ORDERED DIRECT IMPLICATIONAL BASIS OF A FINITE CLOSURE SYSTEM

K. ADARICHEVA, J. B. NATION, AND R. RAND

Abstract. Closure system on a finite set is a unifying concept in logic programming, relational data bases and knowledge systems. It can also be presented in the terms of finite lattices, and the tools of economic description of a finite lattice have long existed in lattice theory. We present this approach by describing the so-called D-basis and introducing the concept of *ordered direct basis* of an implicational system. A direct basis of a closure operator, or an implicational system, is a set of implications that allows one to compute the closure of an arbitrary set by a single iteration. This property is preserved by the D-basis at the cost of following a prescribed order in which implications will be attended. In particular, using an ordered direct basis allows to optimize the *forward chaining procedure* in logic programming that uses the Horn fragment of propositional logic. One can extract the D-basis from any direct unit basis Σ in time polynomial in the size $s(\Sigma)$, and it takes only linear time of the cardinality of the D-basis to put it into a proper order. We produce examples of closure systems on a 6-element set, for which the canonical basis of Duquenne and Guigues is not ordered direct.

1. Introduction

In K. Bertet and B. Monjardet [5], it is shown that five implicational bases for a closure operator on a finite set, found in various contexts in the literature, are actually the same. The goal of this paper is to demonstrate that standard lattice-theoretic results about the "most economical way" to describe the structure of a finite lattice may be transformed into a basis for a closure system naturally associated with that lattice.

The coding of a finite lattice in the form of a so-called OD-graph was first suggested in [14]. We will call the basis directly following from this OD-graph a D-*basis*, since it is closely associated with a D-*relation* on the set of join-irreducibles of a lattice (not necessarily finite) that was crucial in the studies of free and lower bounded lattices, see [11]. The definition and the proof that D-basis does define a given closure system are given in section 4.

The D-basis is a subset of a so-called *dependence relation basis* (Definition 6 in [5]). Thus, it is also a subset of the *canonical direct unit basis* that unifies the five bases discussed in [5]. In section 5, we give an example to demonstrate that the reverse inclusion does not hold, thus showing that this newly introduced D-basis is generally shorter than the existing ones.

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Recall that the main desirable feature of bases from [5] is that they be *direct*, which means that the computation of the closure of any subset can be done by attending each implication from the basis only once. This makes the computation of closures a one-iteration process.

While the D-basis is not direct in this meaning of this term, the closures can still be computed in a single iteration of the basis, provided the basis was put in a specific order prior to computation. Moreover, there is a simple and effective linear time algorithm for ordering a D-basis appropriately. Thus, applying the D-basis can be compared to the iteration known in artificial intelligence as the *forward chaining algorithm*, see for example [10].

We introduce the definition of ordered iteration and ordered direct basis in section 6, where we also prove that the D-basis is ordered direct and discuss the algorithmic aspects of ordering it. The further directions of optimization of D-basis are outlined in section 8, where we also introduce the notion of an *ordered direct sequence* built from a given basis of a closure system.

In section 9, we also discuss the so-called E-relation, introduced in [11], which leads to the definition of the E-basis in closure systems *without D-cycles*. In general, the implications written from the E-relation do not necessarily form a basis of a closure system, but in closure systems without D-cycles, the E-basis is ordered direct, is contained in the D-basis, and often shorter than the D-basis. We discuss a polynomial time algorithm for ordering the E-basis.

We explore the connections between D-basis, E-basis and the so-called *canonical basis* introduced by Duquenne and Guigues in [9]. While the canonical basis has the minimal number of implications among all the bases of a closure system, it does not have the feature of D-basis or E-basis discussed in this paper, namely, it cannot be turned into an ordered direct basis. Section 10 of our paper presents examples of closure systems on a 6-element set, for which the canonical basis cannot be ordered. As a result, the time required for one iteration of D-basis wins over at least two iterations of the canonical basis. Further polynomial-time optimizations of both D-basis and the canonical basis are discussed.

We provide in section 11 test results comparing the performance of the D-basis with the Duquenne-Guigues canonical basis and canonical direct unit basis. Section 7 is devoted to discussion and testing the forward chaining algorithm in comparison to the ordered direct basis algorithm.

The next two sections contain the required definitions and establish connections between finite lattices, closure operators, implicational systems, Horn formulas and Horn Boolean functions. The reader may consult the survey [4] for various aspects of closure systems on finite sets.

2. Lattices and closure operators

By a *lattice*, one means an algebra with two binary operations $\wedge, \vee$, called *meet* and *join*, respectively. A lattice is finite when the base set of this algebra is finite. The symbols $\bigwedge, \bigvee$ are used when more than two elements meet or join. We will use the notation 0 for the least element of a lattice, and 1 for its greatest element. Simultaneously, every lattice is a partially ordered set in which every two elements have a least upper bound (which coincides with the join of those elements), and a greatest lower bound (the meet). If $a \leqslant b$ in lattice L, then we denote by $[a, b]$ the interval in L, i.e., the set of all c satisfying $a \leqslant c \leqslant b$.

Recall now the standard connection between a closure operator on a set and the lattice of its closed sets. Given a non-empty set S and the set $P(S) = 2^S$ of all its subsets, *a closure operator* is a map $\phi : P(S) \to P(S)$ that satisfies the following, for all $X, Y \in P(S)$:

(1) increasing: $X \subseteq \phi(X)$;
(2) isotone: $X \subseteq Y$ implies $\phi(X) \subseteq \phi(Y)$;
(3) idempotent: $\phi(\phi(X)) = \phi(X)$.

It would be convenient for us to refer to the pair $\langle S, \phi \rangle$ of a set S and a closure operator on it as a *closure system*.

A subset $X \subseteq S$ is called *closed* if $\phi(X) = X$. The collection of closed subsets of closure operator ϕ on S forms a lattice, which is usually called the *closure lattice* of the closure system $\langle S, \phi \rangle$. This paper deals with only finite closure systems and finite lattices.

