor. Given a G vector bundle ξ over M, the associated map $\mu : M \to G_{\mathbb{R}}(\Xi, k)$ is called the *classifying map* of ξ.

Definition. *Let G be a compact Lie group and ξ a k-dimensional real G vector bundle over a smooth G manifold B. We say that ξ is algebraically realized if there exists a non-singular real algebraic G variety X and an equivariant diffeomorphism $\phi : X \to B$ such that the classifying map of $\phi^*(\xi)$ is equivariantly homotopic to an equivariant entire rational function (for a definition of an entire rational function, see Section 2).*

To simplify language, we also allow the case where B consists of several components (which may have different dimensions) and the dimension of the bundle depends on the component. In this case we require that ξ is algebraically realized if we restrict it to a collection of components of B of the same dimension over which the fibre dimension is constant. Nevertheless, we suppose that the total space of the bundle has a well defined dimension.

Given an orthogonal representation Ω of G, we may consider the space of lines in Ω. It is denoted by $\mathbb{R}P(\Omega)$ and called the real projective space of Ω. With the notation from above, it is also $G_{\mathbb{R}}(\Omega, 1)$. One may pass from one to the other description by noting that a line in Ω can be identified with the endomorphism which represents the orthogonal projection onto this line. Note that $G_{\mathbb{R}}(\Omega, 1)$ is a real algebraic G variety. This construction may be applied fibrewise to an orthogonal G vector bundle $\xi = (E, p, B)$, in which case we denote the resulting fibre bundle (the fibres are real projective spaces) by $\mathbb{R}P(\xi)$ and its total space by $\mathbb{R}P(E)$.

The important property for algebraically realized G vector bundles is

Proposition 1.12 (see [DM2])**.** *Let G be a compact Lie group, $\xi = (E, p, B)$ an orthogonal G vector bundle, and $\mathbb{R}P(E)$ the total space of the associated projective bundle. If ξ is algebraically realized, then $\mathbb{R}P(E)$ is algebraically realized.*

Outline of Proof. Suppose we are given a real algebraic G variety X and an equivariant entire rational function $\mu : X \to G_{\mathbb{R}}(\Xi, k)$. Let X be realized as the zero set of a collection of polynomials in a representation Ω. We write μ as a quotient β/γ of regular maps (for a definition of regular maps see Section 2) where $\beta : \Omega \to \mathrm{End}_{\mathbb{R}}(\Xi)$ and $\gamma : \Omega \to \mathbb{R}$. Denote the total space of $\mu^*(\gamma_{\mathbb{R}}(\Xi, k))$ by Γ. Then we get a description of $\mathbb{R}P(\Gamma)$ from

$$\begin{aligned} \mathbb{R}P(\Gamma) &= \{(b, T) \in B \times G_{\mathbb{R}}(\Xi, 1) \mid (Id - \alpha(b))T = 0\} \\ &= \{(b, T) \in B \times G_{\mathbb{R}}(\Xi, 1) \mid (\gamma(b)Id - \beta(b))T = 0\} \end{aligned}$$

The second description is through polynomial equations, and this shows that $\mathbb{R}P(\Gamma)$ is a real algebraic G variety. As a second step in the argument one shows that $\mathbb{R}P(\Gamma)$ is non-singular whenever X is non-singular. This is a rather long and technical argument which we do not want to repeat here. As a third and last step one observes that the assumptions of the proposition provide (X, μ) up to diffeomorphism, hence the non-singular real algebraic variety $\mathbb{R}P(\Gamma)$ is equivariantly diffeomorphic to $\mathbb{R}P(E)$. This completes the argument. □

We return to Theorem 1.10 outlining the proof for odd order groups. We interpret a calculation of the equivariant bordism ring by Costenoble (see [C]). With a little bit more work we could base our proof also on older work of Stong [S1]. We provide more details in Section 6.

Outline of Proof of Theorem 1.10 (1). Costenoble provides certain manifolds $P_{H,V,i}$ where H is a subgroup of G, V is an irreducible representation of H, and $i = 1, 2, \ldots$. Each $P_{H,V,i}$ is of the form $e_H \times [G \times_H R_{H,V,i}]$ where $R_{H,V,i}$ is the total space of a projective bundle associated to an algebraically realized bundle. Proposition 1.12 shows that $R_{H,V,i}$ is (up to diffeomorphism) a non-singular real algebraic H variety. Inducing the H action to an action of G we obtain the non-singular real algebraic G variety $G \times_H R_{H,V,i}$ (compare Proposition 2.7). It is then crossed with a specific finite G set e_H (these sets are described in Proposition 6.2), which by Proposition 2.6 provides the structure of a non-singular real algebraic G variety on $P_{H,V,i}$. It is then shown that every bordism class can be represented by a polynomial in the $P_{H,V,i}$ with non-equivariant closed smooth manifolds as coefficients. It follows from the solution of the non-equivariant Nash Conjecture and Proposition 2.6 that every bordism class can be represented by a non-singular real algebraic variety. Theorem 1.7 implies that every closed smooth G manifold is algebraically realized, and that is what we wanted to show. $\square$

In the remaining cases of Theorem 1.10 (2) (i. e., the cases where the acting group is not of odd order, the action is semifree, and it has a fixed point) the approach is as follows. We use a localization principle to reduce the problem.

Proposition 1.13 (see [DM2])**.** *Let G be a compact Lie group, and M a closed smooth G manifold. Let τ be a central element of order two in G, and let $\nu(M^\tau)$ be the G normal bundle of the τ fixed set. Then M is algebraically realized if and only if $\nu(M^\tau)$ is algebraically realized.*

Remark. We restate and prove the proposition in Section 5 as other ideas, not central to the present discussion, are involved (see Corollary 5.7).

Proof. Assume that $\nu(M^\tau)$ is algebraically realized. A simple bordism construction (see [DM2]) provides an equivariant bordism between M and $\mathbb{R}P(\nu(M^\tau))$. Proposition 1.12 implies that $\mathbb{R}P(\nu(M^\tau))$ is algebraically realized. It follows from Theorem 1.7 that M is algebraically realized as well. $\square$

It remains to show that $\nu(M^\tau)$ is algebraically realized. We need only the following algebraic bundle realization result. Its proof is based on ideas of Segal [Sg] and Benedetti and Tognoli [BT].

Proposition 1.14 (see [DM2])**.** *Let G be a compact Lie group and ξ a real G vector bundle over a closed smooth manifold on which G acts trivially. Then ξ is algebraically realized.*

Proof of Theorem 1.10 (2). We want to show that every smooth manifold with a semifree action of a compact Lie group G is algebraically realized. Previously we discussed the case where G is of odd order, and the case where the action of G is free. We may thus assume that G has an element of order 2, and that the action of G on M has a fixed point. Let $x \in M$ be a fixed point. Then G acts freely on the unit sphere in the normal slice at x. Milnor showed that every element of order two in G must be central [M3]. Hence G has a non-trivial central element τ of order two. Obviously, $M^\tau = M^G$. Let F be any component of M^G and $\nu(F)$ its G normal bundle in M. It follows from Proposition 1.14 that $\nu(F)$ is algebraically realized, and so is $\nu(M^\tau)$. It follows from Proposition 1.13 that M is algebraically realized. This is what we wanted to show. $\square$

Let us turn our attention to the problem of realizing closed smooth G manifolds with additional structures algebraically. The structure of foremost interest is a G vector bundle, another structure of importance is a principal G fibration whose fibre is a compact Lie group. Above we defined what it means that a G vector bundle is algebraically realized, and the reader sees in the proofs of Theorem 1.10 (1) & (2), where we apply Propositions 1.13 and 1.14, how answers to the bundle realization problem are applied. Our next conjecture is more general than the Equivariant Nash Conjecture. It appears to be of interest in itself, and there are several anticipated applications for positive solutions of it.

Algebraic Bundle Realization Conjecture. *Let M be a closed smooth G manifold and ξ a real G vector bundle over M. Then ξ is algebraically realized.*

In an even stronger form, using one algebraic realization of M for every bundle over M, the conjecture was proved by Benedetti and Tognoli for the trivial group [BT]. As in the approach to the Equivariant Nash Conjecture, there are two aspects to contend with. One of them is bordism theoretic, and we discuss it in Section 5. The other one is the approximation technique expressed in Theorem 1.7, which we need to generalize. Let Y be a real algebraic G variety, and (M, f) a pair which consists of a closed smooth G manifold M and an equivariant map $f : M \to Y$. There is a naturally defined notion of bordism between such pairs (see Section 4).

