

# Vectors and Plane Geometry

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# Preface

During the first week of the semester it is difficult to get started with the course material. Some students have not settled in, some are still changing sections, and some still have to sign up for a course. For this reason, it is reasonable to teach an interesting, relevant topic that is somewhat independent of the course. Some instructors in a calculus course use the first week to review topics from precalculus. Instead, we decided to spend this week on vectors and the geometry of the plane, topics that other sciences and engineering like to see covered early. These notes are meant as lecture notes for a one-week introduction.

There is nothing original in these notes. The material can be found in many places. Many calculus books will have a section on vectors in the second half, but students would not like to start reading there. The material is also contained in a variety of other mathematics books, but then we would not want to force students to acquire another book. For these reasons, we are providing these notes.



# Chapter 1

## The Algebra of Vectors

Some information is completely described by a single number, such as the balance of your checking account at a specific moment. But if you like to record the motion of a billiard ball on a pool table, then you will need three numbers. You need to record the position of the ball in a plane at any given time. An array whose entries are real numbers is an example of a vector, no matter how many entries the array may have. We may add vectors and we may multiply them by numbers, and the rules for this arithmetic offer no surprises.

### 1.1 Definition of a Vector

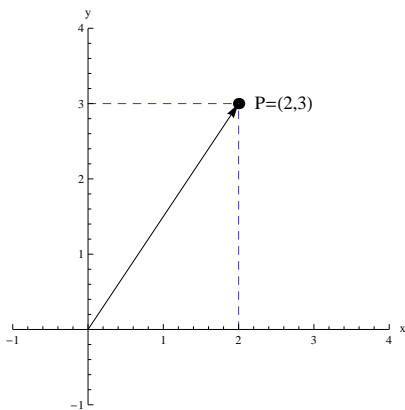


Figure 1.1: Vector in  $\mathbb{R}^2$

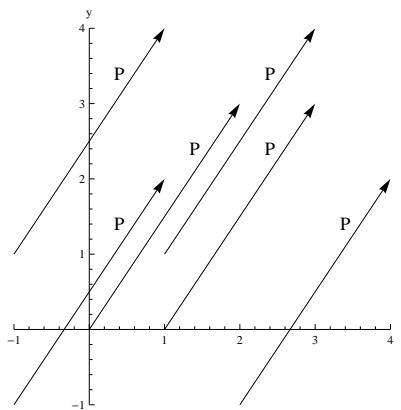


Figure 1.2: Same vectors

We will content ourselves with *vectors* in the cartesian plane  $\mathbb{R}^2$  or in three dimensional space  $\mathbb{R}^3$ . Points  $P$  in the plane are described by pairs  $(a, b)$  of real numbers, where  $a$  and  $b$  stand for the  $x$  and  $y$  coordinates of the point  $P$ . This means, when we project  $P$  perpendicularly on the  $x$ -axis, then we get  $a$ , and when we project  $P$  on the  $y$ -axis, then we get  $b$ , see Figure 1.1.

On occasions, we identify the point or vector  $P$  with an arrow from the origin  $(0, 0)$  of the coordinate system to  $P$ . We may even denote both objects with the same symbol. In this description the vectors become movable. They are characterized by their direction and length. In Figure 1.2 you see the vector  $P$  moved (by parallel translations) to different positions in the plane. Each of the arrows still represents  $P$ .

The vector  $u$  from  $Q = (a_1, b_1)$  to  $P = (a_2, b_2)$  can be written as

$$(1.1) \quad \boxed{u = \overrightarrow{QP} = (a_2 - a_1, b_2 - b_1)}.$$

Its tail is  $Q$  and its tip is  $P$ . For an illustration, see Figure 1.3.

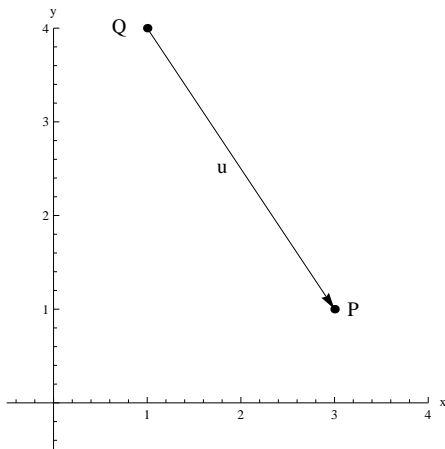


Figure 1.3:  $u = \overrightarrow{QP}$

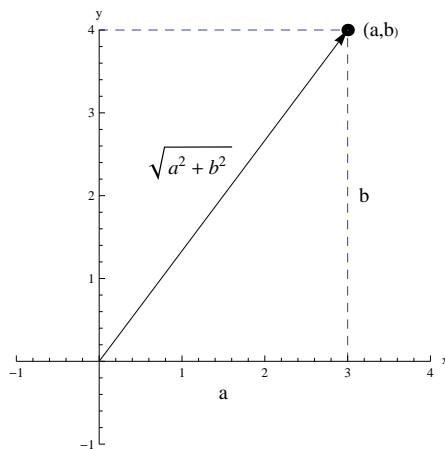


Figure 1.4: Length

**Example 1.1.** The vector from  $Q = (1, 2)$  to  $P = (5, 3)$  is

$$\overrightarrow{QP} = (5 - 1, 3 - 2) = (4, 1).$$

**Exercise 1.** Sketch the vector  $u$  with tail  $(1, 2)$  and head  $(2, -1)$ . In the same set of coordinates, sketch the vector  $u$  with  $(3, 1)$  as its tail, and with  $(4, 2)$  as its head.

The two ways of viewing vectors, points in the plane versus arrows, are related by the formula

$$P = \overrightarrow{OP}$$

where  $O = (0, 0)$  is the origin of the coordinate system. Both, the point and the arrow, are shown in Figure 1.1.

In three dimensional space  $\mathbb{R}^3$  we have three coordinate axes, often called the  $x$ ,  $y$ , and  $z$ -axes. The first two are used in a horizontal  $x$ - $y$ -plane, and the  $z$ -axis is perpendicular to the  $x$ - $y$ -plane pointing upwards. Accordingly, vectors in  $\mathbb{R}^3$  are triples of real numbers.

