A Touch of Calculus

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[1] *Mathematica*, Wolfram Research, Inc., Champaign, Illinois.

Contents

Preface

In these notes we like to give an introduction to calculus that is accessible to students with a college algebra background. We want the students to understand and appreciate the ideas of calculus. They should learn how to do simple computations without being overwhelmed by the technical machinery of the subject. Even more so, students should understand how calculus is applied in real life. We will give mostly applications that are taken from economics and business.

As we develop the theory, we will make sure that the statements are correct. We will not expand on all technical assumptions that make them correct. This should keep the notes readable, and a critical mathematician will have no difficulties filling in details that we omit.

There are a number of ways in which to approach derivatives. The author learned the one taken here from the calculus manuscript that was used at UC Berkeley some time ago.[2]

The author would like to thank Dr. Wayne Lewis for a careful reading of the maniscript. His suggestions covered English and mathematics, and they improved the manuscript greatly.

This manuscript remains work in progress, and the author apologizes for all typos and other errors that will take a while to weed out.

[2] *Calculus I* by Jerrold Marsden and Alan Weinstein, Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1985.

Chapter 1

Tangent Lines and Derivatives

We introduce the idea of a tangent line, first in a geometric example and then formally. Then we define the notion of differentiability at a point and on an interval. Next we compute derivatives.

1.1 An Example

In Figure 1.1 you see the graph of a function[1] f. We marked the point $(2, f(2))$ on this graph, and drew a line L that looks as if it was tangent to the graph at this point. We will be interested in the slope of this line. To determine the slope we pick two points on the line, say $(1.2, .56)$ and $(2.67, 0)$, roughly. The slope of the line is then (rise/run) about

$$m = \frac{-.56}{1.47} \sim -.4.$$

The slope of the tangent line will be called the *derivative* of the function at the point, or its rate of change. In Newton's notation, which is the most common one, it is denoted by $f'(2)$.

Let us write down the equation of the tangent line. We employ the point slope formula[2] of a line. In our case the slope of the line is about $-.4$ and

[1] At times we write $f(x)$ instead of f to denote a function. Including the variable x in the notation can be convenient, and most readers have seen both notations used.

[2] The equation of a line $l(x)$ with slope m through the point (x_0, y_0) is

$$l(x) = m(x - x_0) + y_0.$$

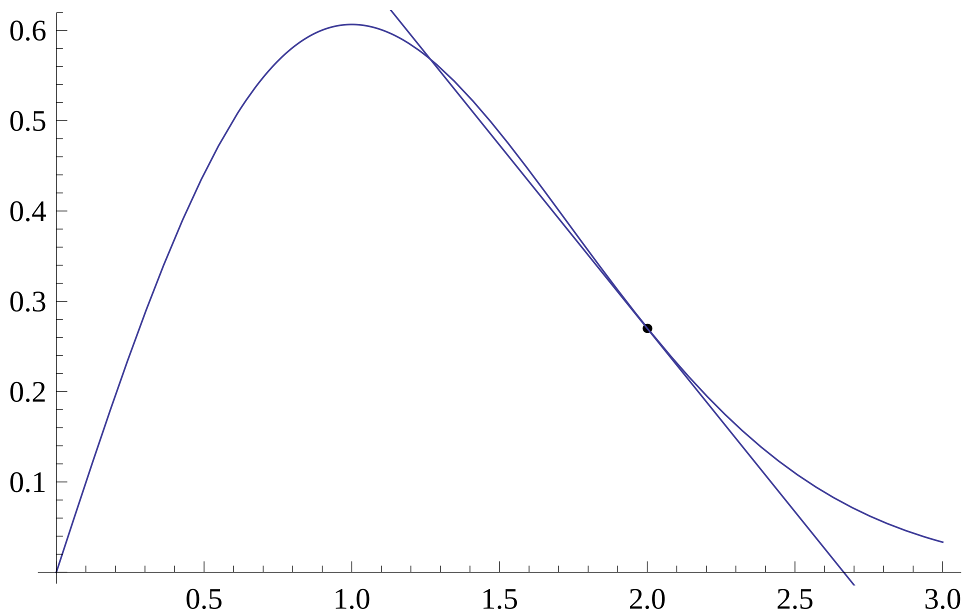

Figure 1.1: Graph and tangent line

it goes through the point $(2, f(2)) \sim (2, .27)$. Plugging these numbers into the equation of the line we find its equation:

$$t(x) = f'(2)(x-2) + f(2) \sim -.4\,(x-2) + .27$$

Exercise 1. *Use the graph in Figure 1.1.*

1. *Mark the point* $(2.5, f(2.5))$ *on the graph, draw the tangent line to the graph through this point, determine the slope of the line, and find the equation of the tangent line.*

2. *Repeat the process with the point* $(1.2, f(1.2))$.

Exercise 2. *Copy a graph from somewhere (a book, a newspaper, the internet) and repeat the previous exercise with a point of your choice.*

1.2 The Tangent Line

So far we relied on the reader's intuition of the tangent line. We may want to define tangent lines precisely.

Definition 1.1. *Let $f(x)$ be a function that is defined on an open interval (a, b), and let c be a point in the interval. We call a line $t(x)$ the* tangent line *to the graph of $f(x)$ at $(c, f(c))$ if $t(x)$ is the best*[3] *approximation of $f(x)$ by a line near this point.*

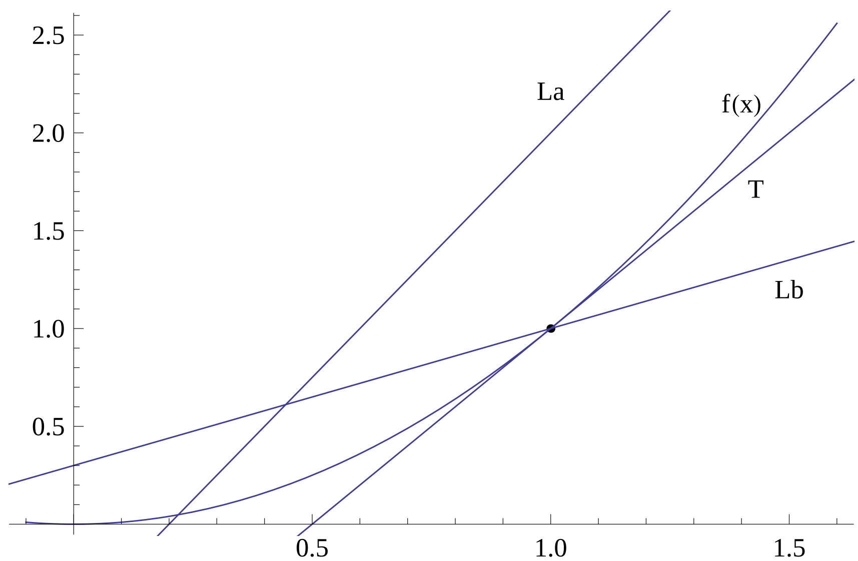

Figure 1.2: Attempts at tangent lines

Example 1.2. We illustrate the definition with an example. In Figure 1.2 you see the graph of the function $f(x) = x^2$. It is the curved line with the label $f(x)$ right next to it. We marked the point $(1, f(1)) = (1, 1)$ on the graph by a dot. In addition, there are three (straight) lines, La, Lb and T.

How does La fare as tangent line? The line La is not near the graph of $f(x)$, and certainly not on an interval around $x = 1$. In fact, at $x = 1$ the function f and the line La have different values, $La(1) \neq f(1) = 1$. We can certainly find lines that are closer to the graph of $f(x)$ than La. One such line would be Lb. At least $La(1) \neq f(1) = 1$. So, La does not have a chance at being the tangent line.

[3] The word *best* should be taken literally. If $l(x)$ is any line, then $t(x)$ is at least as close to $f(x)$ as $l(x)$ for any point x in some open interval that contains c. Expressed as a mathematical formula, if $l(x)$ is any line, then there exists some open interval that contains c, and

$$|f(x) - t(x)| \leq |f(x) - l(x)|$$

for all x in this interval.

How do Lb and T compare as tangent lines? Both lines go through the point $(1, f(1))$. But, as one sees clearly, the graph of T is closer to the one of f than the one of La. In fact, you may see that if you take the graph of T and rotate it around the point $(1, 1)$, then the line will get further away from $f(x)$ than T.

The equation for the tangent line in the example is $T(x) = 2(x-1)+1$. The slope of the line is 2 and thus $f'(1) = 2$ for the function $f(x) = x^2$.

Example 1.3. Find the tangent line to the graph of a linear function

$$f(x) = mx + b.$$

What is $f'(x)$ at any x?

Solution: Apparently a line is its own best approximation and the tangent line to the graph of $f(x)$ is $f(x)$ itself. This means that $f'(x) = m$ for any x.

Example 1.4. Find the tangent line to the graph of $f(x) = 3x - 1$.

Solution: The tangent line will be $l(x) = 3x - 1$ and $f'(x) = 3$.

1.3 Differentiability

Definition 1.5. *Let $f(x)$ be a function that is defined on an open interval*[4] *(a, b), and let c be a point in the interval. We say that $f(x)$ is* differentiable *at c if the graph if $f(x)$ has a tangent line at $x = c$. The slope of the tangent line is called the* derivative *of $f(x)$ at c, or the* rate of change *of $f(x)$ at $x = c$. It is denoted by $f'(c)$.*

There are plenty of functions that are differentiable where ever they are defined. We have encountered a couple so far. For understanding the concept better, here are two functions that are not differentiable at one point.

Example 1.6. In Figure 1.3 you see the graph of $f(x) = |x - .5| - .2$, the absolute value function shifted to the right by .5 and down by .2. There is no tangent line at the vertex. Consider the two dashed lines lines through the vertex. On one side of the vertex one line is closer to the graph than the other. One the other side of the vertex this relation is reversed. None is closer to the graph than the other on an interval around .5. No line is closer than all other lines on an open interval that contains $x = .5$.

[4] The reader may notice that we are using the same notation (a, b) for a point in the plane with coordinates a and b and the open interval between the points a and b on the number line. The context will make sure which meaning is intended.

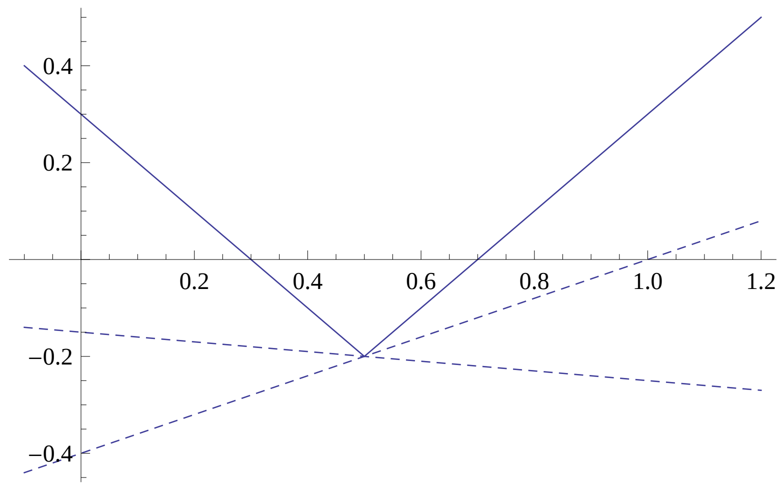

Figure 1.3: No tangent line at the vertex

Example 1.7. In Figure 1.4 you see the graph of a function that has a jump. After a brief contemplation the reader should recognize that there is no line that is close to the graph on an open interval around the point $x = 1$.

So far we have differentiated functions at a single point. If possible, we can differentiate a function $f(x)$ whereever it is defined. The we get a new function $f'(x)$, and its values tell us the rate of change of $f(x)$ at each point.

Definition 1.8. *Let the function $f(x)$ be defined on an open interval (a, b). We say that $f(x)$ is* differentiable *on the interval if it is differentiable at each point x in (a, b). In this case $f'(x)$ is a function that is again defined on the interval (a, b), and it is called the derivative of $f(x)$ on the interval (a, b). (If you spend more time on differentiation, then you will learn how to generalize this idea.)*

Example 1.9. Let us again illustrate the idea with an example. In Figure 1.5 you see the graph of a function $f(x)$ (solid line). In the same set we graphed its derivative $f'(x)$ (dashed line). In addition we drew the tangent lines at a few points (short straight line segments). The reader should check that the slopes of the line segments are the values of $f'(x)$ at the indicated points.