Definition 1.15. *For a pair (M, f) as above we say that it has an algebraic representative (in its bordism class) if it is equivariantly cobordant to a pair (X, h) where X is a non-singular real algebraic G variety and h is an equivariant entire rational function. We say that (M, f) is algebraically realized if there is an equivariant diffeomorphism $\varphi : X \to M$ such that $f \circ \varphi$ is equivariantly homotopic to h.*

As a generalization of Theorem 1.7 we expect the following to hold. We call it a conjecture as we have not written down all details.

Conjecture 1.16. *Let Y be a non-singular real algebraic G variety, and $f : M \to Y$ an equivariant map. If (M, f) has an algebraic representative, then (M, f) is algebraically realized.*

Eventually one would even like to have a relative version of 1.16. Suppose we are given a collection of submanifolds $M_1, \ldots, M_k$ of M

in general position. Then we would like that these are mapped to non-singular subvarieties in the algebraic realization of M.

We expect that the last few results will suffice to prove our next conjecture (it is also more general than the Equivariant Nash Conjecture), at least in the case where the acting group is of odd order. In the non-equivariant setting the conjecture is a theorem of Akbulut and King [AK1], [AK2].

Interior Manifold Conjecture. *The interior of a compact smooth G manifold is algebraically realized.*

Kim and Masuda proved this conjecture for manifolds of dimension two [KM]. They only assume that G is a compact Lie group. Here specific information of surfaces is of help. In the general case the strategy of proof can be borrowed from the work of Akbulut and King, but some essential arguments need to be added, even for groups of odd order. Conceptually, this conjecture seems to be of importance in a general theory of real algebraic transformation groups. Aside from this, its solution for a group G implies the solution of the Fixed Point Conjecture for G. This approach to the Fixed Point Conjecture is one out of two alluded to after Corollary 1.6.

2. Basic Concepts

As expressed in the Equivariant Nash Conjecture, this paper deals with the relation between smooth and real algebraic transformation groups. As the reader may be unfamiliar with one or the other category, we recall the basic definitions in both, i. e., objects, morphisms, and isomorphism classes of objects. In addition, we recall a few facts which clarify the basic concepts.

In the introduction we defined real algebraic varieties. It remains to discuss non-singularity. Let $p_1, \ldots, p_m : \mathbb{R}^n \to \mathbb{R}$ be polynomials and

$$V = \{x \in \mathbb{R}^n \mid p_1(x) = \cdots = p_m(x) = 0\}.$$

As a topological space we will consider V with two topologies, the subspace topologies induced by the Euclidean topology on $\mathbb{R}^n$

and the one induced by the Zariski topology. The closed sets in the Zariski topology on $\mathbb{R}^n$ are the zero sets of polynomials $p : \mathbb{R}^n \to \mathbb{R}$. Most of the time we will use the Euclidean topology without mentioning this explicitly. Whenever we use the Zariski topology we will say so.

Definition 2.1. *The variety $V \subset \mathbb{R}^n$ is said to be non-singular at $x \in V$ if there are polynomials $q_1, \ldots, q_s : \mathbb{R}^n \to \mathbb{R}$ which vanish on V and a Zariski open neighbourhood U of x in $\mathbb{R}^n$ such that*

(1) $V \cap U = U \cap q_1^{-1}(0) \cap \cdots \cap q_s^{-1}(0)$

(2) *the gradients $(\nabla q_i)_x$ are linearly independent for $i = 1, \ldots, s$.*

We say that V is non-singular if V is non-singular at each point $x \in V$, and all connected components of V have the same dimension.

A subset $M \subset \mathbb{R}^n$ is said to be a *smooth submanifold* of dimension m if for all $x \in M$ there exists a neighbourhood U_x of x in $\mathbb{R}^n$ and a smooth function $\phi_x : U_x \to \mathbb{R}^{n-m}$ such that $M \cap U_x = U \cap \phi_x^{-1}(0)$ and the derivative $(D\phi)_x$ of ϕ at x has rank $n - m$. In comparison to the local behaviour of a non-singular variety at the point $x \in M$, ϕ_x is only required to be smooth and need not be polynomial as in Definition 2.1.

In a different approach, one defines the concept of a smooth atlas for a paracompact topological space M, which then defines a smooth structure on M. Together with the chosen structure (if one exists), M is said to be a *smooth manifold*. With the help of the implicit function theorem one may define a unique smooth structure on a smooth submanifold of $\mathbb{R}^m$. With the help of Whitney's embedding theorem one can realize every smooth manifold as a submanifold of some $\mathbb{R}^m$. For more background on the topic of smooth manifolds we refer the reader to a book by Munkres [Mu].

A *smooth action* of a compact Lie group G on a smooth manifold M is a smooth map $\mu : G \times M \to M$, which is also an action of G on M. We call M together with the action a smooth G manifold. According to the Palais-Mostow embedding theorem and smooth G manifold can be embedded into a representation of G.

We turn our attention to the concept of a morphism. In the smooth category we generally consider smooth maps. A smooth map between smooth manifolds is said to be a diffeomorphism if it has a smooth inverse. Diffeomorphic smooth manifolds are generally treated as the same. In the presence of an action of a group G we consider equivariant maps. An equivariant smooth map between smooth G manifolds is said to be an equivariant diffeomorphism if it has a smooth inverse. It is automatic that the inverse is equivariant. Again, equivariantly diffeomorphic manifolds are treated as the same. We make no special provisions for submanifolds of some Euclidean space. They are treated as manifolds without reference to the Euclidean space in which they are embedded.

In the real algebraic category we have the concept of a polynomial map, a rational map, and an entire rational map. Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be real algebraic varieties. A map $f : V \to W$ is said to be *regular* if it extends to a map $F : \mathbb{R}^n \to \mathbb{R}^m$ such that each component of F is a polynomial. We say that f is *rational*[1] if there are regular functions $p : \mathbb{R}^n \to \mathbb{R}^m$ and $q : \mathbb{R}^n \to \mathbb{R}$ such that $f = p/q$, restricted to V, and q does not vanish on V. Finally, f is said to be an *entire rational function* if p and q can be chosen such that q does not vanish anywhere on $\mathbb{R}^n$. Occasionally, we even call a regular function a polynomial.

Before we give the equivariant equivalents we discuss the process of averaging a function (compare e. g. [Br]). Let Ω and Ξ be representations of a compact Lie group G, and $f : \Omega \to \Xi$. Denote the

[1] Our definition is taken out of Ivanov's paper [I], and as he notes, this is not the standard definition. Based on Propositions 2.3 and 2.5, there is no specific need to distinguish between rational and entire rational functions. In the language of some other references our rational or entire rational maps are called entire rational, and for them a rational function from V to W does not need to be defined everywhere on V. Such a definition is motivated by the situation in complex algebraic geometry. For us, rational or entire rational functions should certainly be defined everywhere. To avoid any conflict, we use only entire rational maps, and here every ones definition seems to be the same, in particular after taking 2.5 into account.

Haar measure of G by dg, and let x be a point in Ω. Then

$$A(f)(x) = \int_G g^{-1} f(gx)\, dg.$$

We collect some standard facts concerning the averaging operator A.

Lemma 2.2. *With the notation from above:*

(1) *$A(f)$ is equivariant, and $A(f) = f$ if f is equivariant.*
(2) *If f is a polynomial, then so is $A(f)$.*

Let $V \subset \Omega$ and $W \subset \Xi$ be real algebraic G varieties. An equivariant map $f : V \to W$ is said to be an *equivariant regular function* if it is regular. Given any polynomial extension $F : \Omega \to \Xi$ of f we get an equivariant polynomial extension of f using $A(F)$. This follows from Lemma 2.2. An equivariant function is said to be an *equivariant rational function* if it is rational. Why we need not impose any stronger restriction is obvious from our next proposition.

Proposition 2.3. *Let $V \subset \Omega$ and $W \subset \Xi$ be real algebraic G varieties, and $f : V \to W$ an equivariant rational function. There exist equivariant polynomials $P : \Omega \to \Xi$ and $Q : \Omega \to \mathbb{R}$ such that $f = P/Q$, restricted to V, and Q does not vanish on V.*

An *equivariant entire rational function* is an entire rational function which is also equivariant. There are two additional remarks of interest at this point.