The length  $\|u\|$  of a vector  $u = (a, b)$ , illustrated in Figure 1.4, is

$$(1.2) \quad \boxed{\|u\| = \sqrt{a^2 + b^2}.}$$

It is the Euclidean distance between the points  $(0, 0)$  and  $(a, b)$ , or the length of the line segment that joins these two points.

**Example 1.2.** The length of the vector  $u = (2, 3)$  is

$$\|u\| = \sqrt{2^2 + 3^2} = \sqrt{13}.$$

The length of the vector from  $Q = (1, 2)$  to  $P = (5, 3)$  is

$$\|\overrightarrow{QP}\| = \|(4, 1)\| = \sqrt{4^2 + 1^2} = \sqrt{17}.$$

One may also write vectors as columns, say  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ . This is more natural in some contexts. Row vectors take up less space in type setting. For most purposes, the difference is only notational, not conceptual.

**Exercise 2.** Find the length of the vectors  $u = (1, 4)$ ,  $v = (1, 4, 2)$  and  $w = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ . Find the length of the vector  $\overrightarrow{QP}$  from  $Q = (1, 5)$  to  $P = (3, 2)$ .

## 1.2 Addition and Multiplication with Scalars

We consider two operations. If  $v = (x_1, y_1)$  and  $w = (x_2, y_2)$  are vectors, then we define their sum to be the vector:

$$(1.3) \quad \boxed{v + w = (x_1 + x_2, y_1 + y_2)}$$

If  $c$  is a real number and  $v = (x, y)$ , then we define the scalar product of  $c$  and  $v$  to be the vector

$$(1.4) \quad \boxed{cv = c(x, y) = (cx, cy).}$$

In a more general setting,  $c$  could be taken from a different set of numbers, not the real numbers. To allow this, it is common to call  $c$  a scalar. For us, a real number and a scalar are the same.

One might indicate the multiplication by a dot, and write  $c \cdot v$  instead of  $cv$ , but this is only rarely done. It is convenient to write  $v/c$  instead of  $\frac{1}{c}v$ . For the obvious reasons, we say that vectors are added, or multiplied with a scalar, coordinatewise. The operations can be applied also to vectors in  $\mathbb{R}^3$ , or vectors with any number of coordinates.

**Example 1.3.**

$$(1, 2) + (5, 2) = (6, 4) \quad \& \quad 3(2, 7) = (6, 21).$$

For column vectors with three coordinates we have

**Example 1.4.**

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix} \quad \& \quad 4 \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 20 \end{pmatrix}$$

**Exercise 3.** Let  $u = (1, 2)$  and  $v = (2, -3)$ .

1. Find  $2u$ ,  $u + v$ ,  $u + 2v$ , and  $v - u$ .
2. Find  $P$  if  $\overrightarrow{QP} = 2u + v$  and  $Q = (-1, -1)$ .
3. Find  $Q$  if  $\overrightarrow{QP} = u - v$  and  $P = (5, 2)$ .

In the plane  $\mathbb{R}^2$  it is common to set

$$(1.5) \quad \boxed{\mathbf{i} = (1, 0) \quad \text{and} \quad \mathbf{j} = (0, 1),}$$

and in 3-dimensional space

$$(1.6) \quad \boxed{\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0) \quad \text{and} \quad \mathbf{k} = (0, 0, 1).}$$

Using this notation,

$$(2, -3) = 2\mathbf{i} - 3\mathbf{j} \quad \text{and} \quad \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}.$$

It is common to refer to  $\mathbf{i}$  and  $\mathbf{j}$  as standard basis vectors of  $\mathbb{R}^2$ , or to  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  as the standard basis of  $\mathbb{R}^3$ .

**Exercise 4.** Express  $(9, 5)$  in terms of the standard basis vectors.

The arithmetic of vectors offers no surprise:

**Proposition 1.5.** Suppose that  $u$ ,  $v$ , and  $w$  are any vectors in  $\mathbb{R}^2$  and  $c$  and  $d$  are scalars, then

1.  $v + w$  and  $cv$  are in  $\mathbb{R}^2$ .
2.  $u + (v + w) = (u + v) + w$ .
3.  $v + w = w + v$ .
4. There exists a vector  $0$  so that  $v + 0 = 0 + v = v$ .
5. There exists a vector  $v'$  in  $\mathbb{R}^2$  so that  $v + v' = 0$ .
6.  $1v = v$ .
7.  $c(v + w) = cv + cw$ .
8.  $(c + d)v = cv + dv$ .
9.  $(cd)v = c(dv)$ .

In (1) we are asserting that the sum of vectors is a vector, and so is a scalar multiple of a vector. In (2) we are saying that it does not matter in which order the additions are carried out, the result is the same. This property is called the associative law. In (3) we are saying that the addition of vectors is commutative, we may interchange the summands and the result is unchanged.

In (4) we assert that there is a zero for the addition of vectors. Adding zero does not change a vector. Such an element is also called a neutral element for addition, and it is unique. Obviously,  $0 = (0, 0)$  is the vector both of whose coordinates are zero.

In words, (5) says that every vector  $v$  has an additive inverse  $v'$ . Necessarily, and also in a more general setting, it will be unique. If  $v = (a, b)$ , then  $v' = (-a, -b)$ . It is common to denote the additive inverse of  $v$  by  $-v$ . It is consistent with common arithmetic to set

$$u - v = u + (-v),$$

and call this operation subtraction.

The remaining three properties are called distributive laws.

**Remark 1.** Addition and multiplication of vectors in  $\mathbb{R}^3$  obeys the same laws as the ones spelled out in Proposition 1.5. One may also consider a set  $V$  with two operations, like the addition and scalar multiplication from above, and call it a real vector space if properties (1)–(8) in Proposition 1.5 hold. One important example of such a vector space is the set of all real valued functions that are defined on some domain. One adds such functions by adding the values, and one multiplies them with scalars by multiplying the values with scalars. This algebra on functions is one of the important topics of a precalculus course.