In Figure 1.6 and 1.7 you see two more examples of a function and its derivative displayed in the same set of coordinates. The function is drawn as

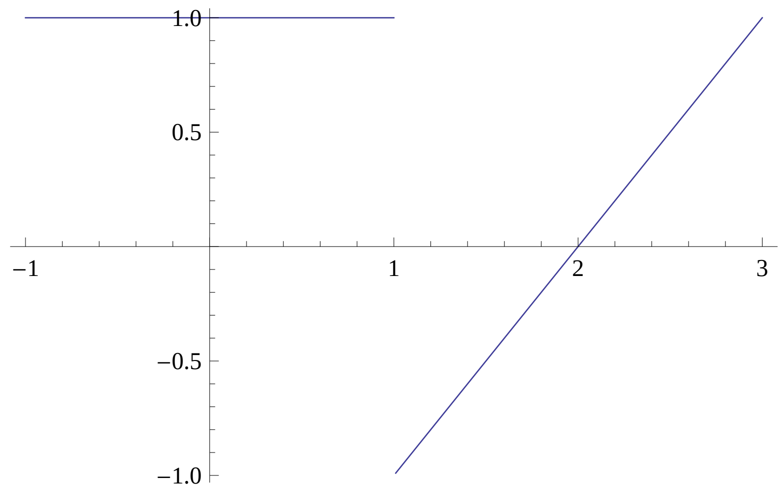

Figure 1.4: No tangent line at the jump

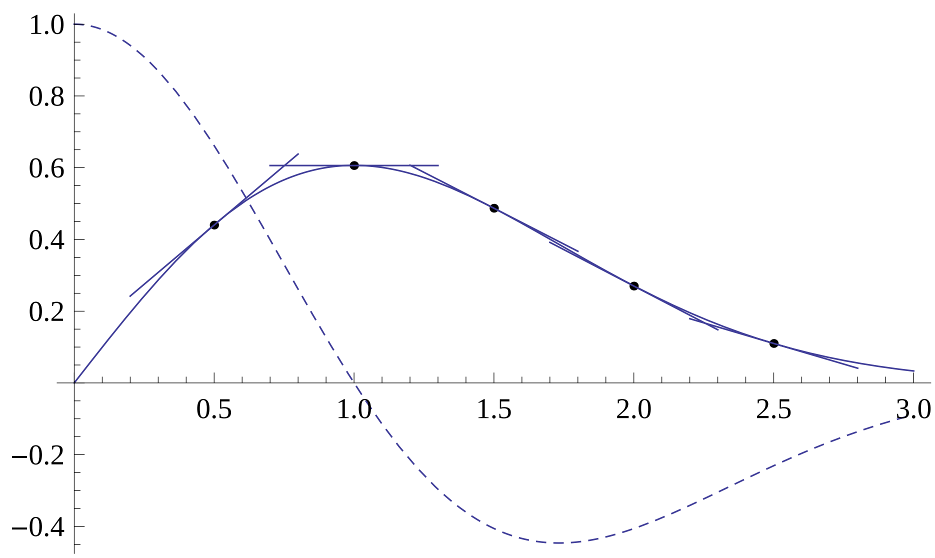

Figure 1.5: Function and its derivative

a solid line and the derivative as a dashed one. Let us look more closely at the first example. On the interval $(-1.3, -.2)$ the function appears to have a positive rate of change. The tangent lines appear to have a positive slope. Initially the function increases fast, and consistent with this, the derivative is large. As we approach $x = -.2$ the rate of change gets smaller. The function does not increase that rapidly anymore. Accordingly, the derivative gets smaller, though it remains positive. By the time we get to $x = -.2$ the tangent line has turned horizontal. Its slope is zero, and you see that the derivative is zero at the point. Between $x = -.2$ and $x = 1.57$ the derivative is negative. You observe that on the same interval the rate of change of the function is negative. First the derivative is slightly negative, then more substantially, and eventually again only slightly. This is reflected in the change of the steepness of the graph of the function. First the rate of change is slightly negative, then more so, and eventually the tangent lines again turn back to being level. After $x = 1.57$ the function has again positive rates of change and the derivative is positive.

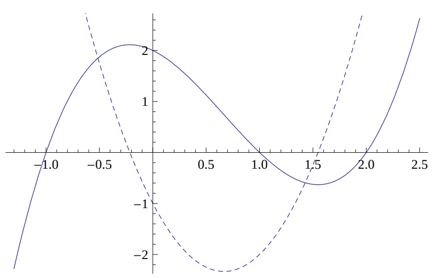

Figure 1.6: Function & Derivative

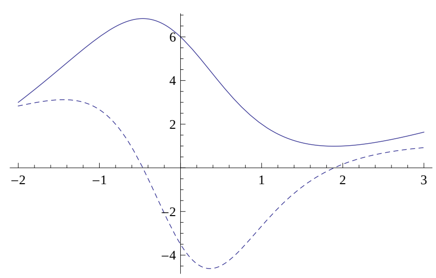

Figure 1.7: Function & Derivative

Exercise 3. *Discuss the relation between the rates of change of the function and the values of its derivative for the example shown in Figure 1.7.*

Remark 1. Differentiable functions have many nice properties, and at times we want to take advantage of this also on closed intervals. We will say that a function is *differentiable* on a closed interval $[a, b]$ if it can be extended to a differentiable function on an open interval (c, d) that contains $[a, b]$.

Exercise 4. *Decide whether the given function* $f(x) = |x|$ *and/or* $f(x) = \sqrt{x}$ *is differentiable on the interval* $[0, 1]$. *Explain your reasoning.*

1.4 Computation of Derivatives

So far we discussed tangent lines and derivatives from a geometric point of view. Now let us compute a couple of derivatives.

Example 1.10. Let us find the derivative of $f(x) = x^2$ at $x = 3$. For this purpose, we express $f(x)$ in powers of $u = (x - 3)$. Then $x = u + 3$. We replace x by $u - 3$, expand the binomial, and reverse the substitution by replacing u with $(x - 3)$. The steps result in the computation:

$$\begin{aligned} x^2 &= (u+3)^2 \\ &= u^2 + 6u + 9 \\ &= (x-3)^2 + 6(x-3) + 9. \end{aligned}$$

We expect the degree zero and one terms to be the tangent line and set $t(x) = 6(x - 3) + 9$. Bringing this expression to the left hand side in above computation, we obtain

$$x^2 - 6(x-3) - 9 = f(x) - t(x) = (x-3)^2.$$

We aply absolute values to both sides of this equation and find that

$$|f(x) - t(x)| = |x - 3|^2.$$

If x is close to 3, then the difference between these two numbers is small, in absolute terms, or $|x - 3|$ is small. If we square a small number, a number whose absolute value is less than 1, then we end up with a small number. So, if $|x - 3|$ is small, then $|x - 3|^2$ is very small. With some effort[5] one deduces for any line $l(x)$ that

$$|f(x) - t(x)| \leq |f(x) - l(x)|$$

for all x in some open interval around $x = 3$. The inequality says that $t(x)$ is closer to $f(x)$ than any other line $l(x)$, and this makes $t(x)$ the best approximation of $f(x)$ by a line, at least near the point $x = 3$. We conclude that $t(x) = 6(x - 3) + 9$ is the tangent line to the graph of $f(x)$ at $x = 3$, and its slope is the derivative, $f'(3) = 6$.

Exercise 5. *Set $f(x) = x^2$. Find $f'(\pi)$. (Hint: Mimic the previous example.)*

[5] A similar argument is also required in the examples that follow.

Example 1.11. Let us differentiate $f(x) = x^2$ at any point $x = a$. As compared to the previous example, we express x^2 in powers of $u = x - a$. We find

$$\begin{aligned} x^2 &= (u+a)^2 \\ &= u^2 + 2au + a^2 \\ &= (x-a)^2 + 2a(x-a) + a^2. \end{aligned}$$

In this case the tangent line will be $t(x) = 2a(x - a) + a^2$. As before the argument is that $|f(x) - t(x)| = (x - a)^2$ is small, and there is no line so that $|f(x) - l(x)|$ is smaller near $x = a$. In particular, $f'(a) = 2a$.

Example 1.12. The is no real difficulty to generalize our method to higher powers of x. Let us differentiate $f(x) = x^5$ at $x = 2$. The reader is invited to fill in the arithmetic. We express x^5 in powers of $x - 2$. With $u = x - 2$ and $x = u + 2$ we calculate:

$$\begin{aligned} x^5 &= (u+2)^5 \\ &= u^5 + 10u^4 + 40u^3 + 80u^2 + 80u + 32 \\ &= (x-2)^5 + 10(x-2)^4 + 40(x-2)^3 + 80(x-2)^2 + 80(x-2) + 32. \end{aligned}$$

As tangent line we find $t(x) = 80(x - 2) + 32$. We observe that

$$\begin{aligned} |f(x) - t(x)| &= |(x-2)^5 + 10(x-2)^4 + 40(x-2)^3 + 80(x-2)^2| \\ &= |(x-2)^3 + 10(x-2)^2 + 40(x-2) + 80|(x-2)^2 \end{aligned}$$

is small near $x = 2$. The reason is that the expression in the absolute value signs is about 80, or not much bigger, if x is close to 2 and $|x - 2|$ is small, and the factor $(x - 2)^2$ makes the entire right hand side small. The slope of the tangent line is our derivative, namely $f'(2) = 80$.

Exercise 6. *Use the procedure from the previous example to calculate the derivative of $f(x) = x^3$ at $x = -1$.*

In Table 1.1 you find the derivatives of the functions that we will need.

A few words of explanation are in order. The derivative of an arbitrary power x^α of the variable holds for all exponents $\alpha \in \mathbb{R}$. For the purpose of these notes it will be sufficient if you understand the meaning of x^α for simple values of α, such as 1, 2, 3, -1, -2, 1/2 1/3, etc. Some of the power functions are defined on the entire real line, certainly those where the exponent is a non–negative integer. For others we may have to exclude $x = 0$ and/or all negative numbers from the domain.

$y(x)$	$y'(x)$	Domain
x^α	$\alpha x^{\alpha-1}$	$(0,\infty)$ or $(-\infty,\infty)$
e^x	e^x	$(-\infty,\infty)$
$\ln(x+a)$	$1/(x+a)$	$(-a,\infty)$

Table 1.1: Some Derivatives

Because of the simplicity of its derivative, we are using the exponential function e^x. It has the Euler number e as its base. Exponential functions a^x with other bases have slightly more complicated derivatives.

The natural logarithm function is the inverse of the exponential function. This means that $\ln(e^x) = x$ and $e^{\ln x} = x$.

In Figure 1.8 and 1.9 you see part of the graphs of the exponential and the logarithm functions.

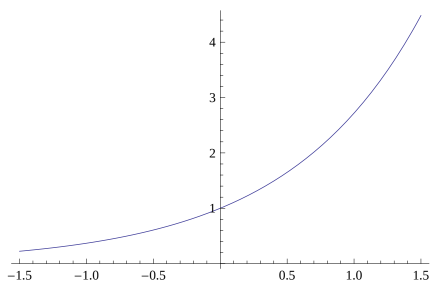

Figure 1.8: e^x

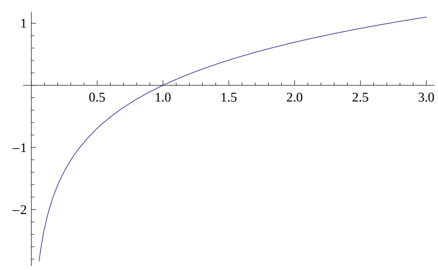

Figure 1.9: $\ln x$

1.5 Marginal Analysis

The concept of marginal analysis can be applied to numerous concepts in economics and elsewhere. As an example, consider the function

$$C(x) = 2000 + 5x + .001x^2$$

You might imagine that $C(x)$ is the cost, in US dollars, to produce x memory chips, for x between 0 and 2,000. The cost to produce the $(a+1)$st unit is the *marginal cost*:

$$C(a+1) - C(a).$$

Assuming that we already produced a units, we find how much it costs to produce one additional unit. E.g., the marginal cost of the 1001st memory chip is

$$C(1001) - C(1000) \sim 2.99$$

Based on our discussion of the derivative of a polynomial, or as an example of techniques to come in Section 1.6.1, we find that

$$C'(x) = 5 - x/500 \quad \& \quad C'(1000) = 3.$$

It is not surprising that $C'(1000) \sim C(1001) - C(1000)$. Fix a value for x and call it a. Let $T_a(x)$ be the tangent line to the graph of $C(x)$ at $x = a$. Then

$$T_a(x) = C'(a)(x - a) + C(a)$$

As $T_a(x)$ is the best possible approximation of $C(x)$ by a line, we have $T_a(x) \sim C(x)$ as long as x is close to a. With $x = a + 1$ we have

$$C(a + 1) \sim T_a(a + 1) = C'(a)((a + 1) - a) + C(a) = C'(a) + C(a).$$

After bringing one term to the other side of the equation, we find that

$$C(a + 1) - C(a) \sim C'(a).$$

These observations motivate us to call $C'(x)$ the *marginal cost function.* It tells us at which rate the cost function is increasing if the rate of production is x.

In comparison, $C(1000) = 6,000$. It costs \$6,000.00 to produce 1000 memory chips, hence \$6.00 per chip. This is an average cost under the assumption that we produce 1,000 chips.

Exercise 7. Repeat all aspects of above discussion for $a = 500$ and for $a = 1500$.

Exercise 8. Suppose that you are in the pet rock business, and your profit is

$$P(x) = 2x - 5\ln(x + 3)$$

US dollars if you sell x rocks.

- What is your profit if you sell 10 rocks?
- How many rocks do you need to sell before you make any profit?
- What is the marginal profit for the 11th rock?
- What is the value of the marginal profit function at $x = 10$? (You will learn that $P'(x) = 2 - 5/(x + 3)$.

1.6 Methods of Differentiation

There are familiar ways in which to construct new functions from old ones. Corresponding to them, there are formulae to compute the derivatives of the new functions from the derivatives of the old ones. All the derivatives that we will need will be obtained from the derivatives of our basic functions in Table 1.1 together with these formulae.

1.6.1 Linearity

Our first formula tells us what happens when we add functions and multiply them with scalars. Let f and g be functions and a and b be scalars. Assuming that f and g are differentiable

$$(af + bg)' = af' + bg'$$

Some readers prefer to include the argument in the functional notation, so that

$$(af + bg)(x) = af(x) + bg(x).$$

Assuming that f and g are differentiable at x one has

$$\boxed{(af + bg)'(x) = af'(x) + bg'(x).} \tag{1.1}$$

In words, the derivative of a sum of functions is the sum of the derivatives of the summands, and the derivative of the scalar multiple of a function is the scalar multiple of the derivative.

Example 1.13. Differentiate $h(x) = 2x^3 - x^2$.

Solution: We apply (1.1) with $f(x) = x^3$, $g(x) = x^2$, $a = 2$ and $b = -1$. According to Table 1.1, $f'(x) = 3x^2$ and $g'(x) = 2x$. We find

$$h'(x) = 2 \cdot 3x^2 + (-1)2x = 6x^2 - 2x = 2x(6x - 1).$$

Example 1.14. Differentiate an arbitrary polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Solution: Applying the ideas from earlier repeatedly we find

$$p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2 a_2 x + a_1.$$

Example 1.15. Differentiate $h(x) = \pi\sqrt{x} + \sqrt{2}/x$.

Solution: We apply (1.1) with $f(x) = \sqrt{x} = x^{1/2}$, $g(x) = 1/x = x^{-1}$, $a = \pi$ and $b = \sqrt{2}$. Using once again Table 1.1 we find $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ and $g' = (-1)x^{-2}$. Substituting all of this into (1.1) we obtain

$$h'(x) = \frac{\pi}{2\sqrt{x}} - \frac{\sqrt{2}}{x^2}.$$

Exercise 9. *Differentiate the following functions.*

(a) $f(x) = x^3 - 5x^2 + 17x - 1$

(b) $g(x) = \sqrt{x} + \sqrt[3]{x}$

(c) $h(x) = \dfrac{2}{\sqrt{x}} + \pi$

(d) $i(x) = \sqrt[5]{x^2}$

(e) $j(x) = x^2\sqrt[3]{x}$

(f) $k(x) = x + e^x$

(g) $l(x) = \dfrac{\pi}{x^3} + \ln x$

(h) $m(x) = \sqrt{x} + \ln x$

(i) $n(x) = 3x - \dfrac{15}{\sqrt{x}}$

1.6.2 Product Rule

The product rule tells us how to differentiate a product of functions. If f and g are functions, then their product is defined by $(fg)(x) = f(x)g(x)$. Assuming that f and g are differentiable at x one has

$$\boxed{(fg)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).} \tag{1.2}$$

We illustrate the use of this formula with some examples.