Proposition 2.4. *Every real algebraic G variety can be expressed as the zero set of a single equivariant polynomial.*

Proposition 2.5. *Every equivariant rational function is entire rational.*

Let $V \subset \Omega$ and $W \subset \Xi$ be real algebraic varieties. An equivariant regular map $f : V \to W$ is said to be an *equivariant regular isomorphism* (of these real algebraic G varieties) if it has an equivariant regular inverse $g : W \to V$, i. e., $g \circ f$ is the identity on V and $f \circ g$ is the identity on W. An *equivariant entire rational isomorphism* is defined similarly. The results of this paper do not depend on the type of isomorphism we use. Whenever we refer to an isomorphism of varieties we will mean a entire rational isomorphism. As for manifolds, we identify real algebraic G varieties if they are equivariantly isomorphic. We will often talk about real algebraic G varieties, leaving it up to the reader to remember that (for any representative in the equivariant isomorphism class) there is a given representation of G in which the variety is the zero set of a family of polynomials.

The following is an easy exercise.

Proposition 2.6.

(1) *The disjoint union of two (non-singular) real algebraic G varieties (of the same dimension) is a (non-singular) real algebraic G variety.*
(2) *The cartesian product of two (non-singular) real algebraic G varieties is a (non-singular) real algebraic G variety.*
(3) *The intersection of two real algebraic G varieties is a real algebraic G variety.*

We recall the basic properties of the induction process. Let H be a subgroup of a group G, and let X be an H space. The induced G space $\operatorname{Ind}_H^G X$ is defined as the orbit space $G \times_H X$. More explicitly, elements in $G \times_H X$ are equivalence classes of elements in $G \times X$, where (g, x) is equivalent to $(gh, h^{-1}x)$ for all $h \in H$. If X is a (closed) smooth H manifold and H is a closed subgroup of a compact Lie group G, then $\operatorname{Ind}_H^G X$ is a (closed) smooth G manifold. This procedure is natural. Given an H equivariant map $f : X \to Y$ of H spaces, the map $Id \times f : G \times X \to G \times Y$ induces a G map $\operatorname{Ind}_H^G f : \operatorname{Ind}_H^G X \to \operatorname{Ind}_H^G Y$ of orbit spaces. If X and Y are smooth H manifolds and f is smooth, then so is $\operatorname{Ind}_H^G f$. If Y is a G space, then we have an induced G equivariant map $G \times_H X \to Y$ which extends f. Here we identify X with $1 \times_H X \subset G \times_H X$. We state a similar result for real algebraic varieties.

Proposition 2.7. *Suppose G is a compact Lie group, and H is a closed subgroup of finite index. If X is a (non-singular) real algebraic H variety, then there is a natural procedure to define a (non-singular) real algebraic G variety structure on $\operatorname{Ind}_H^G X$. Given two real algebraic H varieties X and Y and an H equivariant regular (entire rational) map $f : X \to Y$, there is a naturally defined G equivariant regular (resp. entire rational) map $\operatorname{Ind}_H^G f : \operatorname{Ind}_H^G X \to \operatorname{Ind}_H^G Y$. If Y is a real algebraic G variety then we have an induced regular (entire rational) G equivariant map $G \times_H X \to Y$ which extends f. Here we identify X with $1 \times_H X \subset G \times_H X$.*

A detailed proof of this proposition is given in [DM2].

3. More on the Fixed Point Conjecture

The Fixed Point Conjecture is motivated by results and problems in the smooth as well as the complex algebraic category. We begin our discussion with the smooth case. Conner and Floyd [CF1] were the first to construct fixed point free actions on $\mathbb{R}^n$. These were actions of cyclic groups not of prime power order. The ultimate answer in the smooth category is based on articles by Conner and Montgomery [CM], by Hsiang and Hsiang [HH], and by Edmonds and Lee [EL]. The smoothing of their actions with the help of the Mostow-Palais embedding theorem is a standard argument. The answer is (the trivial group is considered to be a group of prime power order):

Theorem. *A compact Lie group G with connected component G_0 has a fixed point free action on some $\mathbb{R}^n$ if and only if G/G_0 is not of prime power order or G_0 is not abelian.*

The actions in Theorems 1.3 and 1.4 are of interest as they provide the first actions of this type in the more rigid real algebraic category. The solution of the Fixed Point Conjecture would provide the equivalent to above theorem in the real algebraic category. We like to point out that the examples by Conner and Floyd show that there is a real difference between this and the smooth category. According to the necessity part of the Fixed Point Conjecture, their cyclic actions on $\mathbb{R}^n$ cannot exist in the real algebraic category.

Let us return to the problem of constructing real algebraic fixed point free actions on varieties diffeomorphic to Euclidean space. First one may consider the

Conjecture. *A group acts smoothly and without a fixed point on a disk if and only if it acts smoothly and with exactly one fixed point on a homotopy sphere.*

The sufficiency part ($\Longleftarrow$) is trivial, and the necessity part ($\Longrightarrow$) goes back to Montgomery and Samelson [MS]. They asked whether there are one fixed point actions on spheres. Positive solutions were provided by Stein [St], Petrie [P1], and Morimoto [Mo]. Based on Euler characteristic calculations in [DKS] and the techniques in [DP] we expect that the conjecture does not only hold for odd order abelian groups as shown by Petrie [P1], but also for odd order groups more generally. One may then apply Theorem 1.10 (1) to obtain a real algebraic action on a homotopy sphere with exactly one fixed point and apply Lemma 1.9 to get a real algebraic action without a fixed point on a variety diffeomorphic to Euclidean space. This is the other approach to the Fixed Point Conjecture alluded to after Corollary 1.16.

Next we explain the (complex) algebraic background of the Fixed Point Conjecture. We provide only some of the relevant definitions, but hope that the reader gets at least a fairly accurate understanding of the problems. For further reading we suggest the given references. We suppose that G is a reductive group [Kr1], such as a finite group, the non-zero complex numbers $\mathbb{C}^*$ with multiplication as group operation, or the group $O(2, \mathbb{C})$ consisting of orthogonal matrices with entries in $\mathbb{C}$. The following two conjectures were of much interest in algebraic transformation groups.

Equivariant Serre Conjecture[2]**.** *Every complex algebraic G vector bundle over a representation space is a product bundle.*

Note that this conjecture holds for topological vector bundles because a representation space contracts to a point. An algebraic bundle is defined just like a topological one, only that one requires all data (such as the transition functions) to be algebraic, and local triviality is assumed over Zariski open sets.

Linearity Conjecture [Ka]**.** *Every algebraic action of a reductive group on $\mathbb{C}^n$ is conjugate to a linear action.*

Suppose $\mu, \eta : G \times \mathbb{C}^n \to \mathbb{C}^n$ are algebraic actions. They are said to be *conjugate* to each other (in the complex algebraic category) if there exists an algebraic automorphism $\psi : \mathbb{C}^n \to \mathbb{C}^n$ such that the following diagram commutes

$$\begin{array}{ccc} G \times \mathbb{C}^n & \xrightarrow{\mu} & \mathbb{C}^n \\ {\scriptstyle Id\times\psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ G \times \mathbb{C}^n & \xrightarrow{\eta} & \mathbb{C}^n \end{array}$$

In the topological category such a conjecture is certainly not true. As we mentioned earlier, there are even cyclic actions on $\mathbb{R}^n$ without a fixed point, while every linear action leaves at least the origin fixed. A negative answer to the Fixed Point Conjecture in the corresponding complex algebraic setting, and on $\mathbb{C}^n$, would have been a striking counter example to the Linearity Conjecture.

The situation is as follows. Let E be the total space of an algebraic vector bundle ξ over a representation space. According to the solution of the Serre Conjecture by Quillen and Suslin, E is (algebraically isomorphic to) $\mathbb{C}^n$. The vector bundle structure defines

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[2] It appears that Bass is the first person to attach this name to the conjecture, and in analogy to the situation without group action, the name seems to be justified. Still, Serre never made this conjecture.

an additional action of $\mathbb{C}^*$ on E, so that $G \times \mathbb{C}^*$ acts on the total space of the bundle. If ξ is nontrivial as a G vector bundle, then Kraft showed that the action of $G \times \mathbb{C}^*$ on E is not conjugate to a linear action [Kr2]. In a fundamental paper Schwarz produced counter examples to the Equivariant Serre Conjecture for groups such as $O(2, \mathbb{C})$, hence counter examples to the Linearity Conjecture for groups such as $O(2, \mathbb{C}) \times \mathbb{C}^*$ [Sw2]. Continuous families of such examples for finite groups were given by Masuda and Petrie [MP].