Conversely, we can associate with every finite lattice L a particular closure system $\langle S, \phi \rangle$ in such a way that L is isomorphic to a closure lattice of that closure system. Consider $J(L) \subseteq L$, a subset of *join-irreducible elements*. An element $j \in L$ is called join-irreducible, if $j \neq 0$, and $j = a \vee b$ implies $a = j$ or $b = j$. We define a closure system with $S = J(L)$ and the following closure operator:

$$\phi(X) = [0, \bigvee X] \cap J(L)$$

It is straightforward to check that the closure lattice of ϕ is isomorphic to L.

Example 1. *Consider a simple example illustrating a closure system built from the lattice $L = \{0, a, b, c, 1\}$, for which $0 < a < b < 1$, $0 < c < 1$, $a \vee c = b \vee c = 1$ and $a \wedge c = b \wedge c = 0$. Then $S = J(L) = \{a, b, c\}$. The closed subsets are $[0, x] \cap J(L)$ for $x \in L$, which are $\emptyset$, $\{a\}$, $\{c\}$,$\{a, b\}$ and $\{a, b, c\}$. Knowing all closed subsets, one can define a closure of X, or $\phi(X)$, as the smallest closed set containing X. For example, $\phi(\{b\}) = \{a, b\}$.*

There are infinitely many sets and closure operators whose closure lattice is isomorphic to a given L. On the other hand, the one just described is the unique one with two additional properties:

(1) $\phi(\emptyset) = \emptyset$;
(2) $\phi(\{i\}) \setminus \{i\}$ is closed, for every $i \in S$.

Condition (2) just says that each $\phi(\{i\})$ join irreducible. Note that (1) is a special case of (2), and that (2) implies the property

(3) $\phi(\{i\}) = \phi(\{j\})$ implies $i = j$, for any $i, j \in S$.

We will call a closure system with properties (1), (2) above a *standard closure system*. Closure systems with (2) are called $(T\frac{1}{2})$ closure spaces in Wild [15].

A closure system satisfying property (3) is said to be *reduced*. Note that (3) implies $|\phi(\emptyset)| \leqslant 1$. Reduced closure systems correspond to a representation of a lattice L as a closure system on a set S with $J(L) \subseteq S \subseteq L$ and $\phi(X) = [0, \bigvee X] \cap S$. A natural example is the set of principal congruences in the congruence lattice of a finite algebra. Every standard closure system is reduced, and reduced closure systems form a useful intermediate ground between standard and general systems.

It is straightforward to verify that the standard system is characterized by the property that the set S is of the smallest possible size. In other words, one cannot reduce S to define an equivalent closure system. On the other hand, the reduced systems might have excessive elements in S.

Example 2. *Consider again lattice* $L = \{0, a, b, c, 1\}$ *from Example 1. It will represent the closure lattice on* $S_1 = \{a, b, c, d\}$*, where the closed sets are* $\emptyset$*,* $\{a\}$*,* $\{c\}$*,*$\{a, b\}$ *and* $\{a, b, c, d\}$*. Thus, in this representation* $J(L) \subset S_1$*, and property (2) fails:* $\phi(\{d\}) \setminus \{d\} = \{a, b, c\}$ *is not closed. On the other hand, property (3) holds, thus, it is a reduced closure system. Apparently,* S_1 *can be reduced by element d, to get an equivalent representation of Example 1.*

If the closure system $\langle S, \phi\rangle$ is not reduced, one can modify it to produce an equivalent one that is reduced. Moreover, there is an effective algorithm for doing so. Thus, for all practical purposes, one can work with a reduced closure system $\langle U, \mu\rangle$ replacing a given one $\langle S, \phi\rangle$. Slightly more effort yields an equivalent standard closure system $\langle V, \nu\rangle$. The transition is described as follows.

If $\phi(\emptyset) = A \subseteq S$ in $\langle S, \phi\rangle$, then define $T = S \setminus A$, and redefine a closure operator: $\tau(Y) = \phi(Y) \setminus A$, for all $Y \subseteq T$. The closure system $\langle T, \tau\rangle$ satisfies property (1). As (1) is required for a standard closure system, but not for a reduced system, this step may be omitted if only the latter is sought.

Next define an equivalence relation $\approx$ on T by $x \approx y$ if and only if $\tau(x) = \tau(y)$. Then factor out $\approx$, letting $U = T/\approx$ and $\mu(Y) = \tau(Y)/\approx$ for $Y \subseteq U$. Alternately, we could define U to be a set of representatives for $T/\approx$ and μ to be the restriction of τ. Either way, one easily checks that μ is a well-defined closure operator on U, and that the closure lattice of $\langle U, \mu\rangle$ is isomorphic to that of $\langle S, \phi\rangle$. At this point, $\langle U, \mu\rangle$ is reduced. Moreover, we can recover the original system $\langle S, \phi\rangle$ by expanding the equivalence classes and adding back in $\phi(\emptyset)$. If desired, we can now continue to produce an equivalent standard closure system.

Let $V = \{u \in U : \mu(\{u\}) \setminus \{u\} \text{ is closed}\}$, that is, $u \notin \mu(\mu(\{u\}) \setminus \{u\})$, and for $Z \subseteq V$ let $\nu(Z) = \mu(Z) \cap V$. It is straightforward to verify that $\langle V, \nu\rangle$ is a closure system satisfying (1) and (2), and that the lattice of closed sets of $\langle V, \nu\rangle$ is isomorphic to that of $\langle U, \nu\rangle$.

For the sequel, we will consider primarily reduced closure systems. Given an arbitrary closure system, not necessarily reduced, the above reduction can be considered as a setup process to allow us to apply the D-basis and related methods.