Example 1.16. Differentiate the function $h(x) = xe^x$.

Solution: We set $f(x) = x$ and $g(x) = e^x$. The derivatives of the factors are $f'(x) = 1$ and $g'(x) = e^x$. The producte rule tells us that

$$h'(x) = 1 \cdot e^x + x \cdot e^x = (1 + x)e^x.$$

Example 1.17. Differentiate the function $h(x) = (x^3 + 2)\ln x$.

Solution: We set $f(x) = (x^3 + 2)$ and $g(x) = \ln x$. The derivatives of the factors are $f'(x) = 3x^2$ and $g'(x) = \frac{1}{x}$. If follows from the product rule that

$$h'(x) = 3x^2 \ln x + \frac{x^3 + 2}{x}.$$

Example 1.18. Differentiate $h(x) = 5e^x \ln x$.

Solution: We set $f(x) = 5e^x$ and $g(x) = \ln x$. The derivatives of the factors are $f'(x) = 5e^x$ and $g'(x) = \frac{1}{x}$. If follows from the product rule that

$$h'(x) = 5e^x \ln x + \frac{5}{x}\, e^x$$

We could have more than two factors, and in this case we can apply the product rule repeatedly. Suppose f, g, and h are differentiable at x. Then

$$\begin{aligned} (fgh)'(x) &= f'(x) \cdot ((gh)(x)) + f(x) \cdot (gh)'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x). \end{aligned}$$

Example 1.19. Differentiate $F(x) = (x^2 - 1)e^x \ln x$.

Solution: Set $f(x) = x^2 - 1$, $g(x) = e^x$, and $h(x) = \ln x$. Then $F(x) = (fgh)(x) = f(x)g(x)h(x)$, $f'(x) = 2x$, $g'(x) = e^x$, and $h'(x) = 1/x$. Substituting all of this into our formula for the deriviative of a product with three factors, we find:

$$F'(x) = 2xe^x \ln x + (x^2 - 1)e^x \ln x + \frac{x^2 - 1}{x} e^x.$$

Example 1.20. Derive the formula for the derivative of $p(x) = x^n$ for all natural numbers $n = 0, 1, 2, \ldots$.

Solution: We like to show that $p'(x) = nx^{n-1}$.

If $n = 0$, then $p(x) = 1$, the constant function. Its derivative is the zero function. If $n = 1$, then $p(x) = x$, which is a line with slope 1. At any point, the tangent line is $p(x)$ itself, so that $p'(x) = 1$. This is exacty what the formula says.

Set $n = 2$. We express $p(x) = x^2 = x\,x$ as a product and apply the product rule. Then

$$p'(x) = x'\,x + x\,x' = 1\,x + x\,1 = 2x.$$

Set $n = 3$. We express $p(x) = x^3 = x^2\,x$ as a product and apply the product rule. We also use our result for $n = 2$. Then

$$p'(x) = (x^2)'\,x + x^2\,x' = 2x\,x + x^2\,1 = 2x^2.$$

Set $n = 4$. We express $p(x) = x^4 = x^3\,x$ as a product and apply the product rule. We also use our result for $n = 3$. Then

$$p'(x) = (x^3)'\,x + x^3\,x' = 3x^2\,x + x^3\,1 = 4x^3.$$

We hope that you recognize the pattern. The formula for $n = 4$ together with the product rule will imply the one for $n = 5$. If we know the formula for one natural number, then we obtain it for the next one. The formula will hold for all natural numbers. This mathematical technique is called induction.

Exercise 10. *Differentiate the following functions.*

(a)	$f(x) =$	$x^3(x-1)^7$	(b)	$g(x) =$	$(x^2-4)e^x$
(c)	$h(x) =$	$x \ln x$	(d)	$i(x) =$	$2e^x \ln x$
(e)	$j(x) =$	$\sqrt[3]{x}\; e^x$	(f)	$k(x) =$	e^{2x}
(g)	$l(x) =$	$\ln x^2$	(h)	$m(x) =$	$\sqrt{x} + \ln x$

1.6.3 Quotient Rule

The quotient rule tells us how to differentiate the quotient of functions. If f and g are functions, then their quotient is defined by $(f/g)(x) = f(x)/g(x)$. Assuming that f and g are differentiable at x and that $g(x) \neq 0$, then we have

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}} \tag{1.3}$$

The denominator in (1.3) can also be written as $(g(x))^2$. We illustrate the use of the quotient rule with a few examples.

Example 1.21. Differentiate $h(x) = \frac{x}{x^2+3}$.

Solution We set $f(x) = x$ and $g(x) = x^2 + 3$. Then $f'(x) = 1$ and $g'(x) = 2x$. We substitute all of this into the formula in (1.3) and find

$$h'(x) = \frac{1 \cdot (x^2+3) - x \cdot 2x}{(x^2+3)^2} = \frac{3 - x^2}{(x^2+3)^2}.$$

Example 1.22. Differentiate the function $h(x) = \frac{e^x}{x^2}$.

Solution We apply the quotient rule in (1.3) setting $f(x) = e^x$ and $g(x) = x^2$. Then $f'(x) = e^x$ and $g'(x) = 2x$. We find

$$h'(x) = \frac{e^x x^2 - e^x 2x}{(x^2)^2} = \frac{x-2}{x^3} e^x.$$

Exercise 11. *Repeat the previous problem by setting $h(x) = x^{-2}e^x$ and applying the product rule. Check whether your answer agrees with the one above.*

Example 1.23. By definition, a *rational function* is the quotient of two polynomials. If $p(x)$ and $q(x)$ are poynomials, then $r(x) = \frac{p(x)}{q(x)}$ is called a rational function. Linearity of the derivative allowed us to differentiate every polynomial. With the help of the quotient rule we can differentiate every rational function. Set

$$r(x) = \frac{x^3 - 2x}{x^4 + 1}$$

and differentiate this function.

We find that

$$r'(x) = \frac{(3x^2 - 2)(x^4 + 1) - (x^3 - 2x)4x^3}{(x^4 + 1)^2} = \frac{-x^6 + 6x^4 + 3x^2 - 2}{(x^4 + 1)^2}.$$

Exercise 12. *Differentiate the functions*

$$\text{(a) } f(x) = \frac{x+1}{x-1} \qquad \text{(b) } g(x) = \frac{\ln x}{x^2 + 1} \qquad \text{(c) } h(x) = \frac{x^2 e^x}{1 + \ln x}.$$

1.6.4 Leibnitz's Notation

The prime notation for the derivative is atttributed to Newton. It works perfectly when we write $f'(x)$ or $\ln' x$. It does not work well if we like to write down the derivative of the exponential function e^x or the polynomial $x^7 - 6x^4 + 9$, though $(e^x)'$ and $(x^7 - 6x^4 + 9)'$ would do. There is a different notation that, on occasion, is more convenient. It was introduced by Leibnitz[6].

Suppose $y = f(x)$ is a function of x. Then

$$\boxed{f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}\,y.}$$

Newton's notation is somewhat ambiguous. Let $V = \pi r^2 h$ be the volume of a circular drum whose base has radius r and whose height is h. We could calculate dV/dr or dV/dh, the rate of change for the volume as we change r or h. Leibnitz's notation provides room for the name of the variable.

[6]Gottfried Wilhelm Leibnitz (1646-1716) is one of the founders of calculus. I suggest that you google Leibnitz and read about his prolific life. Leibnitz's thesis advisor was Jacob Thomasius, who is also the greatgreatgreat$\cdots$ grand thesis advisor of the author of these notes. Frequently the name is spelled without the 't' as Leibniz. At times, he is referred to as Freiherr Gottfried Wilhelm von Leibnitz, but no documents have been found that state his appointment to nobility.

Sometimes there is no good place where to put the prime, But

$$\frac{d}{dx}\left(x^7 - 6x^4 + 9\right), \quad \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right), \text{ and } \quad \frac{d}{dx}e^x$$

are easy to read and understand expressions.

1.6.5 Chain Rule

Under appropriate conditions, we can compose functions. This means that we first apply one and then the other function. E.g., $h(x) = \sqrt{x^2+1}$ is a composition of two function. The first function, call it g, sends x to $g(x) = x^2 + 1$. The second function f sends u to $f(u) = \sqrt{u}$. The composition $h = f \circ g$ is defined by $h(x) = (f \circ g)(x) = f(g(x))$.

The *Chain Rule* tells us how to calculate the derivative of a composition of functions. Suppose that the function g is differentiable at x and f is differentiable at $g(x)$. Set $h(x) = f(g(x))$. Then

$$\boxed{h'(x) = (f \circ g)'(x) = f'(g(x))\, g'(x).} \tag{1.4}$$

In words, we differentiate the outer function f with respect to its variable and evaluate the outer derivative f' at $g(x)$. We differentiate the inner function g and find $g'(x)$. The product of $f'(g(x))$ and $g'(x)$ is the derivative of $f \circ g$ at x.

Example 1.24. Let us calculate the derivative $h(x) = \sqrt{x^2+1}$.

Solution: Earlier we decomposed $h(x)$ as the compostion of the inner function $g(x) = x^2 + 1$ and the outer function $f(u) = \sqrt{u}$. Their derivatives are $g'(x) = 2x$ and $f'(u) = 1/(2\sqrt{u})$. Put together we obtain

$$h'(x) = \frac{d}{dx}\sqrt{x^2+1} = \frac{1}{2\sqrt{x^2+1}}\, 2x = \frac{x}{\sqrt{x^2+1}}.$$

Example 1.25. Differentiate $f(x) = \ln(x+3)$.

Solution: We use $u = g(x) = x + 3$ as the inner function. The inner derivative will then be 1. As outer function we use $\ln u$, so that the outer derivative will be $1/u$. Taken together we find

$$\frac{d}{dx}\ln(x+3) = \frac{1}{x+3}.$$

Example 1.26. Differentiate $h(x) = (x-5)^7$.

Solution: We could expand the expression for $h(x)$, but we won't. We use $u = g(x) = x - 5$ as inner function and the 7-th power as outer function, $f(u) = u^7$. Then $g'(x) = 1$ and $f'(u) = 7u^6$. Hence $h'(x) = 7(x-5)^6$.

Using Leibnitz's notation with $u = g(x)$ the formula for the chain rule becomes

$$\boxed{\frac{d}{dx}h(x) = \frac{d}{dx}(f \circ g)(x) = \frac{df}{du}(g(x))\,\frac{dg}{dx}(x).} \tag{1.5}$$

If we omit arguments and abuse notation somewhat we may write

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

One may not cancel the dg's to get a proof of the chain rule, but its proof is based on this idea.

Exercise 13. *Differentiate the following functions.*

(a) $f(x) = (3x-7)^{12}$

(b) $g(x) = \dfrac{2}{1+x^2}$

(c) $h(x) = e^{3x+1}$

(d) $i(x) = \dfrac{1}{(2x^2-5)^2}$

(e) $j(x) = \sqrt[3]{2x+5}$

(f) $k(x) = e^{2x}$

(g) $l(x) = \ln(x^2+5)$

(h) $m(x) = (x^4 - 2x + 7)^5$

We may apply differentiation creatively. By definition, the exponential function and the logarithm functions are inverses of each other. This means that

$$\boxed{\ln(e^x) = x \quad \text{and} \quad e^{\ln y} = y} \tag{1.6}$$

for any real number x and any positive real number y. Differentiate both sides of the first equation. The result is

$$\frac{1}{e^x}\frac{d}{dx}e^x = 1.$$

We differentiated the left hand side using the chain rule, with outer function ln and inner function the exponential function. Bringing the e^x in the denominator to the right hand side of the equation results in

$$\frac{d}{dx}e^x = e^x. \tag{1.7}$$

In brief, we deduced the derivative of the exponential function from the derivative of the logarithm function. We could have saved us a line in Table 1.1.

Exercise 14. *Use the second equation in* (1.6) *and the formula for the derivative of the exponential function in* (1.7) *to deduce the formula for the derivative of the natural logarithm function.*

1.7 Differentiation Practice

In Table 1.1 we provided three derivatives. Combined with the rules of differentiation we can differentiate a lot of functions. All it takes is some practice. Here are some problems in which you may need to combine several of the rules.