### 1.3 Geometric Interpretation of Operations.

The vector operations have geometric interpretations. If  $u$  and  $v$  are vectors in the plane, thought of as arrows with tips and tails, then we can construct the sum  $w = u + v$  as shown in Figure 1.5. We arrange it so that the tip of  $u$  is the tail of  $v$ . Then  $w$  is the vector whose tail is the tail of  $u$  and whose tip is the tip of  $v$ . In other words we concatenate the two vectors.

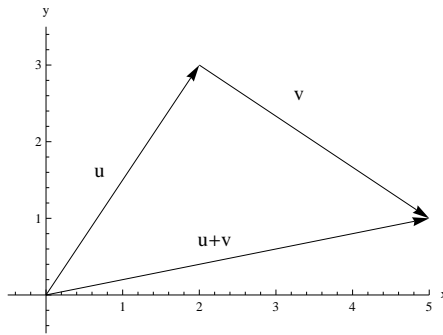


Figure 1.5: Vector addition

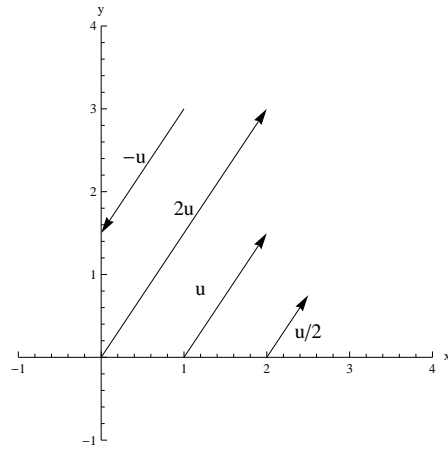


Figure 1.6: Scalar multiples

Suppose  $u$  is a vector and  $c$  a scalar. If  $c$  is positive, then  $cu$  has the same direction as  $u$  and  $c$  times its length, so  $\|cu\| = c\|u\|$ . If  $c$  is negative, then  $cu$  has length  $|c| \cdot \|u\|$ , and its direction is opposite to that of  $u$ . Examples are shown in Figure 1.6.

**Exercise 5.** Let  $u = (-2, 1)$  and  $v = (3, 2)$ ,

1. Sketch  $u$  and  $v$  as arrows with tail at  $(1, 1)$ .
2. Construct the sum of  $u$  and  $v$  geometrically.
3. Sketch  $2u$  as an arrow with tail at  $(1, -2)$ .
4. Sketch  $-u$  as an arrow with tail at  $(1, -1)$ .

In physics vectors are often used to describe forces, and forces add as vectors do. This principle is applied in the following example.

**Example 1.6.** A water balloon with a mass of 10kg (containing about two and a half gallons of water) hangs from two wires (WL to the left and WR to the right) as shown in Figure 1.7. It is suspended from the point  $C = (3, 2)$ . Calculate the tension (force) in the wires that results from the gravitational pull on the balloon.

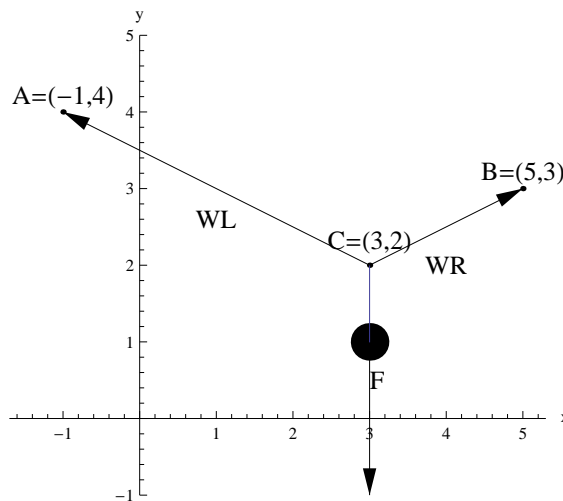


Figure 1.7: Water balloon

Let us clarify the units for our computation. If a force of 1 Newton acts on a mass of 1kg, then the resulting acceleration is  $1\text{m}/\text{sec}^2$ . Gravitation would result in an acceleration of  $9.8\text{m}/\text{sec}^2$ . Thus, the gravitational force on the balloon will be 98 Newton.

Let  $C$  be the point where the three wires are joined. We suppose that  $C = (3, 2)$ . Three forces act on  $C$ , the vertical force  $F$  resulting from the

gravitational pull on the water balloon, the tension  $TL$  in the wire to the left, and the tension  $TR$  in the wire to the right. The three forces cancel each other out, as the system is at rest:

$$(1.7) \quad F + TL + TR = 0.$$

We know that  $F = (0, -98)$ . The forces (tension) in the wires break up into a horizontal and vertical component:

$$TL = (TL^h, TL^v) \quad \text{and} \quad TR = (TR^h, TR^v).$$

We use the geometry of the situation to calculate the components. Let  $\alpha$  be the angle between the positive horizontal axis (through the point  $C$  and pointing to the right) and the wire  $WR$ . The length of the wire  $WR$  is  $\sqrt{5}$ , so that

$$\cos \alpha = \frac{2}{\sqrt{5}} \quad \text{and} \quad \sin \alpha = \frac{1}{\sqrt{5}}.$$

We observe that the angle  $\beta$  between the wire  $WL$  and the horizontal is  $\pi - \alpha$ . Let us denote the tension in  $WR$  by  $TR$ , and the one in  $WL$  by  $TL$ . Then

$$TL = \left( -\frac{2}{\sqrt{5}}\|TL\|, \frac{1}{\sqrt{5}}\|TL\| \right) \quad \text{and} \quad TR = \left( \frac{2}{\sqrt{5}}\|TR\|, \frac{1}{\sqrt{5}}\|TR\| \right).$$

Substituting these calculations into (1.7), we find

$$\begin{aligned} (0, 0) &= (0, -98) + \left( -\frac{2}{\sqrt{5}}\|TL\|, \frac{1}{\sqrt{5}}\|TL\| \right) + \left( \frac{2}{\sqrt{5}}\|TR\|, \frac{1}{\sqrt{5}}\|TR\| \right) \\ &= \left( -\frac{2}{\sqrt{5}}\|TL\| + \frac{2}{\sqrt{5}}\|TR\|, -98 + \frac{1}{\sqrt{5}}\|TL\| + \frac{1}{\sqrt{5}}\|TR\| \right) \end{aligned}$$

The first coordinate being zero tells us that  $\|TL\| = \|TR\|$ . Setting the second coordinate equal to zero, we find

$$-98 + \frac{2}{\sqrt{5}}\|TR\| = 0,$$

or that  $\|TR\| = 49\sqrt{5}$ . The tension in each wire is  $49\sqrt{5}$  Newton.