3. The bases of closure systems, Horn formulas and Horn Boolean functions

If $y \in \phi(X)$, then this relation between an element $y \in S$ and a subset $X \subseteq S$ in a closure system can be written in the form of implication: $X \to y$. Thus, the closure system $\langle S, \phi\rangle$ can be replaced by the set of implications:

$$\Sigma_\phi = \{X \to y : y \in S, X \subseteq S \text{ and } y \in \phi(X)\}$$

Conversely, any set of implications Σ defines a closure system: the closed sets are exactly subsets $Y \subseteq S$ that respect the implications from Σ, i.e., if $X \to x$ is in Σ, and $X \subseteq Y$, then $x \in Y$.

Two sets of implications Σ and Σ' on the same set S are called *equivalent*, if they define the same closure system on S. The term *basis* is used for a set of implications Σ' satisfying some minimality condition; thus there may be different types of bases.

Note that, in general, one can consider implications of the form $X \to Y$, where Y is not necessarily a one-element subset of S. Following [5], we will call basis Σ a *unit implicational basis* if $|Y| = 1$ for all implications $X \to Y$ in Σ. We will mostly be concerned with unit implicational bases, except for the discussion of the canonical basis of Duquenne-Guigues and its comparison with D-basis and E-basis. Given any unit basis, we can always collapse the implications with the same premise into one with all conclusions combined into a single set. This will be called an *aggregated* basis.

For a set of implications $\Sigma = \{X_1 \to Y_1, \dots, X_m \to Y_m\}$, define the *size* by $s(\Sigma) = \sum_{j=1}^{m}(|X_j| + |Y_j|)$. This is one convenient measure of the complexity of an implicational system.

In general, implications $X \to y$, where $X \subseteq S$ and $x \in S$, can be treated as the formulas of propositional logic over the set of literals S, equivalent to $y \vee \bigvee_{x\in X} \neg x$. Formulas of this form are also called *definite Horn clauses*. More generally, Horn clauses are disjunctions of negations of several literals and at most one positive literal. The presence of a positive literal makes a Horn clause *definite*. A *Horn formula* is a conjunction of Horn clauses.

What is called a *model* of Horn clause in logic programming literature corresponds to a closed set of the closure operator defined by this clause. Indeed, by the definition, a model is simply a tuple $m \in 2^S$ of zeros and ones assigned to literals from S, such that the formula is true (=1) on this assignment. If the formula is a definite Horn clause $X \to y$, then m corresponds to a subset Y of S that is closed for a closure operator on S defined by $X \to y$. In fact, m is just the characteristic function of Y.

There is also a direct correspondence between Horn formulas and Horn Boolean functions: a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is called a (*pure* or definite) *Horn function*, if it has some CNF representation given by a (definite) Horn formula Σ. The dual definition is sometimes used in the literature, so that a Horn function is given by some formula in DNF, whose negation is a Horn formula [6]. Using either definition, one can translate many results on Horn Boolean functions to the language of closure operators, see more details in [5].

Consider a set Σ of Horn clauses over some finite set of literals $S = \{x_1, \dots, x_n\}$. If some Horn clause α in Σ is not definite, i.e., is of the form $\bigvee_{x\in X} \neg x$, $X \subset S$, and it does not use all literals from S, then we could define the set of definite clauses $\Sigma_\alpha = \{X \to y : y \in S \setminus X\}$. It is easy to observe that the set of models of Σ_α consists of all models of $\bigvee_{x\in X} \neg x$ and one additional model, which is a tuple of all ones, representing the set S itself. If some clause $\beta \in \Sigma$ is not definite and uses all the literals from S, then we define $\Sigma_\beta = \emptyset$ (another possibility, $x_1 \to x_1$). Again, the set of models Σ_β, i.e., all tuples of zeros and ones, extends the models of β by single tuple of all ones. If follows that the set of definite clauses Σ', where each non-definite clause α from Σ is replaced by a set of clauses Σ_α, has the set of models that extends the set of models of Σ by a single tuple of all ones. This includes the case when Σ has no models, i.e., when it is *inconsistent*.

This observation allows us to reduce the solution of various questions about sets of Horn clauses to sets of *definite* Horn clauses. Thus, it emphasizes the importance of the study of closure operators on S.

One of the important questions in logic programming is whether one clause ϕ is a *consequence* of the set (or conjunction) of clauses Σ. Denoted by $\Sigma \models \phi$, this means that every model of Σ is also a model of ϕ. If ϕ and formulas in Σ are Horn clauses, then, translating this question to the language of closure systems, one reduces it to checking whether every closed set of a closure system defined by Σ respects ϕ.

4. The D-basis

In this section we are going to define a basis that translates to the language of closure systems the defining relations of a finite lattice developed in the lattice theory framework. One can consult [11] for the corresponding notion of a minimal cover and D-relation used in the theory of free lattices and lower bounded lattices.

Given a *reduced* closure system $\langle S, \phi\rangle$, let us define two auxiliary relations. The first relation is between the subsets of S: we write $X \ll Y$, if for every $x \in X$ there is $y \in Y$ satisfying $x \in \phi(y)$. In Example 1, for instance, we have $\{a\} \ll \{b\}$, $\{a, c\} \ll \{b, c\}$ and $\{c\} \ll \{a, c\}$. Note that $X \subseteq Y$ implies $X \ll Y$. We also write $X \sim_{\ll} Y$, if $X \ll Y$ and $Y \ll X$. This is true for $X = \{a, b, c\}$ and $Y = \{b, c\}$ in Example 1.

Several observations are easy.

Lemma 3. *The relation $\sim_{\ll}$ is an equivalence relation on $P(S)$.*

We will denote a $\sim_{\ll}$-equivalence class containing X by $[X]$. Note that for any two members $X, Y \in [X]$, we have $\phi(X) = \phi(Y)$. There is a natural order $\leqslant_c$ on $\sim_{\ll}$-classes: $[X] \leqslant_c [Y]$ if $X \ll Y$.