Exercise 15. *Differentiate the following functions and evaluate the derivative at the given point a.*

(a) $f(x) = 2x - 1, \ a = 5$

(b) $f(x) = x^2 + x, \ a = 1$

(c) $f(x) = 4x^4 - \frac{1}{x}, \ a = 2$

(d) $f(x) = \sqrt{x}, \ a = 5$

(e) $f(x) = (x+2)\sqrt{x}, \ a = 2$

(f) $f(x) = 4x + 2x^2 - x^3, \ a = 1$

(g) $f(x) = \frac{x+1}{x-2}, \ a = 7$

(h) $f(x) = \frac{x-1}{(x+1)^2}, \ a = 2$

(i) $f(x) = \frac{\sqrt{x}}{x-1}, \ a = 2$

(j) $f(x) = e^x, \ a = 4$

(k) $f(x) = x \cdot e^x, \ a = 2$

(l) $f(x) = \frac{e^x}{x+1}, \ a = 1$

(m) $f(x) = \frac{\sqrt{x}}{e^x}, \ a = 2$

(n) $f(x) = \frac{x}{e^x - 2x}, \ a = 1$

(o) $f(x) = \frac{\sqrt{x}-1}{\sqrt{x}+7}, \ a = 2$

(p) $f(x) = (2x^2 + 7)^{\frac{1}{3}}, \ a = 1$

(q) $f(x) = \frac{x}{\sqrt{x^2 - 2}}, \ a = 3$

(r) $f(x) = \ln(7x - x^2), \ a = 1$

(s) $f(x) = (x + \ln(x))^{17}, \ a = 1$

(t) $f(x) = \frac{e^a - 1}{\sqrt{x^2+1}}, \ a = 1$

1.8 Second Derivatives

On occasion one would like to differentiate twice. One would like to know the derivative of the derivative. This is simple, one just differentiates twice. Newton's and Leibnitz's notation for the second derivative of a function $f(x)$, if defined, are

$$f''(x) = \frac{d^2}{dx^2}\, f(x).$$

Here are a few examples. Some computational steps are left to the reader.

$$
\begin{aligned}
\frac{d^2}{dx^2}\,(x^3+x^2+x+1) &= \frac{d}{dx}\,(3x^2+2x+1) \;=\; 6x+2 \\
\frac{d^2}{dx^2}\,(x^3-5x) &= \frac{d}{dx}\,(3x^2-5) \;=\; 6x \\
\frac{d^2}{dx^2}\,e^{5x} &= \frac{d}{dx}\,5e^{5x} \;=\; 25e^{5x} \\
\frac{d^2}{dx^2}\,\ln(3x) &= \frac{d}{dx}\left(\frac{1}{x}\right) \;=\; \frac{-1}{x^2} \\
\frac{d^2}{dx^2}\,xe^{3x} &= \frac{d}{dx}\,(1+3x)e^{3x} \;=\; (3+9x)e^{3x}
\end{aligned}
$$

Exercise 16. *Differentiate the following functions twice.*

(a) $f(x) = (3x-7)^{12}$ (b) $f(x) = \dfrac{2}{1+x^2}$

(c) $f(x) = e^{3x+1}$ (d) $f(x) = \dfrac{1}{(2x-5)^2}$

(e) $f(x) = \sqrt[3]{2x+5}$

(g) $f(x) = \ln(x^2+5)$ (f) $f(x) = e^{(x^2)}$

Chapter 2

Applications of the Derivative

2.1 Shape of a Graph

We will discuss a number of features of a graph that are significant in applications. We will point them out in a specific graph as well as introduce them formally.

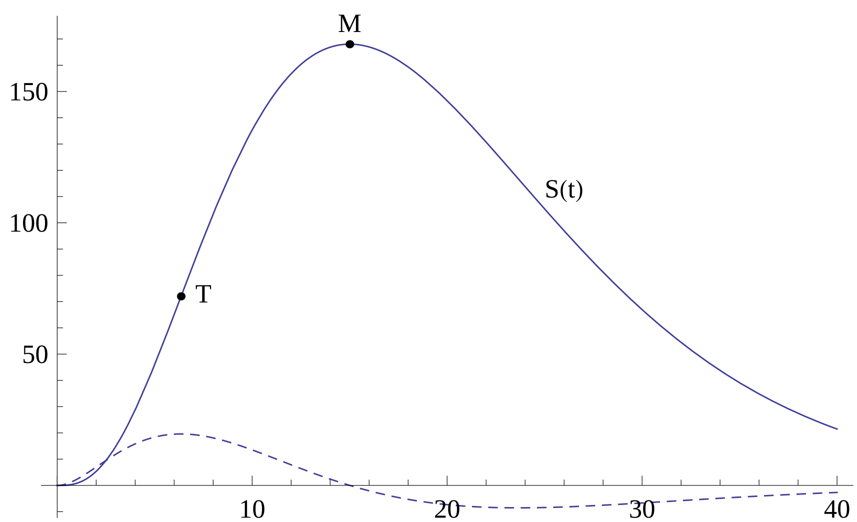

Figure 2.1: Flu Epidemic

The solid graph in Figure 2.1 shows how many people are sick with the flu. This is a function of time (days since the epidemic started). Let us call it $S(t)$. The number of sick people is given per 1000. The dotted graph shows the derivative, $S'(t)$.

2.1.1 Monotonicity

For a while more and more people get sick, or $S(t)$ is increasing. This is true for t between 0 and 15. Spelled out mathematically,

Definition 2.1. *We say that a function $f(x)$* increases*, resp.* decreases*, on an interval I if whenever $t_0 < t_1$ and $t_1,\ t_2 \in I$ then $f(t_1) \leq f(t_2)$, resp., $f(t_1) \geq f(t_2)$.*

Using mathematical symbols instead of common language:

$$t_1, t_2 \in I \text{ and } t_1 < t_2 \implies f(t_1) \leq f(t_2),\ \text{resp., } f(t_1) \geq f(t_2). \tag{2.1}$$

To discuss the *monotonicity properties* of a graph means to find intervals on which the function increases, resp., decreases. Returning to the function $S(t)$ in Figure 2.1, we observe that it decreases from $t = 15$ on, for as long as the graph provides us information.

We may use the first derivative of a function to detect monotonicity.

Proposition 2.2. *Suppose that the function $f(x)$ is defined and differentiable on an open interval (a, b). If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ increases on (a, b). If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ decreases on (a, b).*

The proposition is easy to agree with. If you go uphill for a while, then eventually you are on higher ground than you were to begin with. If water flows downhill for awhile, then eventually it will end up further down. So Escher's pictures are deceptive[1].

Exercise 17. *Verify for the function shown in Figure 2.1 that the function $S(t)$ is increasing on intervals on which $S'(t)$ is positive, and that $S(t)$ is decreasing on intervals on which $S'(t)$ is negative.*

[1] If you are not familiar with sequences of waterfalls that return to their starting point, then you can enjoy them on the internet.

2.1.2 Local Extrema

Apparently the point M on the graph is of interest. The epidemic appears to be worst on day 15. About 168 people are sick per 1000. At this time, or compared to other days, this is the largest number of sick people. We say that $S(t)$ has a local maximum at $t_0 = 15$. In some open interval around 15 the value of S at $t_0 = 15$ is the largest[2]. Formally,

Definition 2.3. *We say that $f(x)$ has a* local maximum, *resp.,* local minimum *at a point c if $f(x)$ is defined in an open interval (a,b) that contains c and*

$$\boxed{f(c) \geq f(x) \text{ resp., } f(c) \leq f(x)}$$

for all x in (a,b). Local extremum refers to a local maximum or minimum.

Remark 2. Primarily we are interested in the x–value where local extrema occurs. Eventually, we would also like to know the local exremum, i.e., the value of the function at that point.

We like to use calculus to detect local extrema. It is rather apparent that

Proposition 2.4. *If the function $f(x)$ is differentiable on an interval (a,b) and if it has a local extremum at a point c in (a,b), then the derivative of $f(x)$ has to vanish at c, i.e., $f'(c) = 0$.*

This motivates us to make the following definition.

Definition 2.5. *Let $f(x)$ be defined on an interval (a,b). We call a point c in (a,b) a* critical point *of $f(x)$ if $f(x)$ is differentiable at c and $f'(c) = 0$, or if $f(x)$ is not differentiable at c*

Using this language,

Proposition 2.6. *If a function has a local extremum at c then c is a critical point.*

Both Propositions, 2.4 and 2.6, are necessary conditions for a function to have a local extremum at a point. They do not tell us that the function has a local extremum at the point under consideration. The functions in Figure 2.2 and 2.3 both have a critical point at $x = 0$, but only the first one has a local extremum. The assertion is that if there is a local extremum

[2]Sure, during next year's flu season things could get worse, but right now we only worry about values of t near 15.

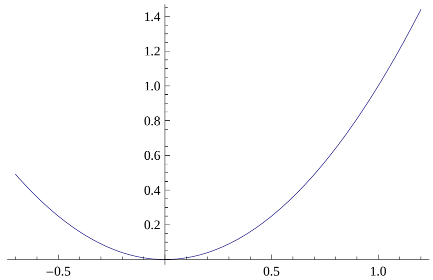

Figure 2.2: $f(x) = x^2$

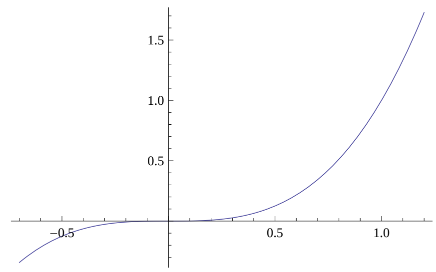

Figure 2.3: $f(x) = x^3$

at the point, then the point must be a critical point. If you are looking for local extrema, then study the critical points.

The *Second Derivative Test* provides us with a *sufficient* condition:

Proposition 2.7. *Suppose that $f(x)$ is defined on an open interval (a, b), and that $f(x)$ is differentiable at the point $c \in (a, b)$. If $f'(c) = 0$ and $f''(c) \neq 0$ then $f(x)$ has a local extremum at c. If $f''(c) < 0$, then the local extremum is a maximum. If $f''(c) > 0$, then the local extremum is a minimum.*

If $f'(c) = 0$ and $f''(c) = 0$ is not defined, then anything can happen. As shown in Figures 2.4 and 2.5, if $f(x) = x^4$ or $f(x) = x^5$, then $f'(0) = f''(0) = 0$. In the first case there is a local minimum at $x = 0$, in the second case there is no local extremum at $x = 0$.

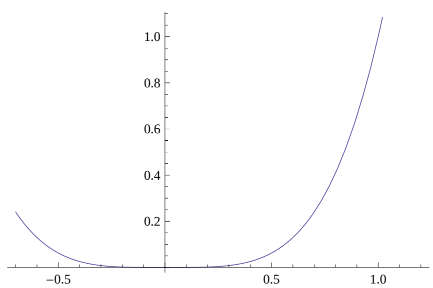

Figure 2.4: $f(x) = x^4$

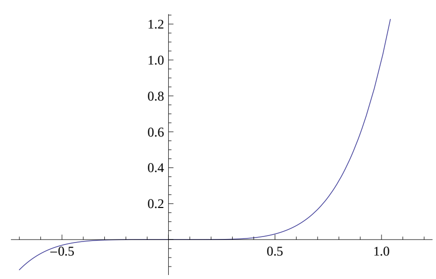

Figure 2.5: $f(x) = x^5$

Let us dramatize the distinction between the necessary and sufficient condition[3]. Throughout, f is defined and twice differentiable on an interval (a,b) and c is a point in the interval.

$$\begin{aligned} f \text{ has local extremum at } c &\implies f'(c)=0\\ f \text{ has local extremum at } c &\not\Longleftarrow f'(c)=0\\ f'(c)=0 \;\&\; f''(c)\neq 0 &\implies f \text{ has local extremum at } c\\ f'(c)=0 \;\&\; f''(c)\neq 0 &\not\Longleftarrow f \text{ has local extremum at } c \end{aligned}$$

2.1.3 Concavity

Let us return to the discussion of the spread of the flu. Let us look at the first week of the spread of the disease, see Figure 2.6. Looking at the graph of either function, $S(t)$ or $S'(t)$, you see that the flu spreads at an increasing rate. The rate at which more people come down with the flu increases, or $S'(t)$ is increasing and the graph of $S(t)$ goes up ever more steeply. This makes sense insofar as more carriers of the germs can infect more healthy people. The medical professionals are concerned that the disease is spinning out off control.

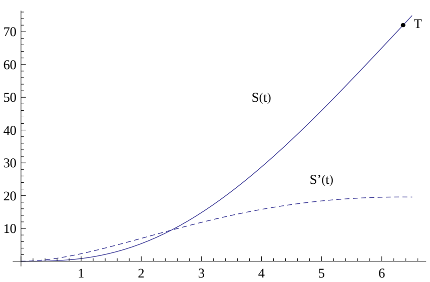

Figure 2.6: Increasing Rate

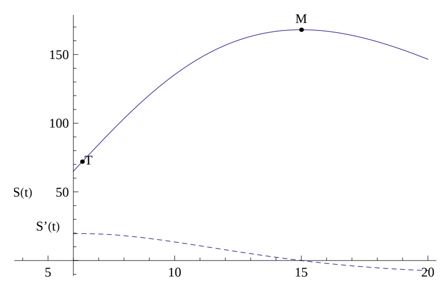

Figure 2.7: Decreasing Rate

As we get into the second week of the outbreak of the flu you see that the rate at which it streads starts to decrease. The flu still spreads, the total number of sick people is still increasing, but it does so at a decreasing rate. The dotted graph in Figure 2.7, which signifies the rate at which the flu is spreading has started to come down. Barring unforseen events, health officials show signs of relief. The flu has come under control. If the rate at which the flu spreads continues to decline, then it will eventually turn

[3]For those unfamiliar with the notation, $A \implies B$ means that A implies B and $A \not\Longrightarrow B$ means that A does not imply B.

negative, at which point the number of sick people will also start to come down.

In retrospect, looking back at the graph in Figure 2.1, the important point on the graph is T. Until this moment the flu is spreading faster and faster; once we are past T the spreading slows, comes to a halt once we get to M, and thereafter it goes away. It is at the point T that health officials see the light at the end of the tunnel.

Motivated by this application we discuss the mathematical background.

Definition 2.8. *Let the function $f(x)$ be defined on an interval I. We say that the function is* concave up *on I if the line segment joining any two points of the graph lies above the graph. We say that $f(x)$ is* concave down *on I if the line segment joining any two points of the graph lies below the graph.*

Alternatively we can consider the property locally.

Definition 2.9. *Let the function $f(x)$ be defined on an interval I. We say that the function is* concave up at a point c in I *if there is a line*[4] *$L(x)$, so that $L(c) = f(c)$ and $L(x) \leq f(x)$ for all x in some open interval (a, b) that contains c. We say that the function is* concave down at a point c in I *if there is a line $L(x)$, so that $L(c) = f(c)$ and $L(x) \geq f(x)$ for all x in some open interval (a, b) that contains c.*

Calculus allows us the determine where a function is concave up or down. For the first characterization we have:

Proposition 2.10. *If a function is twice differentiable on an interval I and $f''(x) > 0$ for all x in I, then $f(x)$ is concave up on I. If $f''(x) < 0$ for all x in I, then $f(x)$ is concave down on I.*

For the local desription we have

Proposition 2.11. *If $f''(c) > 0$, then f is concave up at c. If $f''(c) < 0$, then f is concave down at c.*

Definition 2.12. *Points at which the a function changes concavity (from up to down or from down to up) are called* inflection points.

It is relevant to point out

[4]It is costomary to call $L(x)$ a support line.