**Exercise 6.** Repeat Example 1.6 with  $A = (-1, 5)$ , while  $B$  and  $C$  are left unchanged.

**Exercise 7.** Repeat Example 1.6 if  $A$  and  $B$  are as in the example, wire  $WL$  has length 5, and wire  $WR$  has length 3.

**Exercise 8.** Repeat Example 1.6 if the lengths of the wires  $WL$  and  $WR$  are 7 and 9, while the angles that the wires make with a horizontal are  $\alpha = 32^\circ$  on the left and  $\beta = 44^\circ$  on the right.

## 1.4 Parametrized curves

Let us consider vector valued functions, functions whose domain is a subset of  $\mathbb{R}$  and whose range is contained in  $\mathbb{R}^2$ . An example would be

$$(1.8) \quad S(t) = (x(t), y(t)) = (\cos t, \sin t).$$

For each real number  $t$ , we observe that  $S(t)$  lies on the unit circle (the circle of radius 1) because  $x^2(t) + y^2(t) = 1$ , and as  $t$  increases we trace out the circle in a counter-clockwise direction. The circle is an example of a curve in the plane, while the function  $S(t)$  in (1.8) is called a parametrization of the curve.

**Exercise 9.** Let  $a \geq b > 0$  be positive numbers. Define a parametrized curve by setting

$$S(t) = (a \cos t, b \sin t).$$

Show that  $S(t)$  traces out an ellipse with major and minor axes  $2a$  and  $2b$  in a counter-clockwise way.

**Exercise 10.** Consider the parametrized curve

$$S(t) = 3 \left( \sin \left( \frac{\pi \sin t}{6} \right), -\cos \left( \frac{\pi \sin t}{6} \right) \right).$$

Show that  $S(t)$  traces out the motion of the tip of a pendulum of length 3 that swings between the angles of  $-\pi/6$  and  $\pi/6$ . Caution: We are not asserting that we modeled the velocity correctly, only the trace.

Given two distinct points  $P$  and  $Q$  in the plane, we may consider the line through these two points. We define it as the set

$$(1.9) \quad L = \{Q + t(P - Q) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}.$$

The vector  $D = P - Q = \overrightarrow{QP}$  is called a direction vector for the line. To obtain the points of  $L$  we add arbitrary multiples of  $D$  to  $Q$ . E.g., if  $t = 1$ ,

then  $Q + t(P - Q) = P$ . Lines are curves in the plane, and the line in (1.9) has the parametrization

$$S(t) = Q + t(P - Q).$$

**Exercise 11.** *The line  $L$  in (1.9) is equal to the set*

$$(1.10) \quad \boxed{L = \{aP + bQ \in \mathbb{R}^2 \mid a, b \in \mathbb{R} \text{ and } a + b = 1\}}$$

The equation in (1.10) is more symmetric in  $P$  and  $Q$ , and in this sense more appealing.

**Exercise 12.** *Let  $P$  and  $Q$  be points in  $\mathbb{R}^2$ . Show that  $R = \frac{1}{2}(P + Q)$  is the midpoint between  $P$  and  $Q$ . In other words,  $R$  is equidistant to  $P$  and  $Q$ , or  $\|\overrightarrow{RQ}\| = \|\overrightarrow{RP}\|$ .*

*If this exercise sounds too abstract, then do it first in a concrete case, say  $Q = (2, 3)$  and  $P = (6, 5)$ . Make sure to calculate the coordinates of  $R$  and plot all three points. Measure and calculate  $\|\overrightarrow{RQ}\|$  and  $\|\overrightarrow{RP}\|$ . Secondly, do the exercise for  $Q = (a_1, b_1)$  and  $P = (a_2, b_2)$ .*

**Exercise 13.** *Let  $R = aP + bQ$  be any point on the line through  $P$  and  $Q$ , so  $a + b = 1$ . Find  $\|\overrightarrow{RQ}\|/\|\overrightarrow{RP}\|$  in terms of  $a$  and  $b$ .*

**Exercise 14.** *Let  $Q = (2, 3)$  and  $P = (6, 5)$ . Find the slope intercept formula for the line through  $Q$  and  $P$ .*

As we stated above, two distinct points define a line. We can make a stronger statement:

**Exercise 15.** *A line is determined by any two of its points.*

The assertion in the Exercise 15 sounds rather abstract. Let us make it concrete. Suppose  $P$  and  $Q$  determine a line  $L$ , as in (1.9). Suppose  $P'$  and  $Q'$  are distinct points of  $L$ , and  $L'$  is the line through  $P'$  and  $Q'$ . Then  $L = L'$ .

Let us take a different point of view. We will show that a line in the plane is given by an equation of the form

$$(1.11) \quad \boxed{ax + by + c = 0,}$$

where  $a$  and  $b$  are not both zero. So the line consists of all points  $(x, y) \in \mathbb{R}^2$  which satisfy the equation. The equation in (1.11) is called the *general equation of a line* in the plane.

The equation of a line will not be unique. The set of solutions of (1.11) does not change if we multiply the equation by any non-zero number.