Lemma 4. *The relation $\leqslant_c$ is a partial order on the set of $\sim_{\ll}$-equivalence classes.*

Each class $[X]$ is ordered itself with respect to set containment.

In Example 1, we have that $\{a, b, c\} \sim_{\ll} \{b, c\}$, and no more subsets are $\sim_{\ll}$-equivalent to $\{a, b, c\}$. Thus, $[\{b, c\}]$ consists of two subsets, and $\{b, c\} \subseteq \{a, b, c\}$ is the minimal (with respect to the order of containment) subset in that equivalence class of $\sim_{\ll}$. Also $\{a, c\} \ll \{b, c\}$, whence $[\{a, c\}] \leqslant_c [\{b, c\}]$.

Lemma 5. *If $\langle S, \phi\rangle$ is reduced, then each equivalence class $[X]$ has a unique minimal element with respect to the containment order.*

Proof. Let us assume that there are two minimal members X_1 and X_2 in $[X]$. Without loss of generality we assume that there is $x \in X_1 \setminus X_2$. Since $X_1 \ll X_2 \ll X_1$, we have $x \in \phi(x_2)$ and $x_2 \in \phi(x_1)$, for some $x_1 \in X_1$, $x_2 \in X_2$. We cannot have $x = x_1$, because, if so, then $\phi(x) = \phi(x_2)$, which implies $x = x_2$, since our closure system is reduced. This would contradict to the choice of x.

Thus, $x \neq x_1$, and $x \in \phi(x_1)$. But then we can reduce X_1 to $X' = X_1 \setminus x \subset X_1$, which is still a member of $[X]$ since $X_1 \ll X' \subset X_1$. This contradicts the minimality of X_1 in $[X]$. □

The second relation we want to introduce in this section is between an element $x \in S$ and a subset $X \subseteq S$, which will be called a *cover* of x. (In lattice theory, the terminology *nontrivial join cover* is used.) We will write $x \lhd X$, if $x \in \phi(X)$ and $x \notin \phi(x')$, for any $x' \in X$. This notion is illustrated in Example 1 by $b \lhd \{a, c\}$. Note that it is not true that $a \lhd \{b, c\}$, because $a < b$, so that $a \in \phi(b)$ for the corresponding standard closure operator.

We will call a subset $Y \subseteq S$ a *minimal cover* of an element $x \in S$, if Y is a cover of x, and for every other cover Z of x, $Z \ll Y$ implies $Y \subseteq Z$. So a minimal cover of x is a cover Y that is minimal with respect to the quasi-order $\ll$, and minimal with respect to set containment within its $\sim_{\ll}$-equivalence class $[Y]$, as per Lemma 5.

To illustrate this notion, let us slightly modify Example 1. Rename element 0 by d and add a new 0 element: $0 < d$, resulting in a lattice L_1 with $J(L_1) = J(L) \cup \{d\}$.

We will have $Y = \{a, c\}$ as a minimal cover for b. Indeed, the only other cover for b is $Z = \{a, c, d\}$, for which we have $Z \ll Y$ and $Y \subseteq Z$.

Lemma 6. *For a reduced closure system, if $x \lhd X$, then there exists Y such that $x \lhd Y$, $Y \ll X$ and Y is a minimal cover for x. In other words, every cover can be $\ll$-reduced to a minimal cover.*

Proof. Consider $P_x = \{[X] : x \lhd X\}$, a sub-poset in the $\leqslant_c$ poset of $\sim_{\ll}$-classes. If it is not empty, choose a minimal element in this sub-poset, say $[Y]$, and let Y be the unique minimal element in $[Y]$ with respect to containment, which exists due to Lemma 5. Then $Y \ll X$ and $x \lhd Y$. It remains to show that, for every other cover Z of x, $Z \ll Y$ implies $Y \subseteq Z$. Indeed, since $Z \ll Y$, we have $[Z] \leqslant_c [Y]$. But $[Y]$ is the minimal element in P_x, hence, $[Z] = [Y]$. It follows that $Y \subseteq Z$, since Y is the minimal element of $[Y]$ with respect to containment order. □

We finish this section by introducing the D-basis of a reduced closure system.

Definition 7. *Given a reduced closure system $\langle S, \phi \rangle$, we define the D-basis Σ_D as a union of two subsets of implications:*

(1) $\{y \to x : x \in \phi(y) \setminus y,\ y \in S\}$*;*
(2) $\{X \to x : X$ *is a minimal cover for* $x\}$.

Part (1) in the definition of the D-basis will also be called the *binary part* of the basis, due to the fact that both the premise and the conclusion of implications in (1) are one-element subsets of S.

For the closure system $\langle J(L), \phi \rangle$ associated with the lattice L in Example 1, the D-basis consists of two implications: $b \to a$ and $\{a, c\} \to b$.

Lemma 8. *Σ_D generates $\langle S, \phi \rangle$.*

Proof. We need to show that, for any $x \in S$ and $X \subseteq S$ such that $x \in \phi(X)$, the implication $X \to x$ follows from implications in Σ_D.

If $x \in \phi(x')$, for some $x' \in X$, then $X \to x$ follows from $x' \to x$ that is in Σ_D. So assume that $x \notin \phi(x')$, for any $x' \in X$. Then $x \lhd X$. According to Lemma 6, there exists $Y \ll X$ such that $x \lhd Y$, and Y is a minimal cover for x. Then $Y \to x$ is in Σ_D. Besides, for each $y \in Y$ there exists $x_y \in X$ such that $y \in \phi(x_y)$. Therefore, $x_y \to y$ is in Σ_D as well. Evidently, $X \to x$ is a consequence of $Y \to x$ and $\{x_y \to y : y \in Y\}$. □

5. Comparison of the D-basis and the dependence relation basis

One of the bases discussed in [5] is the *dependence relation basis*. For a closure system $\langle S, \phi \rangle$, not necessarily reduced, the dependence relation basis is

$$\Sigma_\delta = \{X \to y : y \in \phi(X) \setminus X \text{ and } y \notin \phi(Z) \text{ for all } Z \subset X\}.$$

Since $Z \subseteq X$ implies $Z \ll X$, a minimal cover (as defined above) is automatically minimal with respect to containment. Thus we have the following connection.