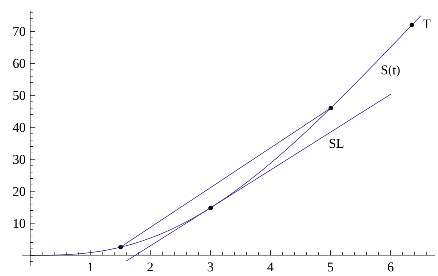

Figure 2.8: Concave Up

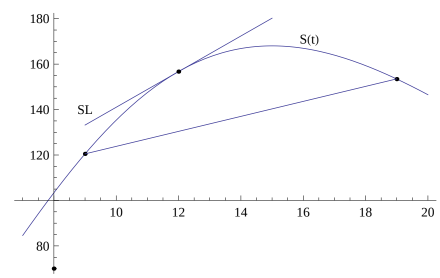

Figure 2.9: Concave Down

Proposition 2.13. *If $f(x)$ is differentiable and concave up or down at c, then the support line is the tangent line at c.*

In Figure 2.8 you see again the early days of the flu function, during which it is concave up. One secant line joining two points on the graph is indicated. In between the points the line is above the graph. There is also a support line (SL). It has the point at day 3 in common with the graph, and otherwise it is below the graph. In Figure 2.9 you see the later days of the flu graph when it is concave down. Again you also see one of the secant lines that lies below the graph and one support line that lies above the graph.

Exercise 18. *For the graph shown in Figure 2.10, find the intervals on which the function is increasing, resp., decreasing, and the local extrema. Find the intervals on which the function is concave up, resp., down, and the inflection points.*

2.2 Absolute Extrema

In many applications you are interested in the absolute minimum or maximum of a function, and not simply the local ones. You are looking for the largest and smallest values, and where they occur. If you consider the graph in Figure 2.10, you would be correct to say that on the interval $[-1.4, 3.2]$, the function assumes it maximum of about 3.1 at approximately $x = -.2$ and its absolute minimum of about -4.2 at -1.4. To avoid any misunderstanding, we make the concept precise.

Definition 2.14. *Suppose that the function $f(x)$ is defined*[5] *on D. We say that $f(x)$ attains its absolute maximum at $a \in D$ if $f(a) \geq f(x)$ for all x in*

[5] Expressed differently, the domain of the function is D.

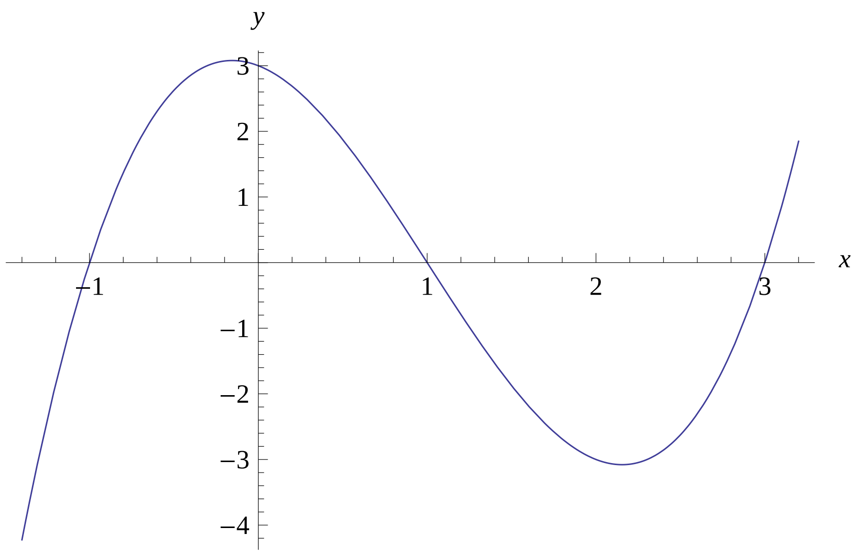

Figure 2.10: Sample Graph

D. We then call $f(a)$ the absolute maximum of f on D. We say that $f(x)$ attains its absolute minimum at $a \in D$ if $f(a) \le f(x)$ for all x in D. We then call $f(a)$ the absolute minimum of f.

In general, a function does not need to have an absolute minimum or maximum. For example, the function $f(x) = 1/x$ does not have an absolute minimum or maximum on $(0, \infty)$. In some important cases one is assured of their existence, The *Absolute Value Theorem* says:

Theorem 2.15. *A function $f(x)$, that is defined and differentiable*[6] *on a closed interval $[a, b]$, assumes its absolute minimum and maximum on the interval.*

Stated differently, for a differentiable function on a closed interval there is a point in this interval where the value of f is largest, and there is a point where the value is smallest. Let us consider an example.

Example 2.16. Take a string of length 100 cm and cut it into two pieces. Use one piece as the perimeter of a square and one as the perimeter of a circle. How should you cut the string so that the combined area of the circle is maximal and how should you cut it so that it is minimal?

[6]See Remark 1.

Solution: Throughout, the units of length are centimeters and the units of area are square centimeters. A square with a side length a has a perimeter of length $4a$ and an area a^2. A disk of radius r has a perimeter of $2\pi r$ and an area of πr^2. As we started out with a string of length 100 cm we find that

$$4a + 2\pi r = 100 \quad \text{or} \quad r = \frac{1}{2\pi}(100 - 4a),$$

and the combined area of the square and the disk is

$$A(a) = a^2 + \pi r^2 = a^2 + \frac{1}{4\pi}(100 - 4a)^2.$$

The variable a varies between $a = 0$, in which case all of the string is used as the perimeter of the disk, and $a = 25$, when all of it is used as the perimeter of the square. The graph of $A(a)$, the combined area of the square and disk, as a function of the side length of the square, is shown in Figure 2.11.

As you see, the combined area has an absolute maximum when $a = 0$, i.e., when we use all of the string for the disk. In this situation the area is $A(0) \sim 795.77$.

The combined area has a local and an absolute minimum when a is approximately 14 cm. Let us obtain the result computationally.

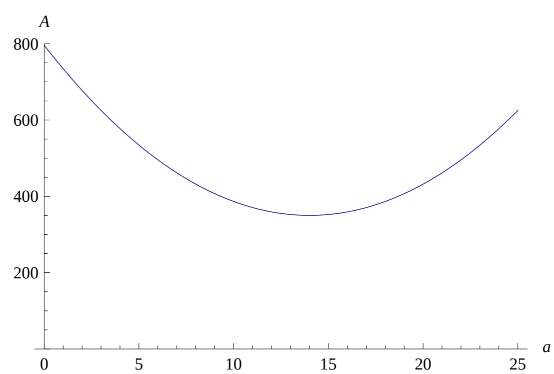

Figure 2.11: Disk and Square

The reader should verify the steps of the calculation. To find the critical points of $A(a)$ we differentiate the function:

$$A'(a) = 2a - \frac{2}{\pi}(100 - 4a).$$

We find a critical point where $A'(a) = 0$, or $a = 100/(4+\pi) \sim 14$. We could just say that $A(a)$ is a parabola that is open upwards, and its vertex and minimum is the critical point that we just found. We may also compute the second derivative, $A''(a) = 2 + 8/\pi > 0$. The function is concave up on the entire interval, and the critical point is a local minimum. A variety of arguments tell us that the critical point gives us the absolute minimum, the easiest of which is the comparison of the values at the end points of the interval and the critical point. When $a = 100/(4 + \pi)$, then $A \sim 350$ square centimeters, and this is the minimum value for $A(a)$.

We were also asked to find the absolute maximum of the function $A(x)$. It will occur at a critical point of the function or at an endpoint of the interval. At the right endpoint a will be 25 centimeters and the area of the square is 625 square centimeters. This is less than what we get when all of the string is used for a circle bounding a disk, whose area would be $A(0) \sim 795.77$ square centimeters. The value at the only critical point is out of contention. At this point we have a local minimum and certainly not a local maximum. We conclude that the absolute maximum occurs at $a = 0$, when all of the string is used for the perimeter of the disk.

2.3 Discussion of a graph

In the discussion of a graph we try to exhibit its essential features. We expect to learn enough about it so that we can draw the graph rather accurately. They include

- Incercepts and invervals on which the sign of the function is either positive or negative.
- Critical points and intervals on which the function increases or decreases, as well as local extrema.
- Intervals on which the function is concave up or down and inflection points.

Example 2.17. Discuss the function

$$f(x) = x^3 - 3x^2 + 4.$$

As a first step we like to calculate intercepts. We see right away that the graph intersects the y–axis in $(0, 4)$. You may also verify that $x = -1$ is a root, so that $(x + 1)$ is a factor of $x^3 - 3x^2 + 4$. Long division allows us to find the cofactor, and we conclude that

$$x^3 - 3x^2 + 4 = (x + 1)(x^2 - 4x + 4) = (x + 1)(x - 2)^2.$$

This tells us that the function has a zero at $x = -1$, and the graph crosses the axis at this point. There is a double zero at $x = 2$. The graph touches the axis without crossing it.

We use the derivative

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

to find additional information about $f(x)$. The zeros of $f'(x)$ are at $x = 0$ and $x = 2$, and these are the critical points of $f(x)$. Looking where the factors of $f'(x)$ change signs[7], we conclude that $f'(x)$ is positive on the intervals $(-\infty, 0)$ and $(2, \infty)$, and negative on the interval $(0, 2)$. We deduce that $f(x)$ increases on the interval $[-\infty, 0]$ as well as on the interval $[2, \infty]$. The function is decreasing on the interval $[0, 2]$.

The function has a local maximum at $x = 0$, because to the left of this point the function increases, and to the right of it the function decreases, at least for a while. Similarly, we deduce that the function has a local minimum at $x = 0$. To facilitate the graphing, we compute that the local maximum is $(0, 4)$ and the local minimum is $(2, 0)$.

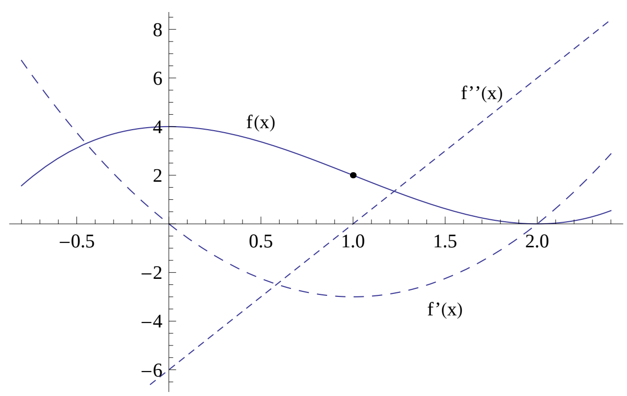

Figure 2.12: Graph of $f(x)$, $f'(x)$, and $f''(x)$.

Next we extract information from the second derivative, $f''(x) = 6x - 6$. This function is a line, it has a zero at $x = 1$, and it changes signs from negative to positive at $x = 1$. This tells us that $f''(x) < 0$ on $(-\infty, 1)$ and that $f(x)$ is concave down on $(-\infty, 1]$. Also, $f''(x) > 0$ on $(1, \infty)$ and $f(x)$ is

[7] Alternatively, we could say that $f(x)$ is a parabola which is open upwards and has zeros at $x = 0$ and at $x = 2$.

concave on $[1,\infty)$. Concavity changes at $x=1$, so that there is an inflection point at $x=1$. The inflection point is $(1,2)$.

By now we have ample information to sketch the graph rather accurately. It is shown as a solid line in Figure 2.12. In the figure we included the graph of $f'(x)$ and $f''(x)$ using long and short dotted lines. The inflection point is indicated by a dot.

Exercise 19. *Discuss the function* $f(x) = x^3 - 3x - 2$.

Example 2.18. Discuss the function

$$f(x) = \frac{x^2-4}{x^2+1}.$$

We start out with a few elementary observations. The function is even, $f(x) = f(-x)$, and its graph is symmetric about the y–axis.

The function can be written as $f(x) = (x-2)(x+2)/(x^2+1)$. The numerator, and with it the function, is zero if $x = \pm 2$. This provides us with the x–inercepts. The y–intercept is $f(0) = -4$.

Let us determine the sign of $f(x)$. We consider the factors $(x-2)$, $(x+2)$, and $1/(1+x^2)$ separately. The factor $(x-2)$ changes signs (from negative to positive) at $x=2$. The factor $(x+2)$ changes from negative to positive at $x=-2$. The product $(x-2)(x+2)$ changes sign from positive to negative at $x=-2$, and it again changes from negative to positive at $x=2$. The denominator $(1+x^2)$ is always positive, and the factor $1/(1+x^2)$ does not alter the sign of $f(x)$. As a result, $f(x)$ is positive for $x \in (-\infty,-2)$, negative for $x \in (-2,2)$ and positive for $x \in (2,\infty)$.

We may also note that $f(x) < 1$ for all x because $x^2 - 4 < x^2 + 1$.

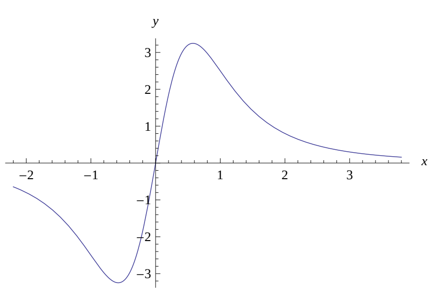

Figure 2.13: First Derivative

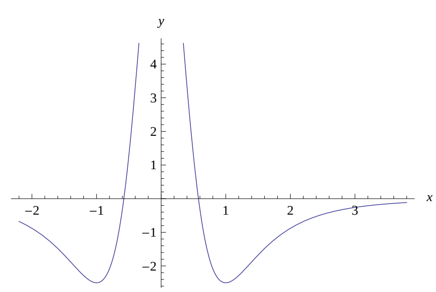

Figure 2.14: Second Derivative

The first and second derivative of the function are:

$$f'(x) = \frac{10x}{(x^2+1)^2} \quad \text{and} \quad f''(x) = \frac{10(1-3x^2)}{(x^2+1)^3}.$$

The reader is encouraged to compute these derivatives with the help of the differentiation formulae. For our discussion we are willing to simply assume them. We provided the graphs of the first and second derivative of $f(x)$ in Figures 2.13 and 2.14. The graphs show more than what we computed, but they are useful when you like to review all of the concepts that enter into the discussion of a graph.

The derivative, $f'(x)$, vanishes at $x = 0$, and this is the only critical point for the function.

Apparently, $f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$. This means that $f(x)$ is decreasing when $x \leq 0$ and increasing when $x \geq 0$. There is also a local minimum at $x = 0$, because up to this point the function decreases and thereafter it increases.