**Proposition 1.7.** *The two approaches to lines expressed in (1.9) and (1.11) are equivalent. More specifically:*

1. *Let  $P$  and  $Q$  be distinct points and  $L$  the line through them as in (1.9). Then there is an equation as in (1.11), so that every point of  $L$  satisfies this equation.*
2. *Given an equation as in (1.11), and let  $P$  and  $Q$  be any two points that satisfy it. Then any point  $R = (x, y) = Q + t(P - Q)$  on the line through  $L$  (as in (1.9)) satisfies this equation.*

**Example 1.8.** Let us illustrate the proposition with examples. For (1), set  $P = (1, 3)$  and  $Q = (4, -6)$ . If both points are to satisfy the equation in (1.11), then

$$a + 3b + c = 0 \quad \& \quad 4a - 6b + c = 0.$$

Taking the difference of the equations, we conclude that  $a = 3b$ . We set  $b = 1$  and conclude that then  $a = 3$  and  $c = -6$ . As equation for the line we use

$$3x + y - 6 = 0.$$

If  $R$  is a point of  $L$ , then

$$R = Q + t(P - Q) = (4, -6) + t[(1, 3) - (4, -6)] = (4 - 3t, -6 + 9t),$$

and this point satisfies our suggested equation for the line, because for all  $t$  in  $\mathbb{R}$ ,

$$3(4 - 3t) + (-6 + 9t) - 6 = 0.$$

We illustrate (2). Consider the equation

$$3x - 2y + 12 = 0.$$

If  $x = 2$ , then  $y = 9$ , so that  $Q = (2, 9)$  is a point on the line. Set  $D = (2, 3)$ , and verify that  $Q + tD = (2 + 2t, 9 + 3t)$  satisfies the equation of the line for every  $t$ . We may use  $P = Q + D = (4, 12)$  as a second point on the line, and as we just saw, any point  $R = Q + tD = Q + t(P - Q)$  will satisfy the equation of the line.

**Exercise 16.** *Prove Proposition 1.7. Hint: Study above example and generalize it.*

**Exercise 17.** *Show that  $(-b, a)$  is a direction vector for the line given by the equation  $ax + by + c = 0$ .*

**Exercise 18.** Write the general equation of a line  $3x + 2y + 1 = 0$  in slope intercept form.

**Exercise 19.** Find two points that satisfy the equation  $3x + 2y + 1 = 0$  and use them to describe the line through them as in (1.10).

**Exercise 20.** Find an equation in general form for the line through the points  $(2, -1)$  and  $(4, 5)$ .

One may use the tools of this section to prove classical results from geometry. The median of a triangle is a line segment that joins a vertex of the triangle with the midpoint of the opposing side. E.g., if the vertices of the triangle are called  $A$ ,  $B$ , and  $C$ , then one of the medians is the line segment that joins  $C$  and  $\frac{1}{2}(A+B)$ . Each triangle has exactly three medians.

**Exercise 21.** Show that the three medians of a triangle are concurrent at one point, they intersect in one point. This point of concurrency, also called the centroid of the triangle, divides the medians into two line segments. The line segment between the vertex and the centroid is twice as long as the line segment between the midpoint of the side of the triangle and the centroid.

To solve the exercise one only needs to interpret the equations

$$\begin{aligned} \frac{1}{3}(A + B + C) &= \frac{2}{3} \cdot \frac{1}{2}(A + B) + \frac{1}{3}C \\ &= \frac{2}{3} \cdot \frac{1}{2}(A + C) + \frac{1}{3}B \\ &= \frac{2}{3} \cdot \frac{1}{2}(B + C) + \frac{1}{3}A \end{aligned}$$

## Chapter 2

# Dot Product, Angles, and Geometry

### 2.1 Theorem of Cosines

The theorem of cosines tells us that one can use length to determine angles:

**Theorem 2.1** (Theorem of Cosines). *Consider a triangle  $ABC$  as in Figure 2.1 with sides of length  $a$ ,  $b$ , and  $c$ , and an interior angle  $\alpha$  at  $A$ . Then*

$$(2.1) \quad \boxed{a^2 = b^2 + c^2 - 2bc \cos \alpha.}$$

Rewritten for our purpose, this formula reads:

$$(2.2) \quad \alpha = \arccos \left( \frac{b^2 + c^2 - a^2}{2bc} \right)$$

*Proof.* As shown in Figure 2.1, we denote the intersection point of the altitude through the vertex  $C$  and the opposite side by  $D$ . Let  $w$  be the length of this altitude,  $u$  the length of the line segment  $AD$  and  $v$  the length of the line segment  $DB$ . Then

$$w = b \sin \alpha, \quad u = b \cos \alpha, \quad \text{and} \quad v = c - u = c - b \cos \alpha.$$

We compute, using the Theorem of Pythagoras twice, that

$$\begin{aligned} a^2 &= w^2 + v^2 \\ &= b^2 \sin^2 \alpha + (c - b \cos \alpha)^2 \\ &= b^2 \sin^2 \alpha + c^2 - 2bc \cos \alpha + b^2 \cos^2 \alpha \\ &= b^2 + c^2 - 2bc \cos \alpha \end{aligned}$$

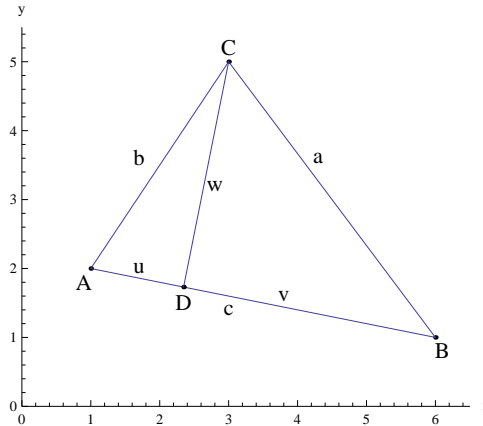


Figure 2.1: Theorem of Cosines

This is exactly what we claimed. The reader should modify the details of the proof if  $\alpha$  is obtuse.  $\square$

## 2.2 Definition of the Dot Product and Length

The dot product, which we are about to define, does not only provide a clean, analytical approach to the topic of length and angles, it also allows us to define and compute angles in higher dimensions.

**Definition 2.2.** Let  $u = (x_1, y_1)$  and  $v = (x_2, y_2)$  be vectors in the plane. Their dot product is

$$u \cdot v = (x_1, y_1) \cdot (x_2, y_2) = x_1x_2 + y_1y_2.$$

One may generalize this definition to vectors with any number of coordinates.