4. Some Elementary Cases of Theorem 1.10

In this section we treat the elementary cases of Theorem 1.10, i. e., those cases which are obtained as immediate consequences of bordism theoretic results and Theorem 1.7.

A group G is said to act *freely* on a space X if $G_x = \{g \in G \mid gx = x\} = \{1\}$ for all $x \in X$. The set of equivariant bordism classes represented by closed smooth G manifolds with free action of G is denoted by $\mathfrak{N}_*^G[\text{free}]$. Two manifolds M_1 and M_2 with free action of G are said to be *freely cobordant* (or in the same class in $\mathfrak{N}_*^G[\text{free}]$) if there exists a compact smooth manifold W with a free action of G such that ∂W is the disjoint union of M_1 and M_2. As a generalization of a result of Stong [S1, Proposition 14.1] we prove:

Proposition 4.1. *Let G be a compact Lie group not of odd order, and suppose that G acts smoothly and freely on a closed manifold M. Then M bounds equivariantly.*

We need some preparation for the proof of the proposition. The following corollary proves the Equivariant Nash Conjecture for manifolds with free group action.

Corollary 4.2. *Let G be a compact Lie group, and suppose that G acts smoothly and freely on a closed manifold M. Then M is algebraically realized.*

Proof. If G is not of odd order, then M is the boundary of a compact smooth G manifold. We apply Theorem 1.7 to conclude that M is algebraically realized.

The case where G is of odd order is covered by Theorem 1.10 (1), but the proof is so easy, we just give it. Let $\mathfrak{N}_*$ be the set (ring) of bordism classes of closed manifolds. It is well known that $\mathrm{Ind}_1^G : \mathfrak{N}_* \to \mathfrak{N}_*^G[\text{free}]$ is surjective [CF2], [S1]. By a result of Milnor [M1] every class in $\mathfrak{N}_*$ can be represented by a non-singular real algebraic variety. By Proposition 2.7 every class in $\mathfrak{N}_*^G[\text{free}]$ can be represented by a non-singular real algebraic G variety. Theorem 1.7 implies once more that every closed smooth G manifold with free action is algebraically realized. □

Next we state a generalization of a result by Stong [S2, p. 779] which we apply in the proof of Proposition 4.1 as well as the proof of Theorem 1.10 (3). Suppose $H \lhd G$. We say that the H fixed point structure of a closed smooth G manifold bounds if M^H is the boundary of a G manifold W and the normal bundle $\nu(M^H)$ of M^H in M (it is a G vector bundle) extends to a G vector bundle over W.

Proposition 4.3 (see [DM2]). *Suppose the center of G contains a non-trivial 2-torus $H \cong (\mathbb{Z}_2)^k \neq 1$, and G acts smoothly on a closed manifold M such that the H fixed point structure bounds. Then M is the boundary of a compact G manifold, and hence M is algebraically realized.*

The second assumption is trivially satisfied if $M^H = \emptyset$. The second part of the claim is an immediate consequence of the first part using Theorem 1.7. The reader may consider the proof the first part of the claim as an interesting generalization of the proof given in the introduction for Theorem 1.10 in the special case of a free involution. The reader may also consult Stong's paper [S2] who proves the proposition in case $G = H$ and generalize his argument. In fact, this is done in [DM2].

Our next proof uses some results about compact Lie groups, their classifying spaces, and some results from bordism theory. An easily accessible discussion of Lie groups is given in Bredon's book [Br]. Classifying spaces for compact Lie groups are discussed in most books on fibre bundles such as [Hu]. There is a natural construction which assigns to any compact Lie groups G a classifying space BG [M4]. One can construct a contractible space EG with free action of G and use its orbit space EG/G as BG. Consider the set of principle G fibrations over M. These are fibrations whose fibre and structure group are G. Considering these fibrations up to fibre homotopy equivalence we obtain $\mathrm{Prin}_G(M)$. The orbit map $\pi : EG \to BG$ defines a principal G fibration γ_G which is universal in the following sense. There is a $1-1$ correspondence (compare 1.11)

$$\mathrm{Prin}_G(M) \leftrightarrow [M, BG] \tag{4.4}$$

The map from right to left is given by the pullback construction. To $\mu : M \to BG$ one associates $\mu^*(\gamma_G)$. Given a principal G bundle over M, the associated map $\mu : M \to BG$ is called the classifying map of ξ. E. g., The classifying space for S^1 may be taken as $\mathbb{C}P^\infty$, the infinite dimensional complex projective space.

All of the results from bordism theory which we need in our proof are summarized in a book book by Conner and Floyd [CF2]. It describes bordism theory as a generalized homology theory. We recall some bordism theoretic notation used in our proof.

Let G be a compact Lie group and X a G space. An element of $\mathfrak{N}_*^G(X)$ is represented by a pair (M, f) where M is a closed smooth G manifold and $f : M \to X$ is an equivariant map. Two such pairs (M_i, f_i), $i = 1, 2$, are said to be cobordant (i. e. they are in the same equivalence class) if there exists a compact smooth G manifold W and an equivariant map $F : W \to X$ such that the boundary of W is the disjoint union of M_1 and M_2 and F restricts to f_i on M_i. With addition of classes defined as the disjoint union of representatives, $\mathfrak{N}_*^G(X)$ is an abelian group. It is a graded group, where (M, f) represents a class in $\mathfrak{N}_n^G(X)$ for $n = \dim M$. A missing superscript G indicates that the acting group is the trivial group, or equivalently, no group is acting. A missing target space X indicates that we use a point as target space, in which case we can neglect the map and target space without any loss of information.

Proof of Proposition 4.1. We follow Stong's proof adding one more idea (see [S1, p. 75]). Let T be a maximal torus of G with normalizer NT and Weyl group $W = NT/T$. Let W_2 be the 2-Sylow subgroup of W, and let N_2T be the subgroup of NT given by the exact sequence

$$1 \to T \to N_2T \to W_2 \to 1.$$

In the proof we use the following commutative diagram (for reasons of space we abbreviate N_2T by N_2 and 'free' by 'fr' in this diagram):

$$\begin{array}{ccccccc}
\mathfrak{N}_* \otimes H_*(BG) & \xrightarrow{\cong} & \mathfrak{N}_*(BG) & \xleftarrow{\cong} & \mathfrak{N}^G_{*+d}[\mathrm{fr}] & \xrightarrow{\alpha} & \mathfrak{N}^G_{*+d} \\
\uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle \mathrm{Ind}_f} & & {\scriptstyle \mathrm{Ind}}\uparrow \\
\mathfrak{N}_* \otimes H_*(BN_2) & \xrightarrow{\cong} & \mathfrak{N}_*(BN_2) & \xleftarrow{\cong} & \mathfrak{N}^{N_2T}_{*+d_1}[\mathrm{fr}] & \xrightarrow{\beta} & \mathfrak{N}^{N_2T}_{*+d_1}
\end{array}$$

Here $d = \dim G$ and $d_1 = \dim N_2T$. Homology is understood with $\mathbb{Z}_2$ coefficients, and the tensor product is taken over $\mathbb{Z}_2$. There are two vertical induction maps denoted by Ind, and two maps induced by the inclusion $N_2T \to G$. The first pair of horizontal isomorphisms is obtained from the Künneth formula in bordism theory. The second pair of isomorphisms is induced by the map which assigns to a free action of H on a manifold M the classifying map of the principal H fibration $M \to M/H$, applied with $H = G$ in the top row and $H = N_2T$ in the bottom row. The maps α and β are forgetful maps. We want to show that $\alpha = 0$. This is an immediate consequence of

(1) $\beta = 0$
(2) Ind_f is surjective

To show (1) it suffices to show that there exists a central nontrivial element τ of order two in N_2T. Then (1) follows from Proposition 4.3.

There is an action of W_2 on T given by

$$W_2 \times T \to T \quad \text{with} \quad (g,t) \mapsto gtg^{-1}.$$

As elements of order 2 in T are again mapped to elements of order 2, there is an induced action of W_2 on the 2-torus $T_2 = (\mathbb{Z}_2)^k$ in T. Here $k = \dim T$. As this action has the unit element $e \in T_2$ as a fixed point, there must be another, non-trivial W_2 fixed point in T_2. We call it τ and note that τ lies in the center of N_2T. This concludes the proof of (1).