The second derivative is zero if and only if

$$(1-3x^2) = (1-\sqrt{3}x)(1+\sqrt{3}x) = 0.$$

The factors are changing signs at $x = \pm 1/\sqrt{3}$, and the product changes signs from negative to positive and then back to negative at these two points. We conclude that $f''(x)$ is negative on the intervals $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$, and that $f(x)$ is concave down on each of these intervals. On the other hand, $f''(x)$ is positive on the interval $(-1/\sqrt{3}, 1/\sqrt{3})$, and $f(x)$ is concave up on the interval. As we have changes of concavity at $x = \pm 1/\sqrt{3}$, these two points are the inflection points of the function.

In Figure 2.15 we gathered all of the results that we computed previously and displayed it graphically. The inflection points are indicated as dots.

Exercise 20. *Discuss the following function. For convenience, the first and second derivatives are given.*

$$f(x) = \frac{x}{x^2+1}, \text{ with } f'(x) = \frac{1-x^2}{(x^2+1)^2} \text{ and } f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

2.4 Inventory Model

We are going to discuss a simple inventory model and use calculus to minimize the cost to maintain the inventory. The ordering quantity that minimizes the inventory cost is referred to as the *economical ordering quantity*, or EOQ.

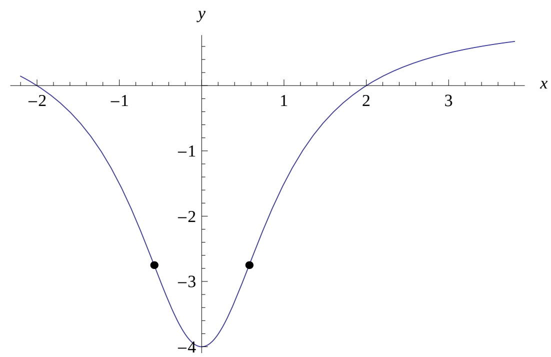

Figure 2.15: Graph of $f(x)$

Suppose that you own a grocery store and you stock rice. It comes in 10 kg bags (about 22 lb). The demand is constant all year round. You plan to order the same number of bags whenever you place an order. You place the order so that the new order comes in at the time when your inventory is depleted.

The catch is that there is a certain charge whenever you place an order. It covers the processing fee for the order, the expense to deliver it, and other related expenses.

This is not all. Keeping the bags of rice in storage entails an expense as well. There may be a fixed cost to have any storage, and there will be a per bag expense. The latter may include, but not be limited to, the expense of having money tied up, or having to borrow it, spoilage, and insurance.

We need to introduce notation before we can express ourselves. Let us say that the annual demand is N bags of rice and we order x bags at a time. We let $I(t)$ denote the number of bags of rice in storage at time t. Denote the cost for placing one order by p. By assumption, this expense does not depend on the order size. Let h be the cost of keeping one bag of rice in storage for one year. It is referred to as the *holding cost* or *storage cost*.

In Figure 2.16 you see the development of the inventory over the course of one year for an annual demand of 3,000 bags of rice and an order size of 750 bags. There are 4 orders per year. Each time the inventory goes up to a value of 750 bags, and then it gets depleted at a constant rate over a course of 3 months.

The inventory cost $C(x)$ has two components[8], the *ordering cost* and the

[8]There may be additional expenses that do not depend on the order size. They would

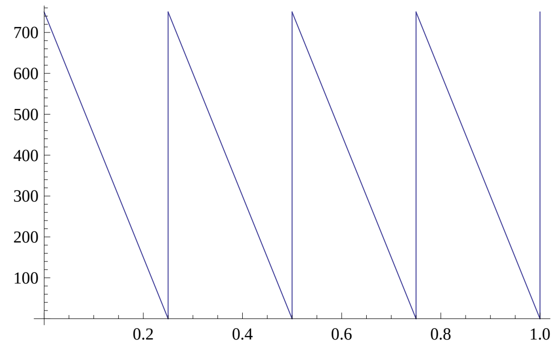

Figure 2.16: Inventory over the course of a year.

storage cost. We are placing N/x orders at an expense of Np/x dollars per year. On the average, $x/2$ bags of rice are in storage. The annual cost for this is $xh/2$ dollars per year. With this, the inventory cost is

$$C(x) = \frac{Np}{x} + \frac{xh}{2}.$$

If the annual demand is 3,000 bags, if we place 4 orders a year, if it costs \$200.00 to place an order, and if the annual holding cost is \$2.00 per bag, then the storage cost (in dollars) will be

$$C(750) = 800 + 750 = 1,550.$$

Exercise 21. *Compare the ordering cost when you place 3, 4, or 6 orders per year.*

Exercise 22. *Repeat the previous exercise with a holding cost of \$4.00 per bag and year.*

With our specific numbers, the different costs, as functions of the order quantity, are shown in Figure 2.17. The short dashed line is the graph of the holding cost. The longer dashes show the order cost. The solid line is the sum, the graph of the total inventory cost.

We wish to find the order quantity x that minimizes the inventory cost. For this purpose we differentiate $C(x)$:

$$C'(x) = -\frac{Np}{x^2} + \frac{h}{2}.$$

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contribute a constant summand to $C(x)$. The outcome of our discussion is not affected by such a term, as the reader should verify.

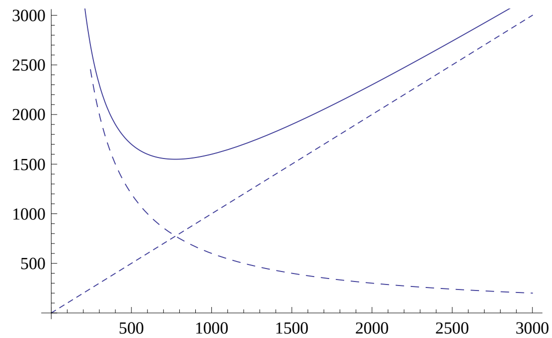

Figure 2.17: Various Costs

If $C'(x) = 0$ then (only the positive root is of relevance):

$$\frac{Np}{x^2} = \frac{h}{2} \quad \text{or} \quad x = \sqrt{\frac{2Np}{h}}$$

The function has only one critical point on the positive x–axis. It is apparent that the critical point is a local minimum. One may also check this by calculating $C''(x) = 2Np/x^3$, which will be positive for positive x. The Second Derivative Test tells us that the function has a local minimum at the point.

The function is differentiable and decreasing up to the critical point and increasing thereafter. This implies that the local minimum is also the global one on $(0, \infty)$. In particular, the economical ordering quantity is:

$$\boxed{EOQ = \sqrt{\frac{2Np}{h}}}$$

For those who are curious, the minimal cost to maintain the inventory is $C(EQO) = \sqrt{2Nph}$.

In the numerical example, $EOQ \sim 775$ and $C(EQO) \sim \$1549.20$. In essence, the strategy of ordering four times a year was almost optimal.

Exercise 23. *Work out the EOQ and the inventory cost if the annual demand is 800 cans of yellow corn, the holding cost per can is \$.30 per year, and it costs \$20.00 to place an order.*

2.5 Elasticity

At times, you only care about the revenue that your business generates. Say you sell pet rocks that you collect for free on the beach. If you set the price too high, then nobody buys them and you make no money. If you give them away for free, then your business may flourish, but you still make no money. Your best strategy will lie somewhere in between.

Exercise 24. *Describe additional business situations where the profit depends entirely on the revenue.*

As a first step, we need to understand the relation between the price of a commodity[9] and the demand for it. We describe such a relation as a function. If we sell x items when the price per item is p, then we might write

$$x = f(p) \quad \text{or} \quad p = g(x) \quad \text{or} \quad h(x,p) = 0. \tag{2.2}$$

As an example consider:

$$x = 100(150 - p) \quad \text{or} \quad p = 150 - \frac{x}{100} \quad \text{or} \quad 100p + x - 15,000 = 0. \tag{2.3}$$

The reader should verify that each of the equations is simply a reformulation of the other ones.

To make it concrete, let us say that we are selling high fashion jeans, and that their price p is in dollars.

The first equation in (2.2) or (2.3) is called the *demand equation*. It describes the demand as a function of the price. The second one tells us how to price the item so that we will have a demand of x items. The third equation expresses the relation between price and demand. The graph in Figure 2.18 shows all three of the functions. Once the independent variable is on the horizontal axis, once on the vertical one. As for the third equation, the points on the line satisfy the equation.

We were interested in the revenue, which is the product of the number of items sold and the per item price:

$$\boxed{R = xp}$$

If we view the demand as a function of the price, then the revenue is a function of the price, $R(p) = x(p)p$. Alternatively, $R(x) = xp(x)$ if we

[9] Commodity is the generic term for any marketable item produced to satisfy wants or needs.

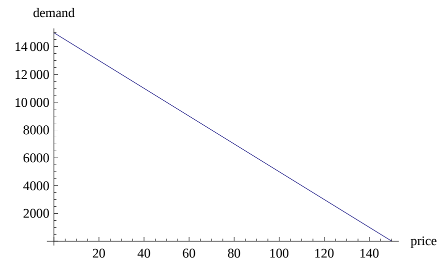

Figure 2.18: Price & Demand

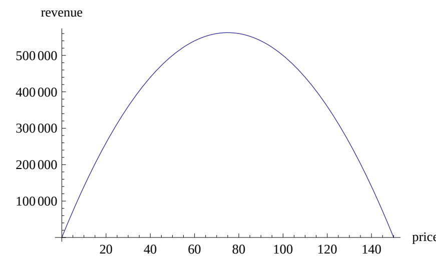

Figure 2.19: Revenue

make the price dependent on the number of items that we like to sell. In Figure 2.19 you see the revenue as a function of the price.

In economics and business one considers the *elasticity of demand*. For a given demand function $x = f(p)$ one sets[10]

$$\boxed{E(p) = -\frac{pf'(p)}{f(p)}.}$$

Proposition 2.19. *Under reasonable assumptions*[11] *on the function $f(p)$ the quantity $E(p)$ will be positive and*

- *If $0 < E(p) < 1$, then the revenue increases as we increase the price. We say that the demand is* inelastic.
- *If $E(p) > 1$, then the revenue decreases as we increase the price. We say that the demand is* elastic.

Let us return to the example. In this case $x = f(p) = 100(150 - p)$. To assure a positive demand we suppose that $0 < p < 150$.

Exercise 25. *Calculate the demand and the revenue when the price is \$50.00 and \$51.00. Does the revenue increase or decrease as you increase the price from \$50.00 to \$51.00?*

You may check that $f'(p) = -100$. Then (please do the arithmetic)

$$E(p) = \frac{100p}{100(150-p)} = \frac{p}{150-p}.$$

[10] Throughout our discussion we assume that $f(p)$ is a differentiable, positive function.

[11] E.g., in the price range under consideration the demand should be positive and decrease as the price increases.

We note that $E = \frac{p}{150-p} < 1$ if and only if $p < 150-p$ and $p < 75$. Similarly, $E > 1$ if and only if $p > 75$. The demand will be inelastic if $p < 75$ and elastic if $p > 75$. In Figure 2.19 you can see that the revenue increases, as a function of the price, if the price is less that \$75.00 and it decreases once it exceeds \$75.00.

Exercise 26. *The current toll for the use of the North-West Toll Road is \$2.50. A study conducted by the highway department determined that q cars will use the road per day if the toll is p dollars, where*

$$q = 60,000e^{-.5p}.$$

1. *Compute the elasticity of demand when $p = 2.5$ dollars.*

2. *Is the demand elastic or inelastic at $p = 2.5$ dollars?*

3. *Will the transit authority collect more or less money if the toll is raised to \$2.60?*

2.5.1 Discussion of the formula for E

Let us study why the formula for the elasticity is right. By the product rule

$$\frac{dR}{dp} = \frac{d}{dp}[pf(p)] = p'f(p) + pf'(p) = f(p) + pf'(p).$$

The revenue is increasing if $R'(p) = f(p) + pf'(p) > 0$. This is the same as saying that $pf'(p) > -f(p)$. Because we assumed that $f(p) > 0$, this is equivalent to

$$\frac{pf'(p)}{f(p)} > -1 \quad \text{or} \quad E(p) = -\frac{pf'(p)}{f(p)} < 1.$$

In summary, $R'(p) > 0$ if and only if $E(p) < 1$.

Exercise 27. *Show that $R'(p) < 0$ if and only if $E(p) > 1$.*

Let us give another explanation for E. Given any differentiable function $h(p)$, we can consider the *relative rate of change* $\alpha(p)$ of $h(p)$:

$$\boxed{\alpha(p) = \frac{h'(p)}{h(p)}.}$$

As an example, consider a country with a population of 10 million people. If the population grows at a rate of 20 thousand people annually, then the population grows at a relative rate of

$$\frac{20,000}{10,000,000} = \frac{.2}{100} = .2\%.$$

We return to the revenue discussion. Let the relative rate of change for the demand $f(p)$, as a function of the price p, be $\alpha(p) = f'(p)/f(p)$. Let he relative rate of change for the price be $\beta(p) = p'/p = 1/p$. By definition,

$$E(p) = -\frac{\alpha(p)}{\beta(p)} = -\frac{pf'(p)}{f(p)}$$

The negative sign in the formula accounts for replacing increasing with decreasing. If $E > 1$, the the demand decreases at a faster relative rate than the price increases, and this means that the revenue drops. In Proposition 2.19 we called this behavior of the demand elastic. If the demands drops at a slower relative rate than the price increases, i.e., $E(p) < 1$, then the revenue increases as the price increases. In Proposition 2.19 we said that the demand is inelastic.

Exercise 28. *Which demand curves are revenue neutral? Formulated differently, for which demand curves is the revenue independent of the price?*

Solution: A technical way to formulate the question in the exercise would be to ask: Find demand curves $f(p)$ so that $E(p) = 1$. That is the same as requiring

$$f'(p) = -pf(p).$$

You may check that $f(p) = Ce^{-p^2/2}$ is a solution for each constant C.

Chapter 3

Integration

We will discuss two concepts, the definite and the indefinite integral. Eventually we will relate differentiation and integration in the Fundemtal Theorem of Calculus. In the basic setting the definite integral measures the area under a graph. It is a number. The indefinite integral is a family of functions. If $f(x)$ is defined on an interval, then its definite integral is the family of all functions whose derivative is $f(x)$.

3.1 Antiderivatives

We start out with the basic definition.