**Example 2.3.**

$$\begin{aligned} (2, 1) \cdot (3, -2) &= 2 \cdot 3 + 1 \cdot (-2) = 4 \\ (1, 2, 3) \cdot (3, 1, 5) &= 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 5 = 20 \end{aligned}$$

The dot product has a number of nice properties, which we summarize in the next proposition:

**Proposition 2.4.** Let  $u$ ,  $v$ , and  $w$  be vectors and  $c$  a scalar. Then

1.  $u \cdot u \geq 0$  and  $u \cdot u = 0$  if and only if  $u = 0$ .
2.  $u \cdot v = v \cdot u$ .
3.  $(u + w) \cdot v = u \cdot v + w \cdot v$ .
4.  $(cu) \cdot v = c(u \cdot v)$ .

In words, (1) means that the dot product is positive definite. Property (2) means that the dot product is commutative. In (3) and (4) we are asserting that the dot product is compatible with addition and multiplication in the first coordinate (hence using (2) also in the second coordinate), and one says that the dot product is linear in each of its arguments.

There is an obvious relation between the dot product and length: If  $u$  is any vector, then

$$(2.3) \quad \boxed{\|u\| = \sqrt{u \cdot u}}$$

One may show that if  $u$  and  $v$  are nonzero vectors and  $\alpha$  the angle between them, then

$$(2.4) \quad \boxed{\cos \alpha = \frac{u \cdot v}{\|u\| \cdot \|v\|}}$$

This equation may be used to define the cosine of an angle between two vectors, as long as one has a product with properties as in Proposition 2.4.

**Definition 2.5.** Two vectors  $u$  and  $v$  are said to be *perpendicular* ( $v \perp w$ ) if  $u \cdot v = 0$ . They are said to be *parallel* ( $v \parallel w$ ) if  $|u \cdot v| = \|u\| \cdot \|v\|$ . A vector is called a *unit vector* if its length is 1.

Note, if  $u$  is any nonzero vector, then  $\frac{u}{\|u\|}$  is a unit vector in the direction of  $u$ . Let illustrate the definitions that we just gave with examples.

**Example 2.6.** Let  $u = (1, 2, 5)$ ,  $v = (2, -1, 2)$ , and  $\alpha$  be the angle between the vectors. Then

$$\cos \alpha = \frac{(1, 2, 5) \cdot (2, -1, 2)}{\|(1, 2, 5)\| \cdot \|(2, -1, 2)\|} = \frac{10}{\sqrt{30}\sqrt{9}} = \frac{\sqrt{10}}{3\sqrt{3}}$$

This makes  $\alpha$  about .9165 radians, or 52.5 degrees.

The vectors  $(2, 1)$  and  $(-1, 2)$  are perpendicular as

$$(2, 1) \cdot (-1, 2) = -2 + 2 = 0.$$

The vectors  $(2, 3)$  and  $(6, 9)$  are parallel because

$$|u \cdot v| = |(2, 3) \cdot (6, 9)| = 12 + 27 = 39,$$

and

$$\|(2, 3)\| = \sqrt{13}, \quad \|(6, 9)\| = 3\sqrt{13}, \quad \text{and} \quad \|(2, 3)\| \cdot \|(6, 9)\| = 3 \cdot 13 = 39.$$

If  $u = (2, 1)$ , then  $\|u\| = \sqrt{5}$ , so that  $\frac{1}{\sqrt{5}}(2, 1)$  is a unit vector in the direction of  $(2, 1)$ .

**Exercise 22.** Find the cosine of the angle  $\alpha$  between the vectors  $(2, 5)$  and  $(-1, 3)$ , as well as  $\alpha$  itself (in radians and degrees). Show that the vectors  $(1, 2, 3)$  and  $(5, -1, -1)$  are perpendicular. Show that the vectors  $(1, -3)$  and  $(-2, 6)$  are parallel. Find a unit vector in the direction of  $u = (1, 3, 4)$ , as well as a unit vector in the opposite direction of  $u$ .

**Definition 2.7.** Two lines are said to be parallel if their direction vectors are multiples of each other. They are said to be perpendicular if their direction vectors are perpendicular to each other. More generally, the angle between two intersecting lines is the angle between their direction vectors.

**Exercise 23.** Let  $L$  be the line given by the equation  $3x + 4y = 12$ .

1. Find the line that is parallel to  $L$  and intersects the  $x$ -axis at  $x = 2$ .
2. Find the line that intersects  $L$  perpendicularly in  $(-4, 6)$ .

**Exercise 24.** Find the angle between  $u = (1, 3)$  and  $v = (-1, 2)$ . Find point of intersection and the angle between the lines given by the equations

$$3x + 2y + 5 = 0 \quad \& \quad x - 3y + 1 = 0.$$

*Proof of (2.4).* Consider a triangle whose sides are the vectors  $u$ ,  $v$ , and  $v - u$  as in Figure 2.2. Let  $\alpha$  be the angle between  $u$  and  $v$ .

Set  $b = \|u\|$ ,  $c = \|v\|$ , and  $a = \|v - u\|$ . The Theorem of Cosines says that

$$a^2 = b^2 + c^2 - 2bc \cos \alpha.$$

Using (3) in each argument, (2), and (4) from Proposition 2.4, we compute

$$(v - u) \cdot (v - u) = v \cdot v + u \cdot u - 2u \cdot v.$$

Comparing these two equations we conclude that

$$bc \cos \alpha = u \cdot v,$$

which implies (2.4). □

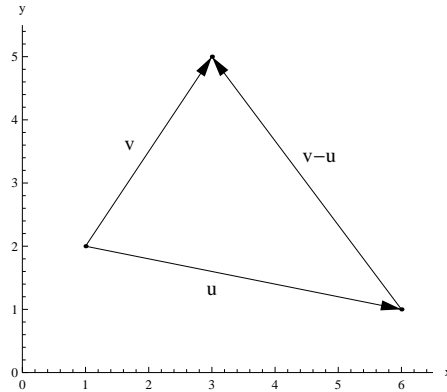


Figure 2.2: Theorem of Cosines

### 2.3 Properties of the Length Function

The properties in Proposition 2.4 combined with (2.3) tell us that for our length function we have

**Proposition 2.8.** *Suppose  $u$  and  $v$  are vectors. Then*

1.  $\|u\| \geq 0$  and  $\|u\| = 0$  if and only if  $u = 0$ .
2.  $\|cu\| = |c| \cdot \|u\|$ .
3.  $|u \cdot v| \leq \|u\| \cdot \|v\|$ .
4.  $\|u + v\| \leq \|u\| + \|v\|$ .