To see (2) we consider the fibration

$$G/N_2T \to BN_2T \xrightarrow{i} BG$$

and the composition

$$H_*(BG, \mathbb{Z}_2) \xrightarrow{tr} H_*(BN_2T, \mathbb{Z}_2) \xrightarrow{i_*} H_*(BG, \mathbb{Z}_2)$$

Here tr denotes the Becker-Gottlieb (Euler characteristic) transfer [G, Theorem C], [BG]. In these papers it is shown that $i_* \circ tr$ is multiplication with the Euler characteristic of the fibre, i. e., with $\chi(G/N_2T)$. It is well known that $\chi(G/NT) = 1$ (see [Br]) and $|W/W_2| \equiv 1 \pmod 2$ because W_2 is the 2-Sylow subgroup of W. It follows that $\chi(G/N_2T)$ is odd, that $i_* \circ tr = Id$, and that i_* is surjective. This completes the proof of (2) as well as the proof of the proposition. □

5. Algebraic Realization of G vector bundles

The underlying idea in the definition of an algebraically realized bundle (given in the introduction) is the one of a strongly algebraic G vector bundle. To define this concept, we remind the reader of the definition of the Grassmannian in the introduction. Originally, the concept of a strongly algebraic vector bundle was considered by Benedetti and Tognoli [BT]. Their definition is somewhat different from ours, but in the non-equivariant setting their and our definition are equivalent. To show this one uses Akbulut and King's solution of the Nash Conjecture.

Definition 5.1. *A strongly algebraic G vector bundle over $\mathbb{R}$ is a pair (X, μ) where X is a real algebraic G variety and $\mu : X \to G_{\mathbb{R}}(\Xi, k)$ is an equivariant entire rational function. Assuming that Ξ is a summand of a representation Ξ' of G, we have an embedding $i : G_{\mathbb{R}}(\Xi, k) \to G_{\mathbb{R}}(\Xi', k)$. In this sense we identify the strongly algebraic G vector bundles (X, μ) and $(X, i\mu)$.*

Remark. There is no apparent *'natural'* way to define the concept of isomorphism for strongly algebraic bundles. One might say that two strongly algebraic G vector bundles $\mu_i : X \to G_{\mathbb{R}}(\Xi, k)$ $(i = 0, 1)$ are isomorphic if there exists an equivariant map $\eta : X \times I \to G_{\mathbb{R}}(\Xi, k)$ such that $\eta_{|X \times i} = \mu_i$, and $\eta_{|X \times t}$ is entire rational for each $t \in I$. This defines an equivalence relation. It is tempting to require that η is defined on $X \times \mathbb{R}$, and that η is entire rational, but then it is not clear that this defines an equivalence relation. (Try to show transitivity!)

Following 1.11, we may interpret a G vector bundle over M as an equivariant map $\mu : M \to G_{\mathbb{R}}(\Xi, k)$ for an appropriately chosen orthogonal representation Ξ and natural number k. Using the notion of an algebraically realized map (introduced in Definition 1.15) we may give an equivalent formulation of the Algebraic Bundle Realization Conjecture, stated in the introduction.

Conjecture 5.2. *Let (M, η) with $\eta : M \to G_{\mathbb{R}}(\Xi, k)$ be a G vector bundle and suppose that M is a closed smooth G manifold. Then (M, η) can be realized algebraically.*

Using Conjecture 1.16, we may reformulate Conjecture 5.2. We use the concept of equivariantly cobordant maps (see Section 4).

Conjecture 5.3. *Let (M, η) with $\eta : M \to G_{\mathbb{R}}(\Xi, k)$ be a G vector bundle and suppose that M is a closed smooth G manifold. Then (M, η) is equivariantly cobordant to a pair (X, μ) where X is a nonsingular real algebraic G variety and μ is an equivariant entire rational function.*

Expressed in bordism theoretic notation (we recalled this notation in Section 4) and using the language introduced in Definition 1.15, Conjecture 5.3 says that every class in $\mathfrak{N}_*^G(G_{\mathbb{R}}(\Xi, k))$ has an algebraic representative.

With this the algebraic bundle realization problem has been reduced to an equivariant bordism problem, but the present knowledge in equivariant bordism theory does not suffice to solve the problem. A solution in a special case is described in the next section.

We conclude this section with the discussion of the tangent bundle and the normal bundle of a non-singular real algebraic G variety X. Both of them turn out to be strongly algebraic, in the appropriate sense. In addition, if H is a subgroup of G, then the normal bundle of X^H in X is strongly algebraic, if restricted to the components of fixed dimension. As a corollary we obtain the necessity part of the localization principle, see Proposition 1.13 in the introduction.

Let X be a non-singular real algebraic G variety which is realized as the zero set of a polynomial (or a finite set of polynomials) in an orthogonal representation Ω of G. Denote the tangent bundle of X by TX and the normal bundle of X in Ω by $\nu(X)$. As above we identify them with their classifying maps, say χ_T and χ_ν from X to $G_{\mathbb{R}}(\Omega, k)$. In the first case k denotes the dimension of X, in the second one the codimension of X in Ω. Based on the given embedding of $X \subset \Omega$, χ_T and χ_ν are well defined (not only up to equivariant homotopy). With this understood

Proposition 5.4. *The tangent bundle and the normal bundle of a non-singular real algebraic G variety are strongly algebraic G vector bundles.*

Proof. The non-equivariant version of the proposition is proved in [AK1, Lemma 2.3], or see [BCR, p. 260]. As all data is equivariant, the entire rational map provided in the reference is automatically equivariant. $\square$

We need a slightly stronger result. Let X and Y be non-singular real algebraic G varieties, realized as zero sets of polynomials in a representation Ω of G, and suppose $X \subset Y$. Consider the normal bundle $\nu(X,Y)$ of X in Y. Its fibre at $x \in X$ consists of vectors $v \in T_xY$ (the tangent space to Y at x) which are perpendicular to the subspace T_xX of T_xY, i. e., $(\chi_{TY}(x))(v) = v$ and $(\chi_{TX}(x))(v) = 0$. The classifying map $\chi_{\nu(X,Y)} : X \to G_{\mathbb{R}}(\Omega, k)$ is an equivariant map. Here k denotes the codimension of X in Y. We show that $\chi_{\nu(X,Y)}$ is entire rational (see also [AK2, Lemma 2.7]):

Proposition 5.5. *With these definitions $(X, \chi_{\nu(X,Y)})$ is a strongly algebraic G vector bundle.*

Proof. We observe

$$\chi_{\nu(X,Y)} = \chi_{TY} - \chi_{TX}$$

We explain this. Let x be a point in X. Then $\chi_{TY}(x)$ is the orthogonal projection from Ω onto T_xY. Similarly, $\chi_{TX}(x)$ is the orthogonal projection from Ω onto T_xX. As T_xX is a subspace of T_xY we have

$$\chi_{TX}(x) \circ \chi_{TY}(x) = \chi_{TY}(x) \circ \chi_{TX}(x) = \chi_{TX}(x)$$

Then $(\chi_{\nu(X,Y)}(x))(v)$ is in T_xY and $(\chi_{TX}(x) \circ \chi_{\nu(X,Y)}(x))(v) = 0$ for any $v \in \Omega$. This shows that $(\chi_{\nu(X,Y)}(x))(v) \in \nu_x(X,Y)$. One verifies easily that

(1) $(\chi_{\nu(X,Y)}(x))^2 = \chi_{\nu(X,Y)}(x)$
(2) $(\chi_{\nu(X,Y)}(x))^t = \chi_{\nu(X,Y)}(x)$
(3) $(\chi_{\nu(X,Y)}(x))(v) = v$ if $v \in \nu(X,Y)$,

using the corresponding properties for χ_{TY} and χ_{TX}. This verifies that $\chi_{\nu(X,Y)}(x)$ is the orthogonal projection onto $\nu_x(X,Y)$. As both, χ_{TY} and χ_{TX}, are equivariant entire rational functions, $\chi_{\nu(X,Y)}$ is equivariant and entire rational as well. This completes the proof. □

Proposition 5.6. *Let Y be a non-singular real algebraic G variety realized in an orthogonal representation Ω of G, and H a subgroup of G. The union C of the m-dimensional components of Y^H is a non-singular real algebraic variety contained in Y and realized in Ω.*

Corollary 5.7. *If M is an algebraically realized smooth G manifold, and H is a subgroup of G, then $\nu(M^H, M)$ is algebraically realized.*

Proof of Corollary. By definition, there exists a non-singular real algebraic G variety Y and an equivariant diffeomorphism $\phi : Y \to M$. Let C be the union of the m-dimensional components of Y^H and $C_M = \phi(C)$, so C_M is the union of the m-dimensional components of M^H. Both, C and C_M, are NH invariant. It follows from Proposition 5.6 that C is non-singular and from Proposition 5.5 that $\nu(C, Y)$ is a strongly algebraic NH vector bundle; we set $G = NH$ and $X = C$ in the application of Proposition 5.5. The pair $(\phi_{|C}, \chi_{\nu(C,Y)})$ provides the algebraic realization of the normal bundle $\nu(C, Y)$ of C_M in M. According to our definition (given in the introduction) $\nu(M^H, M)$ is algebraically realized if M^H decomposes into a disjoint union of collections of components, and the normal bundle of each of the collections is algebraically realized. This we just have shown, hence the proof of the corollary is complete. □

In preparation of the proof of Proposition 5.6 we express a first step as a separate lemma.