Definition 3.1. *Let $f(x)$ be a function that is defined on an open interval*[1] *(a, b). We say that $F(x)$ is an* antiderivative *of $f(x)$ if $F(x)$ is differentiable on (a, b) and $F'(x) = f(x)$. The set of all antiderivatives of $f(x)$ is called the* indefinite integral *of $f(x)$ and it is denoted by*

$$\int f(x)\, dx.$$

Here is a basic example. If $f(x) = 2x$, then $F(x) = x^2$ is an antiderivative of $f(x)$ and

$$\int 2x\, dx = x^2 + c$$

is the indefinite integral. The letter c stands for a constant and $x^2 + c$ is a family of function, one for each value of c.

[1] We allow the a to be $-\infty$ and b to be ∞. One may extend the definition to functions that are defined on other intervals by demanding extensions of the data to open intervals.

Remark 3. Why do we write $+c$? Given one antiderivative, we asserted that one finds all antiderivatives by simply adding a constant. Why is this? If two functions differ by a constant, then they have the same derivative. But we are asserting the opposite. If two functions have the same derivative, then they differ by a constant. This is true on intervals. It is a consequence of Cauchy's Mean Value Theorem[2]

Earlier we made a short table with a few important derivatives. Reversing the differentiations we obtain a table with antiderivatives.

$y(x)$	$\int y(x)\,dx$	Domain	Remark
x^α	$\frac{1}{\alpha+1}x^{\alpha+1}+c$	$(0,\infty)$ or $(-\infty,\infty)$	$\alpha \neq -1$
e^{ax}	$\frac{1}{a}e^{ax}+c$	(∞,∞)	
$1/(x+a)$	$\ln(x+a)+c$	$(-a,\infty)$	

Table 3.1: Some Derivatives

One obtains more antiderivatives if one uses the rules of differentiation. E.g., linearity of the derivative (see (1.6.1)) implies the linearity of antiderivatives:

$$\boxed{\int [af(x)+bg(x)]\ dx = a\int f(x)\,dx + b\int g(x)\,dx} \tag{3.1}$$

Example 3.2. As an explicit example we might write down

$$\int 7e^{2x} - 2x^2\,dx = \frac{7}{2}e^{2x} - \frac{2}{3}x^3 + c.$$

Reversing the chain rule leads to a technique called u–substitution. We won't introduce it, but here is an example where it could be employed.

Example 3.3. Find the indefinite integral of $\sqrt{2-5x}$:

$$\int \sqrt{2-5x}\,dx = \int (2-5x)^{\frac{1}{2}}\,dx = \frac{2}{3}\,\frac{-1}{5}(2-5x)^{\frac{3}{2}} + c = \frac{-2}{15}(2-5x)^{\frac{3}{2}} + c.$$

The reader should verify the correctness by differentiating the right hand side of the equation.

[2]The interested reader may find the theorem stated in any standard calculus book. You can also google it. A proof is typically given only in a junior level analysis course.

It is a good stratey to simply make an educated guess as to what an antiderivative $F(x)$ of a given function $f(x)$ might look like. Then one differentiates $F(x)$ and hopes to end up with $f(x)$. If it does not quite work out, one hopefully sees how to adjust $F(x)$ to get the antiderivative right. Let us give an

Example 3.4. Let us find an antiderivative of $f(x) = x\sqrt{2+x^2}$. Rewrite the function as $f(x) = x(2+x^2)^{\frac{1}{2}}$. A factor $2x$ would look like an inner derivative of the expression under the radical. As a first approximation we try $F(x) = (2+x^2)^{\frac{3}{2}}$. The derivative would be $F'(x) = \frac{3}{2}2x\sqrt{2+x^2}$. We are off by a factor 3. The correct answer should be

$$F(x) = \frac{1}{3}(2+x^2)^{\frac{3}{2}} \quad \text{and} \quad \int x\sqrt{2+x^2}\,dx = \frac{1}{3}(2+x^2)^{\frac{3}{2}} + c.$$

We will gain more familiarity with antiderivatives as we apply the concept.

3.2 Area under a Graph

As the first step towards the definite integral, we like to discuss the area of the region under the graph of a nonnegative function. As an example, use Figure 3.1.

Think of the region as a plot of land. The street front is straight and 150 feet long. The property lines on the left and right are straight and perpendicular to the street front. The back side is a little irregular. Mr. S tries to sell the plot to Mr. B. They have agreed on a price of \$10.00 per square foot, they only need to figure out the exact size of the piece of land. Obviously, Mr. S wants to get as much money as possible, and Mr. B would like to pay as little as possible.

In his opening proposition, Mr. S suggests that at one point the plot is almost 155 feet deep and suggests a price of \$232.500.00. Mr. B counters by saying that at some place the lot is barely 50 feet deep, and the price should not be more than \$75,000.00.

Buyer and seller agree that they can't just pretend that the back of the property is a straight line, parallel to the street. They need to take the irrgular property line into acount. As a first compromise, the seller would be glad to be paid for the shaded region shown in Figure 3.2. The street front is divided into smaller pieces. On the first 30 feet the seller assumes a depth of 152 feet. The second piece is 100 feet long, from tick 30 to tick 130, and the seller likes to be paid for a depth of 88 feet. On the last 20 feet

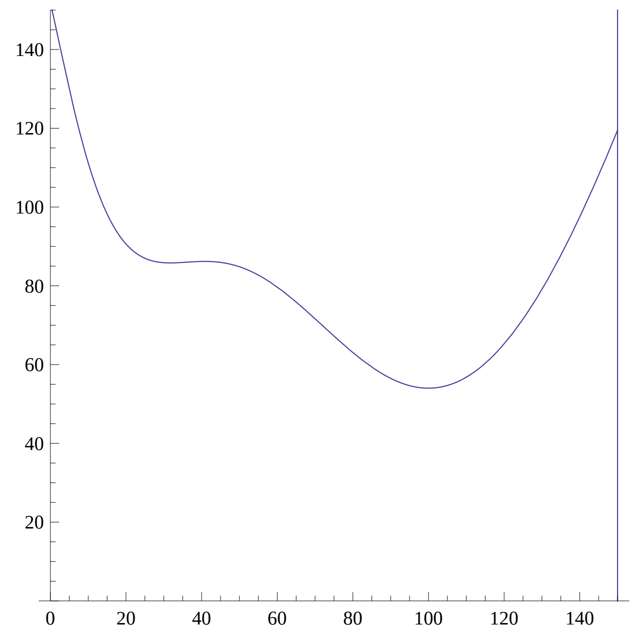

Figure 3.1: Parcel of Land

the seller claims a depth of 120 feet. His revised asking price for the parcel of land is

$$A1 = (30 \times 152 + 100 \times 88 + 20 \times 120) \times \$10.00 = \$157,600.00.$$

The potential buyer makes a similar move, see Figure 3.3. For the first 55 feet he accepts a depth of 82 feet, for the next 70 feet, from tick 55 to 125, he accepts a depth of 54 feet, and for the remaing 25 feet of street front a depth of 70 feet.

His offer jumps to

$$O1 = (55 \times 80 + 70 \times 54 + 25 \times 70) \times \$10.00 = \$99,300.00$$

With a little good will, the difference between the asking price and the offer has been cut into less than one half.

Actually, with little work S and B can narrow down their difference further. Not that long ago they may have hired a couple of students to carry out a few more measurements. They divide the street front into shorter intervals, and on each they under and overestimate the depth of the parcel of land. A result is shown in Figure 3.4.

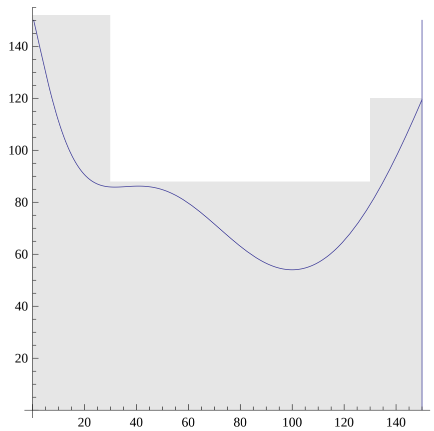

Figure 3.2: Seller's Compromise

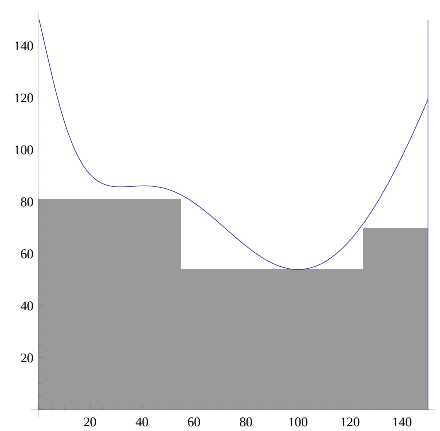

Figure 3.3: Buyer's Compromise

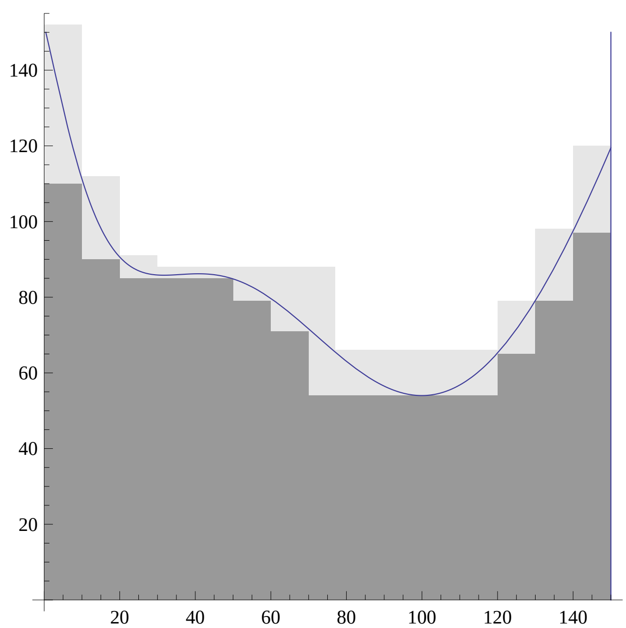

Figure 3.4: Refined Area Estimates

Based of the refined measurements the seller's asking price should drop to \$134,940 and the buyer's offer should increase to \$111,600. Their difference is the lightly shaded area that is not covered up by the darker shading.

Exercise 29. *Make up a reasonable price*[3] *for the parcel of land based on a visually inspecting Figure 3.4. Write a couple of sentences to justify your answer.*

3.2.1 Upper and Lower Sums

Let us review the process. Let $[a, b]$ be an interval and $f(x)$ a function that is defined and nonegative on this interval. We like to determine the area trapped between the x-axis and the graph between a and b. We divide the interval $[a, b]$ into subintervals. For each interval we find a height that exceeds the function on the subinterval. We multiply the width of the subinterval and the height. Adding up all these numbers we get what is called an *upper sum.*

To be concrete, in the example we broke up the interval into intervals:

$$[0, 135] = [0, 30] \cup [30, 130] \cup [130, 150].$$

On the first interval we picked a height of 152, on the second one a height of 88, and on the third one a height of 120. The resulting upper sum is

$$SU = 30 \times 152 + 100 \times 88 + 20 \times 120 = 15760.$$

Similarly, we could pick heights that stay below the graph on the respective subintervals, multiply the widths of the subintervals with these heights, and add up the products. The result is then called a *lower sum.*

In the example, we broke up the interval into intervals:

$$[0, 135] = [0, 55] \cup [55, 125] \cup [125, 150].$$

On the first interval we picked a height of 80, on the second one 54, and on the third 70. The resulting lower sum is

$$SL = 55 \times 80 + 70 \times 54 + 25 \times 70 = 9930.$$

In the motivating example of the bargaining for the price of the parcel of land, it was important that the function is nonnegative. For the construction it is not. So, let us look at a more generic

[3] My best suggestion would be \$121,230.37.

Example 3.5. Find upper and lower sums for the function

$$f(x) = x^3 + x^2 - x$$

for x in the interval $[-1, 1]$. The graph of the function is shown in Figure 3.5.

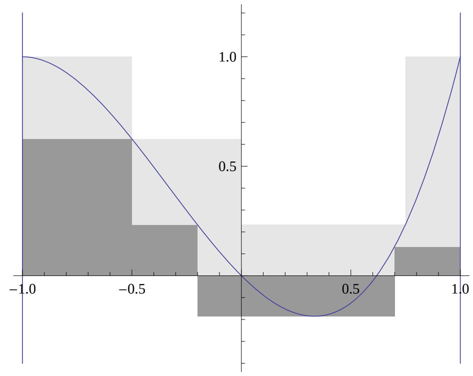

Figure 3.5: Upper and Lower Sums

The rectangles for the computation of the lower sum are shaded darkly, and the height is taken negatively if the rectangle is below the axis. The lower sum is

$$SL = (-.5-(-1))\cdot .625+(-.2-(-.5))\cdot .232+(.7-(-.2))\cdot(-.186)+(1-.7)\cdot .13$$

Each summand is of the form (left endpoint - right endpoint) $\cdot$ height, and you may check that $SL = .2537$.

The rectangles for the computation of the upper sum are shaded lightly, and they are partially covered by the more darkly shaded rectangles for the lower sum. Numerically we have

$$SU = (-.5 - (-1)) \cdot 1 + (0 - (-.5)) \cdot .625 + (.75 - 0) \cdot .234375 + (1 - .75) \cdot 1,$$

or $SU = 1.23828125$. One may remark that area of the visible lightly shaded region is the difference between the upper and lower sum.

There are two important facts:

Theorem 3.6. *Suppose $f(x)$ is a bounded*[4] *function on a closed interval $[a, b]$. There exists a number I so that*

$$SL \leq I \leq SU$$

[4]There are numbers A and B so the $A \leq f(x) \leq B$ for all $x \in [a, b]$. Without this assumption we would not be able to pick the heights in the construction.

for all upper sums SU and all lower sums SL.

For our buyer and seller it means that there is a price for the parcel of land that is at least as high as any offer the buyer is willing to make and at most as high as the lowest offer that the seller will accept. For not–so–well–behaved functions there could still be a gap between the highest price that the buyer is willing to pay and the lowest price that seller is willing to accept. We are interested in situations where the buyer and seller come to an agreement.