Lemma 9. *For a reduced closure system, $\Sigma_D \subseteq \Sigma_\delta$.*

For later reference, the dependence relation δ from Monjardet [13] can be described by $y \delta x$ whenever $x \in X$ for some $X \to y$ in Σ_δ.

In the next example and in the sequel, whenever there is no confusion, we will omit the braces in notations of subsets of some set S: $\{x\}, \{a, b, c\}$, etc. will be denoted simply x, abc, etc.

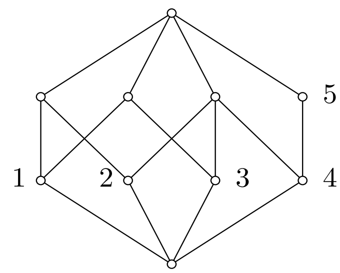

FIGURE 1. Example 10

Example 10. *This example is based on Example 5 from* [5]. *Consider the closure system on* $S = \{1,2,3,4,5\}$ *with the set of closed subsets* $F = \{\emptyset, 1, 2, 3, 4, 12, 13, 234, 45, 12345\}$. *Then* $\Sigma_\delta = \{5 \to 4, 23 \to 4, 24 \to 3,\ 34 \to 2, 14 \to 2, 14 \to 3, 14 \to 5, 25 \to 1, 35 \to 1,\ 15 \to 2, 35 \to 2, 15 \to 3, 25 \to 3, 123 \to 5\}$.

All implications except $5 \to 4$ are of the form $X \to x$, where $x \lhd X$. On the other hand, not all covers X are minimal covers of x. We can check that each of implications $15 \to 2, 35 \to 2, 15 \to 3, 25 \to 3$ does not represent a minimal cover. For example, $2 \lhd 15$, but $14 \ll 15$ and $2 \lhd 14$ is the minimal cover. In particular, D-basis consists of all implications from Σ_δ except the four indicated: $\Sigma_D = \{5 \to 4, 23 \to 4, 24 \to 3, 34 \to 2, 14 \to 2, 14 \to 3, 14 \to 5, 25 \to 1, 35 \to 1, 123 \to 5\}$.

As this example demonstrates, the D-basis can be obtained from Σ_δ simply by removing some unnecessary implications. It turns out that the same can be done for the big range of bases called *direct unit bases*. Moreover, it can be done in polynomial time in the size of the given basis. See Proposition 16 in the next section.

6. DIRECT BASIS VERSUS ORDERED DIRECT BASIS

The bases discussed in Bertet and Monjardet [5] are, in general, redundant: a proper subset of such a basis would generate the same closure system. For example, as we saw in the previous section, Σ_δ from Example 5 was reduced to a smaller basis Σ_D. Example 29 shows that the D-basis can also be redundant; see Remark 30.

While the desire to keep the basis as small as possible might be a plausible task, there is another property of a basis that could be better appreciated in a programming setting. Here we recall the definition of a *direct basis*.

If Σ is some set of implications, then let $\pi_\Sigma(X) = X \cup \bigcup\{B : A \subseteq X \text{ and } (A \to B) \in \Sigma\}$. In order to obtain $\phi_\Sigma(X)$, for any $X \subseteq S$, one would normally need to repeat several iterations of π: $\phi(X) = \pi(X) \cup \pi^2(X) \cup \pi^3(X) \ldots$.

The bases for which one can obtain the closure of any set X performing only one iteration, i.e., $\phi(X) = \pi(X)$, are called *direct*.

It follows from Theorem 15 of [5] that the dependency relation basis Σ_δ is direct. Moreover, this basis is direct-optimal, meaning that no other direct basis for the same closure system can be found of smaller total size. (The *total size* $t(\Sigma)$ is the sum of the cardinalities of all sets participating in its implications. This will be less than $s(\Sigma)$ if some sets are repeated.) In particular, any reduction of Σ_δ will cease to be direct. Thus, there is a apparent trade-off between the number of implications in the basis and the number of iterations one needs to compute the closures of subsets.

The goal of this section to implement a different approach to the concept of iteration. That would allow the same number of programming steps as with the iteration of π, while allowing us to reduce the bases to a smaller size.

Definition 11. *Suppose the set of implications Σ is equipped with some linear order $<$, or equivalently, the implications are indexed as $\Sigma = \{s_1, s_2, \dots, s_n\}$. Define a mapping $\rho_\Sigma : P(S) \to P(S)$ associated with this ordering as follows. For any $X \subseteq S$, let $X_0 = X$. If X_k is computed and implication s_{k+1} is $A \to B$, then*

$$X_{k+1} = \begin{cases} X_k \cup B, & \text{if } A \subseteq X_k, \\ X_k, & \text{otherwise.} \end{cases}$$

Finally, $\rho_\Sigma(X) = X_n$. We will call ρ_Σ an ordered iteration *of Σ.*

Apparently, $\pi_\Sigma(X) \subseteq \rho_\Sigma(X)$, because all implications from Σ are applied to original subset X, while they are applied to potentially bigger subsets X_k in construction for $\rho_\Sigma(X)$. We note though that assuming the order on Σ is established, the number of computational steps to produce $\rho_\Sigma(X)$ is the same as for $\pi_\Sigma(X)$.