Lemma 5.8. *Let Y, G, and H be as in Proposition 5.6. The union C of the m-dimensional components of Y^H is a non-singular real algebraic variety contained in Y and realized in Ω.*

Proof. Consider the classifying map $\chi_{TY} : Y \to G_{\mathbb{R}}(\Omega, k)$ of the tangent bundle of Y. It is entire rational by Proposition 5.4. Restrict it to a entire rational map $\chi_H : Y^H \to G_{\mathbb{R}}(\Omega, k)^H$. As described in Proposition 6.5, $G_{\mathbb{R}}(\Omega, k)^H$ breaks up as a disjoint union of subvarieties $F(V_i)$, where V_i ranges over the different k-dimensional representations of H which are summands of $\mathrm{Res}_H \Omega$. It follows from the more detailled description of the $F(V_i)$ in [DM2] that $T_xY = V_i$ if $\chi_{TY}(x) \in F(V_i)$. Let F_m be the union of those components $F(V_i)$ for which $\dim V_i^H = m$. Then

$$C = Y \cap \Omega^H \cap \chi_H^{-1}(F_m)$$

By asumption Y is a variety, and Ω^H is a variety because it is a linear subspace of Ω. As a union of varieties F_m is one as well, and because χ_H is entire rational, $\chi_H^{-1}(F_m)$ is a variety. Then C is a variety as we described it as an intersection of three varieties (see Proposition 2.6). $\square$

We need to show that C in Lemma 5.8 is non-singular. This requires the process of complexification and the proof of one more lemma. Let Ω be an orthogonal representation of G and $V \subset \Omega$ a real algebraic G variety. Let $\mathfrak{I}(V)$ be the ideal of all regular functions $p : \Omega \to \mathbb{R}$ which vanish on V. It is called the ideal of V. Let $\Omega_{\mathbb{C}} = \Omega \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of Ω. We consider Ω as embedded in $\Omega_{\mathbb{C}}$, calling it the real points of $\Omega_{\mathbb{C}}$. In $\Omega_{\mathbb{C}}$ we consider the unique smallest complex variety which contains V. It is denoted by $V_{\mathbb{C}}$ and its ideal $\mathfrak{I}(V_{\mathbb{C}})$ is $\mathfrak{I}(V) \otimes_{\mathbb{R}} \mathbb{C}$. All of this is explained in [W] (in particular, see Lemma 6 of this reference). The definition of non-singularity of a variety at a point in the (complex) algebraic category is formally the same as the one we gave in the real algebraic category, only that one uses $\mathbb{C}$ instead of $\mathbb{R}$.

We recall (see [W, p. 546]) the definition of the rank of V at a point and of the rank of V. For every function $f : \Omega \to \mathbb{R}$ and $p \in V$ one defines the differential $df(p)$. It is a covector (or a linear map of vectors in Ω). The definition, geometric or through the use of a coordinate system, is

$$df(p) \cdot v = \lim_{t \to 0^+} \frac{1}{t}[f(p + tv) - f(p)] = \sum_i v_i \frac{\partial f(p)}{\partial x_i}$$

The rank $\mathrm{rk}_p(S)$ of a set of functions at a point p is the maximal number of independent differentials $df_i(p), \cdots, df_s(p)$, the f_i being in S. If S is an ideal, the set of all covectors $df(p)$ ($f \in S$) forms a vector space, whose dimension is $\mathrm{rk}(S)$. Clearly $\mathrm{rk}_p(S)$ is also the number of independent differentials from any set of functions generating S. The rank $\mathrm{rk}_p(Q)$ of a point set Q at $p \in Q$ is $\mathrm{rk}_p(\mathfrak{I}(Q))$; the rank of Q is the largest value of $\mathrm{rk}_p(Q)$ for $p \in Q$. In fact, a variety is non-singular if and only if its rank at a point does not depend on the point, or equivalently, if $\mathrm{rk}_p(V) = \mathrm{rk}\,\mathfrak{I}(V)$ for all $p \in V$. Another equivalent condition is that the dimension for the space of differentials $\{df \mid f \in \mathfrak{I}(V)\}$ equals $\mathrm{rk}_p(V)$ for all $p \in V$. The definition of rank has the same meaning for $\mathbb{R}$ and $\mathbb{C}$, and to distinguish it from the real rank we denote it by rk^*. We mention that a complex algebraic variety W is non-singular at a point if it is smooth at this point. This is a classical result, and a proof is outlined in a remark on page 13 of [M2].

Let X be a (real or complex) algebraic variety. We say that it is irreducible if it cannot be expressed as the union of two proper subsets, each of which is Zariski closed in X. Given any variety X, it can be expressed as the union of irreducible Zariski closed subsets $X_1, \dots, X_k$. Requiring that X_i does not contain X_j for $i \neq j$, the X_i are uniquely determined. The X_i are called the irreducible algebraic components of X. All of this can be found in most books on algebraic geometry (e. g., see [Ha]). Every irreducible algebraic component of X is a union of connected components of X in the ordinary point set theoretical sense using the Euclidean topology on X.

Lemma 5.9. *Let $V \subset \Omega$ and $V_{\mathbb{C}} \subset \Omega_{\mathbb{C}}$ be as above. Then*

(1) *$V_{\mathbb{C}}$ is G invariant, hence a complex algebraic G variety.*
(2) *If V is irreducible, then so is $V_{\mathbb{C}}$.*
(3) *If $x \in V$ then V is non-singular at x if and only if $V_{\mathbb{C}}$ is non-singular at x.*

Proof. (1) For each $g \in G$ the space $gV_{\mathbb{C}}$ is a complex algebraic variety. It contains V as V is G invariant. Therefore $V_{\mathbb{C}} \cap gV_{\mathbb{C}}$ is a complex algebraic variety which contains V, and $V_{\mathbb{C}} = gV_{\mathbb{C}}$ because of the minimality assumption.

(2) According to Lemma 7 of [W] there is a $1-1$ correspondence between the irreducible components of V and those of $V_{\mathbb{C}}$. The correspondence is given by 'complexification' and 'taking the real points'. This implies our second claim.

(3) Suppose V is non-singular at x. Lemma 8 of [W] says that $\operatorname{rk}_x(V) = \operatorname{rk}^*_x(V_{\mathbb{C}})$. The maximal complex rank of $V_{\mathbb{C}}$ equals the maximal rank of V because $\mathfrak{I}(V_{\mathbb{C}}) = \mathfrak{I}(V) \otimes_{\mathbb{R}} \mathbb{C}$. Hence the complex rank of $V_{\mathbb{C}}$ is maximal at all $x \in V$ and $V_{\mathbb{C}}$ is non-singular at all $x \in V$.

Conversely, suppose $V_{\mathbb{C}}$ is non-singular at x. It is then an elementary consequence of the Cauchy-Riemann equations that V is non-singular at x. □

Finally we give the

Proof of Proposition 5.6. Let us gather the notation and the ingredients for the proof.

(1) C is the variety consisting of the m-dimensional components of Y^H, and x is a point in C. Let k be the codimension of C in Ω. Then $k = \dim \Omega - m$.
(2) D is the irreducible component of C which contains x. Observe that $\operatorname{rk}_x D \leq \operatorname{rk} D = k$. (The equal sign holds for a dense subset of points $p \in D$, the non-singular points of D, see [W, Theorem 1].)
(3) D^c is the complexification of D. Observe that $\operatorname{rk}^*_x D^c \leq \operatorname{rk}^* D^c = k$.
(4) E^c is the irreducible algebraic component of $(Y_{\mathbb{C}})^H$ which contains x. By a minimality argument, $D^c \subset E^c$.