3.3 The Definite Integral

Definition 3.7. *Suppose $f(x)$ is any bounded function on a closed interval $[a,b]$. We call the function integrable if there is exactly one number I between all lower and upper sums. In this case, we call this unique number I the integral of the function $f(x)$ for x between a and b, and we denote it by*

$$\int_a^b f(x)\,dx$$

There are different ways to define integrals. The approach in Definition 3.7 is called the Darboux integral.

Theorem 3.8. *Differentiable[5] functions on closed intervals $[a,b]$ are integrable.*

Initially, we wanted to find the area of a region under the graph of a nonnegative function. We are now ready to bring the discussion to a conclusion.

Definition 3.9. *Let $f(x)$ be a nonnegative, integrable function on the interval $[a,b]$. Let Ω be the region bounded by the x–axis and the graph for x between a and b. Denote the area of Ω be $A(\Omega)$. Then*

$$\boxed{A(\Omega) = \int_a^b f(x)\,dx.}$$

We constructed upper and lower sums also for functions that are not necessarily nonnegative and the notion of integral applies to functions independent of the sign. What interpretation does the integral have for a function that has both positive and negative values?

[5]Actually, many more functions than differentiable ones are integrable on closed intervals $[a,b]$, but we make no effort to characterize them further.

Theorem 3.10. *Let $f(x)$ be an integrable function on the interval $[a, b]$. Decompose the area Ω between the x–axis and the graph into the part Ω_+ above the x–axis and the part Ω_- below the x–axis. Let A again denote the area. Then*

$$\boxed{\int_a^b f(x)\,dx = A(\Omega_+) - A(\Omega_-)}$$

3.4 The Fundamental Theorem of Calculus

It would be hard to calculate integrals using upper and lower sums, or other related constructions. Calculus has a perfect answer to this question. It is referred to as the Fundamental Theorem of Calculus. It comes in two parts.

Theorem 3.11. *Let $f(x)$ be a differentiable function on a closed interval $[a, b]$, and let $F(x)$ be an antiderivative of $f(x)$. Then*

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).} \tag{3.2}$$

The middle term in (3.2) is a convenient abbreviation for the expression that follows.

Example 3.12. Consider the function $f(x) = x^3 + x^2 - x$ in Example 3.5. It has $F(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}$ as an antiderivative. We find, after a bit of arithmetic that the reader is expected to fill in, that

$$\int_{-1}^{1} x^3 + x^2 - x\,dx = F(1) - F(-1) = \frac{2}{3}.$$

The reader should recognize that the integral is between the lower of .2537 and the upper sum of 1.23828125 that we calculated in Example 3.5. By construction this is expected.

Example 3.13. Find the area under the graph of $f(x) = 1/x$ for x between 2 and 5. See Figure 3.6 for the region whose area we are calculating.

$$\int_2^5 \frac{dx}{x} = \ln(x)\Big|_2^5 = \ln 5 - \ln 2 = \ln(5/2).$$

Logically, the following formulation of the Fundamental Theorem of Calculus is the more basic one, and Theorem 3.11 is a consequence.

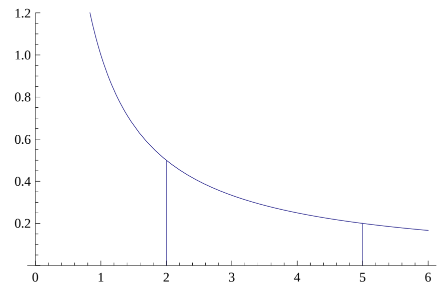

Figure 3.6: Area Example

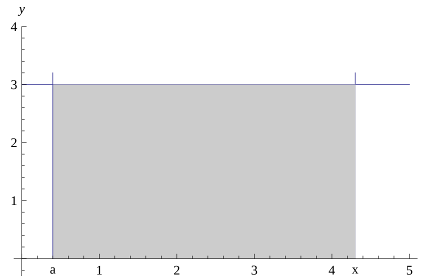

Figure 3.7: Idea of Proof

Theorem 3.14. *Let $f(x)$ be a differentiable function on a closed interval $[a, b]$. Then*

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x).}$$

Idea of Proof. In the simplest case the theorem is obvious. Let $f(x)$ be a constant function, say $f(x) = c = 3$ in Figure 3.7.

By definition, the area of the shaded region is the integral of $f(x)$ for x between a and x, and

$$\int_a^x f(t)\,dt = c(x-a).$$

If we differentiate this expression with respect to the variable x, then we obtain $f(x) = c$, as asserted in this special case.

For more general functions, and not only differentiable ones, the proof is quite demanding, and mathematics majors would not see the details until their junior year. □

Previously we said little about the Euler number e. Let us give one description, though not a very practical one. Earlier you learned that $e^0 = 1$ and $e^1 = e$. Recall that the natural logarithm function is the inverse of the natural exponential function; i.e., the exponential function with base e. Then $\ln 1 = 0$ and $\ln e = 1$. This means that

$$\int_1^e \frac{dx}{x} = \ln x\Big|_1^e = \ln e - \ln 1 = 1.$$

Expressed differently, consider a region under the graph of the function $1/x$, starting at 1. If the area of the region is 1, then it ends at e.

3.5 Surplus

Let us look at another business application. In Figure 3.8 you see a demand curve. The demand is on the horizontal x-axis and the price on the vertical one. As example we are using the weekly demand for a simple digital camera. It is determined by the equation $p = 300e^{-x/50}$. The horizontal line is at $p = 60$, and this is the price that the store is charging. At this price, there will be a demand for about 80 cameras per week.

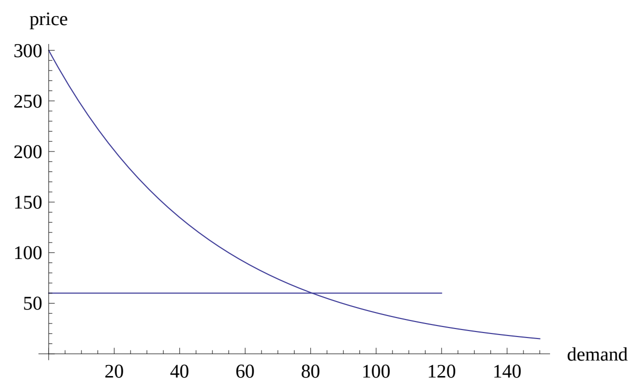

Figure 3.8: Consumer Surplus

Let us talk about happy customers. If the price is \$300.00, then there is demand for one camera. One customer would have been willing to pay \$300.00 for the camera. Obviously, he (or she) is really happy with the great bargain of the \$60.00 camera. He thinks that he saved \$240.00. The 20th customer[6] would be willing to pay only \$201.00. Still he gets the camera for a lot less than what he would have been willing to pay, saving \$141.00. The 60th customer would have been willing to pay about \$90.00 and is under the impression of having saved \$30.00. There won't be customers beyond the 80th one, because they consider the camera as too expensive.

So, how much did the customers save together? This amount is call the *Consumer Surplus*. Mathematically speaking it is the area of the region Ω bounded by the sales price p below, the demand curve $g(x)$ above, and the vertical axis on the left. To the right we go all the way to the point x_0 where the the demand curve and the fixed price line intersect. That's the point where people stop buying. The consumer surplus, abbreaviated as CSP, or

[6]Think of the customers lined up along the horizontal axis in Figure 3.8, with the graph above their head indicating the price they are willing to pay.

the area of the region Ω, is then expressed as an integral:

$$\boxed{CSP = \int_0^{x_0} [g(x) - p]\, dx}$$

In our specific example:

$$\begin{aligned} CSP &= \int_0^{80} \left[300e^{\frac{-x}{50}} - 60\right]\, dx \\ &= \left(-50 \cdot 300e^{\frac{-x}{50}} - 60x\right)\Big|_0^{80} \\ &= 15000[1 - e^{-1.6}] - 4800 \\ &\sim 7171 \end{aligned}$$

Hence, together those who purchased a camera together paid \$7,171 less that what they would have been willing to pay.

Exercise 30. You are into manufacturing high fashion T–shirts, and you estimate that the relation between price p and monthly demand x is given by the curve shown in Figure 3.9. The equation for the curve is:

$$(p - 70)^2 = 8(x + 10).$$

Both variables need to be non–negative to make sense. This limits the demand to 600 T-shirts and the price to \$60.00.

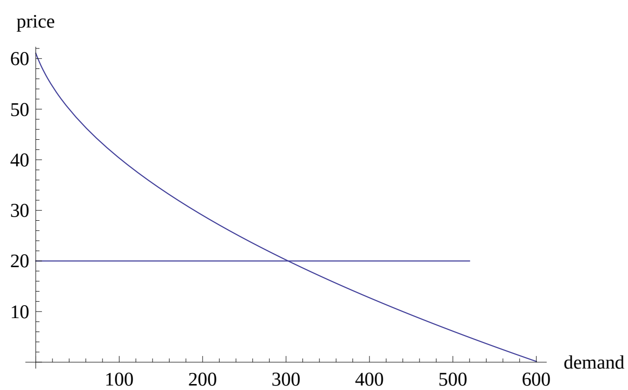

Figure 3.9: Demand Equation

1. What is the demand per month if you set the price at \$30.00 per T-shirt?
2. What is the monthly consumer surplus if you set the price at \$20.00?
3. Which price will maximize your revenue?

Let us study the T-shirt problem from the manufacturer's point of view. The local garment industry may be motivated to manufacture T-shirts if the price is right. There will be a supply curve that relates the price and the monthly supply. Such a supply curve is shown in Figure 3.10. It is part of the curve given by the equation

$$(p + 8)^2 = 2(x + 70).$$

The horizontal line in the graph represents a price of \$20.00 per T-shirt. Looking at the graph, you may think that at a price of \$20.00 there will be a monthly supply of 320 T-shirts. Using the equation, the more precise answer will be 322.

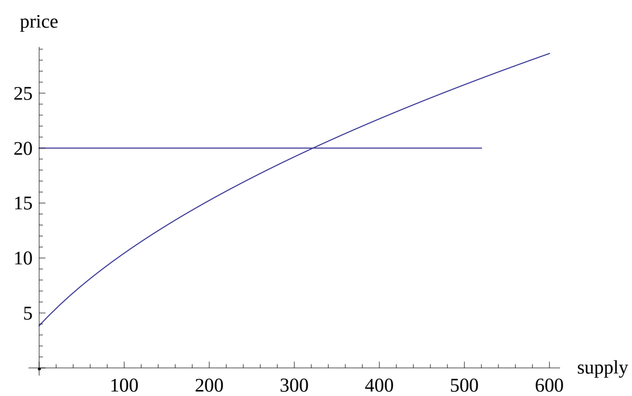

Figure 3.10: Supply Equation

Manufacturers will start to supply the market when they are offered \$4.00 per shirt. If the price rises to \$5.00, the supply will only be 15 T-shirts per month. At \$10.00 a shirt, the monthly supply would increase to about 90 T-shirts.

Suppose that the going rate for a T–shirt is \$20.00. Then some manufacturers will be overjoyed. They were willing to deliver for much less.

The area between the fixed price (\$20.00 in our case) and the supply line between $x = 0$ and the intersection point (at $x = 322$ in our case) is the *suppliers' surplus*. It is the amount of money that the suppliers received in addition to their minimum demand.

More formally, let $h(x)$ describes the price as a function of ths supply, let p be a fixed price, and let x_0 be where these two lines intersect, or $h(x_0) = p$. Then the suppliers' surplus is:

$$\boxed{SSP = \int_0^{x_0} [p - h(x)]\, dx.}$$

In our example $h(x) = -8 + \sqrt{2(x+70)}$ and :

$$\begin{aligned} SSP &= \int_0^{322} \left[20 - (-8 + \sqrt{2(x+70)})\right] dx \\ &= \int_0^{322} \left[28 - \sqrt{2}(x+70)^{\frac{1}{2}}\right] dx \\ &= 28x - \frac{2\sqrt{2}}{3}(x+70)^{\frac{3}{2}}\Big|_0^{322} \\ &\sim 2251 \end{aligned}$$

In summary, the T-shirt producers feel to make a monthly surplus of about \$2,251.00.

To have a functioning business model supply should meet demand, which happens where the supply and the demand line intersect. You can follow the discussion in Figure 3.11. The vertical (price) coordinate of the intersection point tells us how to price the T-shirts so that price meets demand. Explicitly:

$$-8 + \sqrt{2(x+70)} = 70 - \sqrt{8(x+10)}$$

The equation is a little tricky to solve, but not too bad, and the solution is $x = 1700 - 104\sqrt{179} \sim 308.57$. The associated price is $p = 4\sqrt{179} - 34 \sim 19.51$.

We conclude that the perfect price would be \$19.50, to have a nice and appealing number. Then the monthly demand is just over 300 T-shirts.

Exercise 31. *Find the consumer and supplier surplus in the previous example if the price is set so that supply meets demand.*

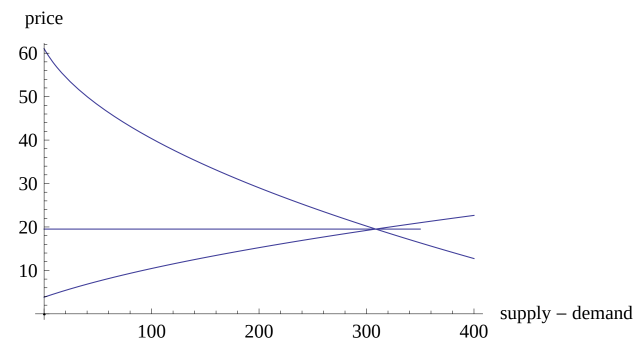

Figure 3.11: Supply meets Demand

3.6 Properties of the Definite Integral

To avoid repetition, we assume that in the following formulae all integrals exist.

Apparently from the definition, one obtains

$$\boxed{\int_a^a f(x)\,dx = 0} \tag{3.3}$$

Like the derivative (see (1.1)) and the antiderivate (see (3.1)) the definite integral is linear:

$$\boxed{\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.} \tag{3.4}$$

The integral preserves inequalies. If $f(x) \le g(x)$ for all x between a and b, then

$$\boxed{\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.} \tag{3.5}$$

One may use this formula to obtain

$$\boxed{\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx} \tag{3.6}$$

It should also be clear from the definition that

$$\boxed{\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.} \tag{3.7}$$

On occasions it is convenient to reverse the direction of an interval. This leads to a reversal of signs, and motivates

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx} \tag{3.8}$$

This formula is consistent with (3.3) and (3.7).