Definition 12. *The set of implications with some linear ordering on it, $\langle \Sigma, < \rangle$, is called an* ordered direct basis*, if, with respect to this ordering, $\phi_\Sigma(X) = \rho_\Sigma(X)$ for all $X \subseteq S$.*

Our next goal is to demonstrate that Σ_D is, in fact, an ordered direct basis. Moreover, it does not take much computational effort to impose a proper ordering on Σ_D.

Theorem 13. *Let Σ_D be the D-basis for a reduced closure system. Let $<$ be any linear ordering on Σ_D such that all implications of the form $y \to z$ precede all implications of the form $X \to x$, where X is a minimal cover of x. Then, with respect to this ordering, Σ_D is an ordered direct basis.*

Proof. Suppose that $X \subseteq S$ and $b \in \phi(X)$. We want to show that b will appear in one of the X_k in the sequence that leads to $\rho(X)$.

If $b \in \phi(a)$ for some $a \in X$, then b will appear in some X_k, when $a \to b$ from Σ_D is applied. So now assume that $b \notin \phi(a)$ for every $a \in X$. Then $b \lhd X$ and, according to Lemma 6, there exists $Y \ll X$ such that $b \lhd Y$ and Y is a minimal cover for y. It follows that for any $y \in Y$ there exists $a \in X$ such that $y \in \phi(a)$. All implications $a \to y$ will be applied prior to any application with the minimal cover. It follows that by the time the implication s_k, say $Y \to b$, is tested against X_{k-1}, we will have $Y \subseteq X_{k-1}$. Hence, $X_k = X_{k-1} \cup \{b\}$. □

We note in this regard, that the D-basis is ordered direct in both forms: in its original unit form, and in the aggregated form. Indeed, it follows from the fact that the only restriction on the order of the D-basis is to have its binary part prior to the rest of the basis.

Corollary 14. *If $\Sigma_D = \{s_1, \dots, s_m\}$ is the D-basis of a reduced implicational system Σ, then it requires time $O(m)$ to turn it into an ordered direct basis of Σ.*

Example 15. *Consider the closure system with $S = \{1, 2, 3, 4, 5, 6\}$ and the family of closed sets $F = \{1, 12, 13, 4, 45, 134, 136, 1362, 1346, 13456, 123456\}$. Then the*

D-basis of this system is $\Sigma_D = \{5 \to 4, 14 \to 3, 23 \to 6, 6 \to 3, 15 \to 6, 24 \to 6, 24 \to 5, 3 \to 1, 2 \to 1\}$. *According to Theorem 13, a proper ordering that turns this basis into ordered direct can be defined, for example, as: (1)* $5 \to 4$, *(2)* $6 \to 3$, *(3)* $3 \to 1$, *(4)* $2 \to 1$, *(5)* $14 \to 3$, *(6)* $23 \to 6$, *(7)* $15 \to 6$, *(8)* $24 \to 6$, *(9)* $24 \to 5$.

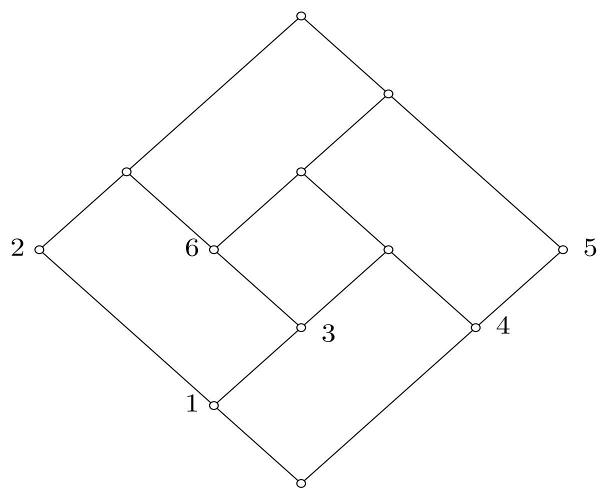

FIGURE 2. Example 15

7. PROCESSING OF ORDERED BASIS VERSUS FORWARD CHAINING ALGORITHM

In this section we look more closely at the algorithmic aspects of finding the closures of input sets using the *ordered basis* approach in comparison to the well-known *forward chaining algorithm*, introduced in [8] for checking the satisfiability of Horn formulas. As we pointed out earlier, these seemingly different tasks can be seen equivalent, when interpreting Horn formulas as implications. The ordered basis algorithm can be applied to any basis that is ordered direct. In our case, we did the comparisons using the D-basis as an input basis for both algorithms.

We will assume that the base set is $S = \{x_1, \dots, x_n\}$, which can be interpreted as propositional variables, and the closure system is given by a unit basis Σ with m implications.

The forward chaining procedure effectively requires two runs of the given basis. In the first (setup) pass, it constructs the *ClauseList*, *Propositions*, and *Consequent* arrays, along with the queue of *True* elements thought of as an input set. Here *ClauseList* is the set of arrays X_i, for each propositional variable x_i, that keep the indices of all clauses (implications) in which x_i appears as a negative literal (or, equivalently, appears on the left-hand side of the implication). *Propositions* is an array of size m: for every implication s_i, it has the number of propositional variables in its left-hand side that remain to be evaluated to true. Also, *Consequent* is an m-element array that, for every clause s_i, points to a proposition x_j that appears on its right-hand side.

In the second run, to actually compute the closure, it pops a True element, say, x_i from the queue, addresses the corresponding X_i in *ClauseList*, and, for each pointer to a clause/implication there, it decrements by 1 the corresponding entry of *Propositions*. Whenever the entry in *Propositions* reaches 0, the corresponding implication is ready to be processed, which means taking the corresponding variable in *Consequent* and putting it into queue of *True* elements.

Since every entry of *Propositions* will, in the worst case, be reduced to zero, the number of steps in computing the closure is bounded by the size $s(\Sigma)$, i.e., the combined length of the implications in the basis. Including the pre-processing steps, the forward chaining algorithm should require $O(s(\Sigma))$ operations to compute the closure. If the closures of multiple sets are to be performed, of course, the setup steps can be abbreviated: only *Propositions* and *True* need be updated for subsequent runs.