As Y is non-singular at x, it follows that $Y_{\mathbb{C}}$ is non-singular at x, and that $Y_{\mathbb{C}}$ is complex analytic, hence smooth, at x. It follows that $(Y_{\mathbb{C}})^H$ is smooth at x, and the real dimension of $(Y_{\mathbb{C}})^H$ at x equals $2m$. Because E^c is irreducible, it follows from the smoothness at x (see the result quoted from [M2] above) that E^c is non-singular at x, and $\operatorname{rk}^*_x E^c = \operatorname{rk}^* E^c = k$. We have two irreducible varieties, D^c and E^C, of the same rank, and $D^c \subseteq E^c$ (see (4)). Then (e. g., see [W, Lemma 2]) $E^c = D^c$. It follows that D^c is non-singular at x, and that D is non-singular at x (see Lemma 5.9 (3)). This is what we wanted to show. □

6. Actions of Groups of Odd Order

In this section we outline the proof of Conjecture 5.3 for odd order groups and prove Theorem 1.10 (1). All of the section is devoted to the proof of

Theorem 6.1. *Let G be a group of odd order, n and k non-negative integers, and Ξ an orthogonal representation of G. Then every class in $\mathfrak{N}_n^G(G_{\mathbb{R}}(\Xi, k))$ has an algebraic representative. In particular (setting $k = 0$), every closed smooth G manifold is equivariantly cobordant to a non-singular real algebraic G variety.*

Proof of Theorem 1.10 (1). This is an immediate consequence of Theorem 6.1 and Theorem 1.7. □

There are two steps in the proof of Theorem 6.1. In the first one we show how to represent an arbitrary class in $\mathfrak{N}_n^G(X)$ in a nice form. In the second one we show how to get algebraic representatives for the classes in $\mathfrak{N}_n^G(G_{\mathbb{R}}(\Xi, k))$.

So far we have considered $\mathfrak{N}_*^G(X)$ as a set, but in the following we need its module structure. Disjoint union of representatives defines an addition on $\mathfrak{N}_*^G(X)$. Cartesian product of representatives defines a product on $\mathfrak{N}_*^G(\text{point}) = \mathfrak{N}_*^G$, so $\mathfrak{N}_*^G$ is a ring and $\mathfrak{N}_*^G(X)$ is a module over $\mathfrak{N}_*^G$. Furthermore, $\mathfrak{N}_0^G$ is a ring and $\mathfrak{N}_*^G$ is an algebra over $\mathfrak{N}_0^G$. Consider the ring $\mathfrak{N}_*$ of bordism classes of closed manifolds. We may think of manifolds without group action as manifolds with trivial action. In this sense, $\mathfrak{N}_*^G(X)$ is also a module over $\mathfrak{N}_*$. We indicate bordism classes by $[\quad]$.

We need certain idempotents (elements e for which $e^2 = e$) in $\mathfrak{N}_0^G$. For any subgroup H of G we define a ring homomorphism $\phi_H : \mathfrak{N}_0^G \to \mathbb{Z}_2$. Let A represent a class in $\mathfrak{N}_0^G$, so A is a finite G set. We set $\phi_H([A]) = |A^H| \mod 2$. It is elementary to show (e. g., see [C, Section 1])

Proposition 6.2. *Let G be a group of odd order. There exist idempotents $e_H \in \mathfrak{N}_0^G$, one for each subgroup H of G, such that*

(1) $\phi_H(e_K) = 1$ *if* $(H) = (K)$.

(2) $\phi_H(e_K) = 0$ *if* $(H) \neq (K)$.
(3) $\sum_H e_H = 1$ *where the summation ranges over subgroups of* G.

Consider formal variables $\gamma_{H,V,i}$. Here H denotes a subgroup of G, V an irreducible representation of H, and $i = 1, 2, \ldots$. Consider also

$$\mathfrak{N}_*[\gamma_{H,V,i}] \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X^H)$$

The first term in this tensor product is a polynomial ring with variables $\gamma_{H,V,i}$ and coefficients in $\mathfrak{N}_*$. The second term is the bordism group of X^H, its elements are represented by maps from closed smooth manifolds to the H fixed point set of X. Both of these are modules over $\mathfrak{N}_*$, and in this sense we form the tensor product. We define a map

$$\Psi_H : \mathfrak{N}_*[\gamma_{H,V,i}] \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X^H) \to \mathfrak{N}_*^G(X)$$

Given a typical generator $P \otimes \alpha$ of $\mathfrak{N}_*[\gamma_{H,V,i}] \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X^H)$ we have to describe $\Psi_H(P \otimes \alpha)$, then Ψ_H is defined via linear extension. We represent $\Psi_H(P \otimes \alpha)$ by map $\varphi_H(P \otimes \alpha) : \Phi_H(P \otimes \alpha) \to X$, which we construct now.

Set

$$R_{H,V,i} = \mathbb{R}P((\eta_{i-1} \otimes_{\mathbb{C}} \underline{V}) \oplus \underline{\mathbb{R}}).$$

Here η_{i-1} is the canonical (complex) line bundle over $\mathbb{C}P^{i-1}$ with trivial action of H. This bundle is tensored with the product bundle $\underline{V}$ over $\mathbb{C}P^{i-1}$ with fibre V. We give V any of its complex structures. The resulting bundle is considered as a real H vector bundle. Then we add the product bundle $\underline{\mathbb{R}}$ with fibre $\mathbb{R}$ and take the total space of the associated projective bundle. The result is $R_{H,V,i}$. Let P be an element in $\mathfrak{N}_*[\gamma_{H,V,i}]$, i. e., a polynomial in the $\gamma_{H,V,i}$ with coefficients in $\mathfrak{N}_*$. In P, substitute $R_{H,V,i}$ for $\gamma_{H,V,i}$, represent each coefficient by a closed manifold in this class, and interpret addition as disjoint union and multiplication as cartesian product. Then we obtain a closed H manifold which we denote by $\Phi'_H(P)$. Next, represent α by a map $f : M \to X^H$ and set

$$\Phi''_H(P \otimes \alpha) = \Phi'_H(P) \times M$$
$$\Phi_H(P \otimes \alpha) = e_H \times (G \times_H \Phi''_H(P \otimes \alpha))$$

The expression in the last line defines the domain for the map representing $\Psi_H(P \otimes \alpha)$. Projection on the second factor composed with f defines

$$\varphi''_H(P \otimes \alpha) : \Phi''_H(P \otimes \alpha) \to X^H$$

There exists a unique G equivariant extension of $\varphi''_H(P \otimes \alpha)$:

$$\varphi'''_H(P \otimes \alpha) : G \times_H \Phi''_H(P \otimes \alpha) \to X$$

Composing the projection on the second factor with $\varphi'''_H(P \otimes \alpha)$ (this reflects the module structure of $\mathfrak{N}^G_*(X)$ over $\mathfrak{N}^G_0$) defines

$$\varphi_H(P \otimes \alpha) : \Phi_H(P \otimes \alpha) \to X$$

This is the map which represents $\Psi_H(P \otimes \alpha)$.

Theorem 6.3. *Let G be a group of odd order. There exists a surjective graded $\mathfrak{N}_*$ module homomorphism*

$$\Psi : \prod_{(H)} \mathfrak{N}_*[\gamma_{H,V,i}] \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X^H) \to \mathfrak{N}^G_*(X)$$

where the product is taken over all conjugacy classes of subgroups of G, V ranges over the non-trivial irreducible representations of H, and $i = 1, 2 \dots$. The (abstract) polynomial generators live in dimension $|\gamma_{H,V,i}| = 2(i-1) + \dim_{\mathbb{R}} V$. On each of the factors Ψ is given by the map Ψ_H defined above.