Property (3) is called the Cauchy–Schwarz inequality and (4) is the triangle inequality. In any triangle the sum of the lengths of two sides is at least as large as the length of the third side.

**Example 2.9.** We illustrate (3) and (4) from Proposition 2.8. Set  $u = (1, 2)$  and  $v = (3, -2)$ . Then

$$|u \cdot v| = |3 - 4| = 1 \leq \sqrt{5}\sqrt{13} = \|u\| \cdot \|v\|$$

and

$$\|(1, 2) + (3, -2)\| = \|(4, 0)\| = 4 \leq \sqrt{5} + \sqrt{13} = \|(1, 2)\| + \|(3, -2)\|.$$

**Exercise 25.** If  $u$  and  $v$  are vectors, show that

$$\boxed{\left| \|u\| - \|v\| \right| \leq \|u - v\| .}$$

*Proof of Proposition 2.8.* The first two properties are easily verified. The third one follows right away from (2.4), because the cosine of any angle is between  $-1$  and  $1$ . This third property can also be deduced from the properties in Proposition 2.4, combined with the definition in (2.3). Both sides in (3) are zero if  $u$  or  $v$  is zero. So suppose that  $u \neq 0$  and set

$$w = v - \frac{v \cdot u}{\|u\|^2} u .$$

We compute

$$\begin{aligned} 0 \leq \|w\|^2 &= \left( v - \frac{v \cdot u}{\|u\|^2} u \right) \cdot \left( v - \frac{v \cdot u}{\|u\|^2} u \right) \\ &= v \cdot v - 2 \frac{(v \cdot u)^2}{\|u\|^2} u \cdot v + \frac{(v \cdot u)^2}{\|u\|^2 \|u\|^2} u \cdot u \\ &= v \cdot v - \frac{(v \cdot u)^2}{\|u\|^2} \end{aligned}$$

Slightly rewritten, we find

$$0 \leq \|u\|^2 \|v\|^2 - (v \cdot u)^2 .$$

Bringing one term to the other side in the inequality and taking square roots, we find (3) from the proposition.

Observe that

$$\begin{aligned} \|u + v\|^2 &= (u + v) \cdot (u + v) \\ &= \|u\|^2 + 2u \cdot v + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

The inequality in the computation is the Cauchy–Schwarz inequality, without the absolute value sign. Taking square roots on both sides of the equation yields the triangle inequality.  $\square$

## 2.4 Applications

Consider vectors  $a = \overrightarrow{PQ}$  and  $b = \overrightarrow{PR}$ , and let  $\theta$  be the angle between them. We would like to decompose  $b$  into a sum of a vector  $\overrightarrow{PS}$  that is parallel to  $a$  and a second vector  $\overrightarrow{SR}$  that is perpendicular to  $a$ , see 2.3. The first vector is called the projection of  $b$  onto  $a$  and denoted by  $\text{proj}_a b$ .

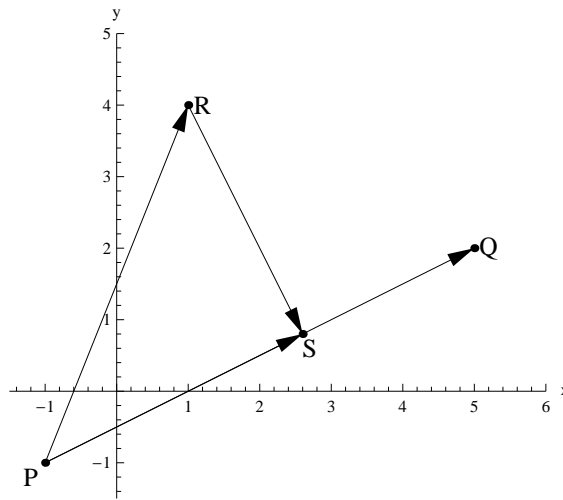


Figure 2.3: Orthogonal Projection

As a first step set

$$\text{comp}_a b = \frac{a \cdot b}{\|a\|} = \|b\| \cos \theta = \pm \|\overrightarrow{PS}\|.$$

The first equal sign defines  $\text{comp}_a b$ , the second one is (2.4), and the third one follows from right angle trigonometry, where the sign depends on whether  $\theta$  is acute or obtuse. We obtain  $\text{proj}_a b$  by multiplying a unit vector  $u_a$  in the direction of  $a$  with  $\text{comp}_a b$ :

$$\text{proj}_a b = \overrightarrow{PS} = \frac{a \cdot b}{\|a\|} u_a = \frac{a \cdot b}{\|a\|^2} a.$$

**Exercise 26.** Let  $a$  and  $b$  be vectors, and  $a \neq 0$ . Show that  $\text{proj}_a b$  and  $a - \text{proj}_a b$  are perpendicular to each other.

**Definition 2.10.** A normal vector  $n$  to a line  $L$  is a nonzero vector that is perpendicular to a direction vector of the line.

**Example 2.11.** If  $L$  is the line through the points  $P = (1, 5)$  and  $Q = (-1, 3)$  then  $D = P - Q = (2, 2)$  is a direction vector for the line, and  $n = (2, -2)$  is a normal vector.

**Exercise 27.** Consider a line given by the equation  $ax + by + c = 0$ . Show that  $n = (a, b)$  is a normal vector to this line.

**Exercise 28.** Consider a line given by the equation  $2x + 3y + 12 = 0$ . Find a direction and normal vector to this line. Graph the line and these two vectors.