Finding the closure of a given set from the D-basis is similarly linear in the basis length. Each implication s_i is recorded as an array X_i where the zero index contains the positive literal and the following indices contain the negative literals. We check each literal against a boolean array of length n which stores a 1 at the index i corresponding to a true literal x_i. If any negative literal corresponds to a 0 (false) in the array, we move on to the following implication. If all the negative literals in a given implication are found to be true, we set the value in the boolean array that corresponds to its positive literal to true. Since in the worst case, we process each literal of every clause exactly once, this procedure takes $O(s(\Sigma))$ operations to compute the closure.

In our testing of ordered-basis versus forward-chaining performance, we evaluated the time spent on finding the closures of random subsets of the domain by computing the number of elementary operations (setting a variable, comparing Boolean values, etc.) performed during the processing. The inputs for both algorithms were identical.

Test data bears out both complexity estimates. In 100,000 runs of randomly generated closure systems and their D-bases from the domain $S = \{1, 2, 3, 4, 5\}$, the average length $|\Sigma_D|$ was 18.99. It took an average of 59.82 simple operations (33.71 when not counting the pre-processing time) to find the closure of an arbitrary input set in forward chaining procedure compared to 34.35 simple operations to compute the closure using the linear processing of the D-basis, with similar values for larger domains (see the table in Figure 3). Taking all computing overhead into account then, it takes very close to $2|\Sigma_D|$ and $3|\Sigma_D|$ operations for the ordered-basis and forward-chaining respectively.

Noticeably, the ordered-basis approach does not actually require the representation of propositions as $S = \{x_1, \ldots, x_n\}$ and implications as $\Sigma = \{s_1, \ldots, s_m\}$, where each proposition has an associated integer value, necessary for indexing and traversing *ClauseList*, *Propositions*, and *Consequent*. Though we take advantage of the integer value in constructing the array of true values, we could easily substitute that array with a simple set of satisfied propositions. By contrast, to use the forward chaining method on a basis without this representation would require significant overhead in hashing each proposition to its corresponding integer.

Additionally, the ordered-basis approach eliminates the need for pre-processing of the basis to store it in the form of *ClauseList* and *Consequent*. This is particularly important when the basis may not fit into main memory. Instead of having to

Length	Basis Size	Forward Chaining	Chaining (pre-processed)	D-basis
4	8.02	29.39	17.38	16.14
5	18.99	59.82	33.71	34.35
6	35.04	105.47	58.60	60.62
7	56.91	169.34	93.30	95.86

Figure 3. Comparison

individually access each array X_i in *ClauseList* when the propositional variable x_i appears at the head of the queue, the ordered-basis approach allows us to parse the basis in conveniently sized pieces.

There is at least one observation how the idea of the ordered basis may improve the performance of existing forward chaining algorithm. Indexing the implications according to the proper order of the D-basis, whenever we add a positive literal to the *True* queue we may additionally maintain the index i of the implication from which it was derived. Then, when we process the literal, we only need to update j-entries of *Propositions* where $j \geq i$, saving us significant processing time for very large sets.

8. Building and optimizing the D-basis

As we saw in Lemma 9 and Example 10, the D-basis Σ_D of any reduced closure system is a subset of the direct unit basis Σ_δ. The next statement shows that, given any direct unit basis, one can extract the D-basis from it in a polynomial time procedure.

Proposition 16. *Let $\langle S, \phi\rangle$ be a reduced closure system. If the direct unit basis Σ for this system has m implications, and $|S| = n$, then it requires time $O((nm)^2) \sim O(s(\Sigma)^2)$ to build the D-basis Σ_D equivalent to Σ.*

Proof. Let $\langle S, \phi\rangle$ be the closure system on set S defined by Σ. By Lemma 9, $\Sigma_D \subseteq \Sigma_\delta$. According to Theorem 15 of [5], Σ_δ coincides with the canonical iteration-free basis introduced by M. Wild in [15]. Hence, by Corollary 17 of [5], Σ_δ is the smallest basis, with respect to containment, of all direct unit bases of $\langle S, \phi\rangle$. Therefore, $\Sigma_D \subseteq \Sigma_\delta \subseteq \Sigma$.

It follows that Σ_D can simply be extracted from Σ by removing unnecessary implications. This amounts to finding the implications $X \to x$, where X will be a minimal join cover of x, among the implications of Σ.

Note that $O(m)$ steps will be needed to separate binary implications $y \to x$ from $X \to x$, where $|X| > 1$. The number of $x \in S$ that appear in the consequence of implications $X \to x$ is at most the minimum of m and n.

For every fixed x, it will take time $O(m)$ to separate all implications $X \to x$, and the number of such implications is at most m. If $X_1 \to x$ and $X_2 \to x$ are two implications in this set, we can decide in time $O(mn)$ whether $X_1 \ll X_2$ or $x \in \phi(y)$ for some $y \in X_2$. If either holds, $X_2 \to x$ does not belong to the D-basis.

To check this, consider the closure systems $\Sigma_i \subseteq \Sigma$, $i = 1, 2$ that consist of all binary implications of Σ, in addition to $X_i \to x$. Also, put an order on Σ_i, where all the binary implications precede $X_i \to x$. Apparently, x is in the closure of X_2, in the closure system defined on S by Σ_1, iff either $X_1 \ll X_2$ or $x \in \phi(y)$ for some $y \in X_2$.

As pointed out in section 7, computation of the closure of any input set, either by the forward chaining algorithm, or by the ordered basis algorithm, is linear in the size of the input, which in this case is essentially the size of the binary part of Σ, or $O(n^2)$.