Remarks on the Proof. The statement in this theorem can be extracted from [C]. Costenoble studies a map

$$\Phi : \mathfrak{N}_*^G(X) \to \prod_{(H)} \left(\mathfrak{N}_*[\gamma_{H,V,i}] \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X^H)\right)^{NH}$$

Our map Ψ is the explicit description of the converse of Φ. Specifically, $\Psi \circ \Phi$ is the identity on $\mathfrak{N}_*^G(X)$. But $\Phi \circ \Psi$ is the identity only if one restricts oneself to the NH invariant part of the factor corresponding to H. Here $WH = NH/H$ (or NH) acts on the set of irreducible representations of H by conjugating them (i. e., the action permutes these representations), and it acts on $\mathfrak{N}_*(X^H)$ via its action on X^H. A few more details are provided in [DM2]. A complete proof can be obtained from a detailed analysis of Costenoble's paper, adding a few lines here and there. □

So that the following makes sense, we note that

$$G_{\mathbb{R}}(\Xi, k)^H = \{L \in G_{\mathbb{R}}(\Xi, k) \mid hLh^{-1} = L \text{ for all } h \in H\}.$$

is a real algebraic variety. We can now complete the proof of Theorem 6.1 with the help of

Proposition 6.4. *Let G be a compact Lie group, Ξ an orthogonal representation of G and k a non-negative integer. Then every class in $\mathfrak{N}_*(G_{\mathbb{R}}(\Xi, k)^H)$ is algebraic.*

Outline of Proof of Theorem 6.1. Consider an element $P \otimes \alpha \in \mathfrak{N}_*[\gamma_{H,V,i}] \otimes \mathfrak{N}_*(G_{\mathbb{R}}(\Xi, k)^H)$ and represent α by a map $f : M \to G_{\mathbb{R}}(\Xi, k)^H$ where M is a non-singular real algebraic variety and f is entire rational (see 6.4). We construct an algebraic representative of $\Psi_H(P \otimes \alpha)$.

One shows that $(\eta_{i-1} \otimes_{\mathbb{C}} \underline{V}) \oplus \underline{\mathbb{R}}$ is a strongly algebraic H vector bundle over $\mathbb{C}P^{i-1}$. This may appear to be rather obvious, but the detailed argument (see [DM2]) uses strongly algebraic complex vector bundles, which we chose not to discuss here. Proposition 1.12 implies that $R_{H,V,i} = \mathbb{R}P((\eta_{i-1} \otimes_{\mathbb{C}} \underline{V}) \oplus \underline{\mathbb{R}})$ is a non-singular real algebraic H variety. It follows from Proposition 2.6 that every polynomial expression in the $R_{H,V,i}$, and in particular $\Phi'_H(P)$, is a non-singular real algebraic H variety. Several applications of Propositions 2.6 and 2.7 imply that $\Phi_H(P \otimes \alpha)$ is a non-singular real algebraic G variety. This is the domain of $\Psi_H(P \otimes \alpha)$.

We show that the map

$$\varphi_H(P \otimes \alpha) : \Phi_H(P \otimes \alpha) = e_H \times (G \times_H (\Phi'_H(P) \times M)) \to G_{\mathbb{R}}(\Xi, k)^H$$

obtained in our construction is equivariant and entire rational. As a composition of a projection and an entire rational map, $\varphi''_H(P \otimes \alpha)$ is entire rational. The last part of Proposition 2.7 implies that $\varphi'''_H(P \otimes \alpha)$ is equivariant and entire rational. As $\varphi_H(P \otimes \alpha)$ is a composition of another projection and $\varphi'''_H(P \otimes \alpha)$, we see that it is equivariant and entire rational. The map $\varphi_H(P \otimes \alpha) : \Phi_H(P \otimes \alpha) \to G_{\mathbb{R}}(\Xi, k)^H$ is the desired algebraic representative of $\Psi_H(P \otimes \alpha)$. $\square$

To prove Proposition 6.4 we describe $G_{\mathbb{R}}(\Xi, k)^H$. This is done in our next proposition. A version of this proposition in the more general context of classifying spaces of G bundles (including the case of vector bundles), though not for their finite approximations which we need here, has been given by Lashof [L]. We need the complex and the quaternion versions of the Grassmannian as well as some notation.

Let Λ stand for $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let Ξ be a representation of G over Λ, in particular, its underlying space is Λ^n for some n. We assume that the action of G preserves the standard bilinear form on Λ^n over Λ. Let $\operatorname{End}_\Lambda(\Xi)$ denote the set of endomorphisms of Ξ over Λ. It is a representation of G. The action is as in the real case discussed previously. Let k be a natural number. Given a matrix representing an element $L \in \operatorname{End}_\Lambda(\Xi)$, L^* is the matrix obtained by transposing L and taking the conjugate of its entries in Λ. We set

$$G_\Lambda(\Xi, k) = \{L \in \operatorname{End}_\Lambda(\Xi) \mid L^2 = L,\ L^* = L,\ \operatorname{trace} L = k\}$$

This description specifies $G_\Lambda(\Xi, k)$ as a real algebraic G variety.

Let χ be an irreducible representation of H over $\mathbb{R}$, and let $D(\chi)$ be the division ring $\mathrm{Hom}_H(\chi, \chi)$ consisting of all H equivariant endomorphisms of χ. It is a result in representation theory that $D(\chi) = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We note that χ is the realification of a representation of H over $D(\chi)$.

Restrict Ξ to a representation of H which is then denoted by $\mathrm{Res}_H \Xi$. We express $\mathrm{Res}_H \Xi$ as a sum, $\mathrm{Res}_H \Xi = \sum a_\chi \chi$ where χ ranges over the irreducible real representations of H. The integers a_χ are defined by this equation. Set $m(\chi, \Xi) = a_\chi$.

Proposition 6.5. *Let G be a compact Lie group, Ξ an orthogonal representation of G, k a non-negative integer and H a subgroup of G. Then*

$$G_{\mathbb{R}}(\Xi, k)^H = F(V_1) \cup \cdots \cup F(V_s)$$

is a disjoint union of subvarieties (each of them is one component), where V_j ranges over the different k-dimensional representations of H which are summands of $\mathrm{Res}_H \Xi$. *As a variety*

$$F(V_i) = \prod_\chi G_{D(\chi)}(D(\chi)^{m(\chi,\Xi)}, m(\chi, V_i))$$

Here χ ranges over the irreducible representations of H and the integers $m(\cdot, \cdot)$ are as defined above.

The proof of this proposition (given in [DM2]) is an exercise in representation theory which we recommend to the reader.

Akbulut and King [AK1] defined the concept of a variety X having totally algebraic homology. In particular, if X has totally algebraic homology, then every element in $\mathfrak{N}_*(X)$ has an algebraic representative. Akbulut and King showed that the real Grassmannians have totally algebraic homology. An analogous argument shows that the complex and quaternion Grassmannians also have totally algebraic homology. In addition, the product of varieties with totally algebraic homology as well as their disjoint union again have totally algebraic homology. The actual definition in [AK1] does not have a useful equivariant equivalent, but the characterization given above expresses exactly the desired property, and it has an obvious equivariant formulation.

Proof of Proposition 6.4. It follows as in [AK1] that each of the factors of $F(V_i)$, hence $F(V_i)$ itself, has totally algebraic homology. An immediate consequence is that every class in $\mathfrak{N}_*(G_{\mathbb{R}}(\Xi, k)^H)$ has an algebraic representative. □

7. Algebraic Groups and Compact Lie Groups

The use of algebraic groups and G modules may seem to be more appropriate to algebraic geometers, while the use of compact Lie groups and representations may seem more natural to topologists. We want to reconcile these two concepts. For groups the result is as follows:

Theorem (see [OV, p. 247])**.** *On any compact Lie group there is a unique real algebraic structure. If L is a compact subgroup of K then, viewed as real algebraic groups, L is an algebraic subgroup of K.*

For representations and modules we have the following easy consequence.

Proposition. *Let G be a compact Lie group and Ω an orthogonal representation of G with underlying space $\mathbb{R}^n$. Then, viewed as an algebraic group, G acts algebraically on $\mathbb{R}^n$, and in this sense Ω can be considered as a real algebraic G module.*

In algebraic transformation groups a different definition of a real algebraic G variety may seem more appropriate.

Definition. *Let V be a real algebraic variety, and let G be a real algebraic group. An action $\theta : G \times V \to V$ is said to be algebraic if θ is regular.*

This definition seems to be weaker than the one given in the introduction. Suppose V is realized as the zero set of a polynomial $p : \mathbb{R}^n \to \mathbb{R}$. According to the definition in the introduction, one would require that θ extends to a linear action $\overline{\theta} : G \times \mathbb{R}^n \to \mathbb{R}^n$. Actually, in the following sense, these definitions are the same.

Theorem. *Let V be a real algebraic variety with an algebraic action of the group G. There exists a real algebraic G variety V' (in the sense of the introduction) and a regular isomorphism $\phi : V \to V'$ which is equivariant.*

For a proof of this theorem see [Kr1, II.2.4] and a remark on the real case in [Sw3, (1.5)].

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