Let us consider a typical application from an elementary physics course.

**Example 2.12.** A car with mass 1,000kg (roughly the mass of a subcompact) stands on a ramp. See Figure 2.4. The angle of incline of the ramp is  $15^\circ$ . Which force is required to keep the car from rolling down the ramp?

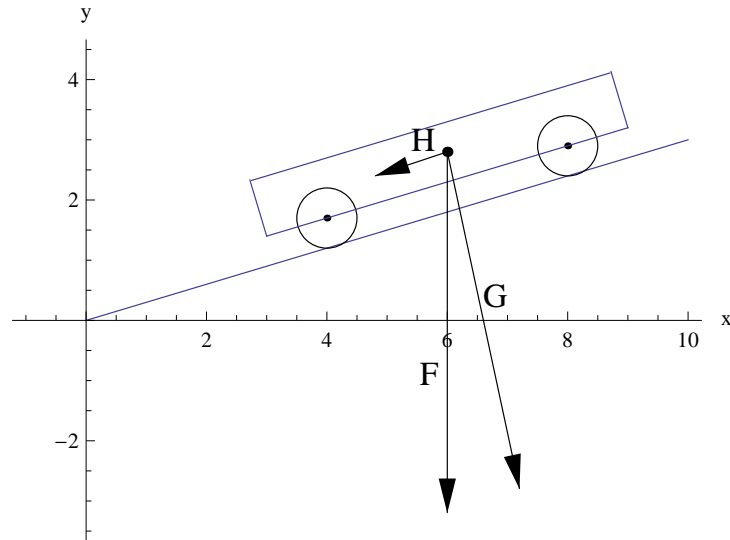


Figure 2.4: Car on ramp

The vertical force  $F$  acting on the car is 9,800 Newton, compare Example 1.6. The force  $F$  decomposes into two components,  $F = H + G$ , where  $H$  is parallel and  $G$  is perpendicular to the ramp. The parallel component is

the one that we need compensate for to keep the car from rolling down the ramp. We calculate its magnitude as the component of  $F$  in the direction of  $H$ .

Set

$$F = (0, -9800) \quad \text{and} \quad D = (1, \tan 15^\circ) = (1, 2 - \sqrt{3}),$$

where  $D$  is a vector parallel to the ramp. We will need a force of

$$|\text{comp}_D F| = \frac{|D \cdot F|}{\sqrt{D \cdot D}} = \frac{(1, (2 - \sqrt{3})) \cdot (0, 9800)}{\sqrt{(1, (2 - \sqrt{3})) \cdot (1, (2 - \sqrt{3}))}} \sim 2536$$

Newton to keep the car from rolling down the ramp.

Let  $L$  be the line given by the equation  $ax + by + c = 0$  and let  $R = (x_0, y_0)$  be a point in the plane. Then the distance between  $L$  and  $R$  is

$$(2.5) \quad \boxed{\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}}$$

**Example 2.13.** The distance between the line  $2x + 3y - 6 = 0$  and the point  $(4, 4)$  is

$$\frac{|2 \cdot 4 + 3 \cdot 4 - 6|}{\sqrt{2^2 + 3^2}} = \frac{14}{\sqrt{13}}$$

**Exercise 29.** Draw the line through the points  $P = (2, 3)$  and  $Q = (4, -1)$ . Set  $R = (5, 5)$ . Sketch this data. Compute and measure the distance between  $R$  and the line.

**Exercise 30.** Find two values for  $a$ , so that the point  $(3, a)$  has distance 2 from the line  $3x - 2y + 6 = 0$ . Sketch the situation and the solution.

**Exercise 31.** Provide a sensible definition for the distance between two parallel lines, and find the distance between the lines  $3x - 2y + 6 = 0$  and  $3x - 2y + 9 = 0$ .

*Proof of (2.5).* For an illustration, see Figure 2.5. Let  $P$  be the orthogonal projection of  $R$  onto the line  $L$  and  $Q$  be any point on the line. Then we find a triangle with vertices  $Q$ ,  $P$  and  $R$ , which has a right angle at  $P$ . The distance between  $R$  and the line  $L$  will be  $|\overrightarrow{RP}|$ , the length of the line segment between  $R$  and  $P$ .

Set  $n = \overrightarrow{RP}$  and  $w = \overrightarrow{RQ} = Q - R$ . The equation of the line is only determined up to multiplication with a nonzero scalar, so we may as well set

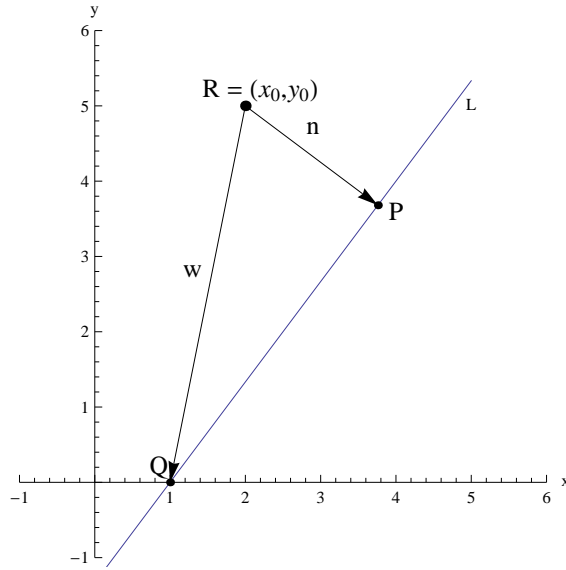


Figure 2.5: Point to line

$n = (a, b)$ . As  $Q = (x_1, y_1)$  is a point on the line, we see that  $ax_1 + by_1 + c = 0$  and  $(a, b) \cdot Q = -c$ . Also  $(a, b) \cdot (-R) = -ax_0 - by_0$ . We compute that

$$\|\vec{RP}\| = |\text{comp}_n w| = \frac{|n \cdot w|}{\|n\|} = \frac{|(a, b) \cdot (Q - R)|}{\sqrt{a^2 + b^2}} = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}},$$

which is exactly what we claimed.  $\square$