At the worst case, about $O(m^2)$ comparisons have to be made, for different covers X_1, X_2 of the same element x, to determine the minimal ones. Hence, the overall complexity is $O(m^2n^2) \sim O(s(\Sigma)^2)$. □

It follows from the procedure of Proposition 16 that the D-basis is obtained from any direct unit basis by removing implications $X \to x$, for which X is not a minimal cover of x and $|X| > 1$. In particular, the binary part of the direct basis, i.e., implications of the form $y \to x$, remain in the D-basis.

We want to discuss a further optimization of the D-basis, as well as the any other basis that has the same binary part as the D-basis. As was observed in section 2, for a reduced closure system $\langle S, \phi \rangle$, the elements of S can be identified with elements of the closure lattice L, in such a way that $J(L) \subseteq S \subseteq L$. This correspondence induces a natural order on S, with $s \leqslant t$ if and only if $\phi(s) \subseteq \phi(t)$. Thus, an implication $y \to x$ belongs to the D-basis iff $x \in \phi(y)$ iff $x \leqslant y$. The binary part of the D-basis then describes the partially ordered set $(S, \leqslant)$.

Recall that, in the language of ordered sets, we say that y *covers* x if $y > x$ and there is no element z such that $y > z > x$.

We can shorten the binary part of the D-basis, leaving only those implications $y \to x$ for which y covers x in $(S, \leqslant)$. This will come at the cost of the need to order the remaining implications. For example, if $x \to y$, $y \to z$, $x \to z$ are three implications from the binary part of some D-basis, then the last implication can be removed, under condition that the first two will be placed in that particular order into the ordered D-basis. More generally, suppose only covering pairs are to be included in the binary part of an ordered basis. Then the ordering of the implications should be such that, if $x > y \geq z > t$ in S with the strict inequalities being covers, then $x \to y$ precedes $z \to t$.

Recall also that if some set of implications Σ' is ordered, then $\rho_{\Sigma'}(X)$, the ordered iteration of Σ', is defined for every $X \subseteq S$, see Definition 11.

Proposition 17. *Let Σ_1 be the binary part of the D-basis of a reduced closure system on a set S. If Σ_1 has k implications and $|S| = n$, then there is an $O(nk+n^2)$ time algorithm that extracts $\Sigma' \subseteq \Sigma_1$ describing the cover relation of join irreducible elements of closure system, and places the implications of Σ' into a proper order. Under this order, $\rho_{\Sigma'}(y) = \rho_{\Sigma_1}(y)$ for every $y \in S$.*

Proof. We have the partially ordered set $(S, \leqslant)$ of size n, whose cover relation has an upper bound k. Thus, it will take time $O(nk + n^2)$ to find the cover relation of this poset, see Theorem 11.3 in [11]. Let $\Sigma' \subseteq \Sigma_1$ be the set of all implications $y \to x$, where y covers x in $(S, \leqslant)$. It remains to put these implications into a proper order. If $(S, \leqslant_1)$ is any linear extension of $(S, \leqslant)$, then one can take any order of Σ' associated with this extension. Starting from the maximal element y

of $(S, \leqslant_1)$, write all implications $y \to x$ from Σ', in any order, then pick a next to maximal element z of $(S, \leqslant_1)$ and write all implications $z \to t$, in any order, then proceed with all elements of $(S, \leqslant_1)$ in the same manner, in descending order $\geq_1$. It remains to notice that there is an $O(n+k)$ algorithm for producing a linear extension a of partially ordered set with n elements and k pairs of comparable elements, see Theorem 11.1 in [11]. □

Now we want to deviate slightly from the notion of ordered direct basis to the notion of *ordered direct sequence of implications*. Suppose Σ is some basis of a closure system $\langle S, \phi \rangle$. The ordered sequence $\sigma = \langle s_1, \dots, s_t \rangle$ of implications from Σ, not all necessarily different, is called *an ordered direct sequence from* Σ, if $\rho_\sigma(X) = \phi(X)$ for every $X \subseteq S$.

The idea of ordered direct sequencing allows some further optimization of the D-basis. If $Z = \langle z_1, \dots, z_k \rangle$ and $T = \langle t_1, \dots, t_s \rangle$ are two ordered sequences, then $Z^\frown T$ denotes their concatenation (the attachment of T at the end of Z).

Lemma 18. *Suppose $\sigma = \Sigma_1^\frown \Sigma_2^\frown \Sigma_3$ is an ordered direct sequence from some basis Σ, where Σ_1, Σ_3 consist of binary implications in proper order of Proposition 17, Σ_2 consists of non-binary implications, and Σ_2 can be put into arbitrary order without changing the ordered direct status. If $(A \to y), (A \to x) \in \Sigma_2$ and $(y \to x) \in \Sigma_1$, then $A \to x$ can be dropped from Σ_2 and replaced by an additional $y \to x$ in Σ_3.*

Proof. We need to show that whenever Y is an input set such that $x \in \phi(Y)$, the replacement of $A \to x$ by $y \to x$ will not affect computation of $\rho_\sigma(Y)$.

Consider the case when $y \notin \phi(Y)$. Then also $A \not\subseteq \phi(Y)$, whence any implication with the premise A will never be applied in computation of $\rho_\sigma(Y)$. The same is true for implications with premise y, so replacement of $A \to x$ by $y \to x$ can trivially be done.

Now suppose that $y \in \phi(Y)$. By assumption, we can take $A \to x$ to be the last implication in the ordering of Σ_2. So consider Y_k, the result of ordered iteration of $\Sigma_1^\frown \Sigma_2 \setminus (A \to x)$ on the input set Y. If $y \in Y_k$, then we can drop $A \to x$ from Σ_2 and place $y \to x$ anywhere in proper order in Σ_3, which will guarantee that x appears in $\rho(Y)$. If $y \notin Y_k$, then there is $z \in Y_k$ such that there exists some sequence in Σ_3 from z to y. By assumption, Σ_3 is in the proper order, hence any implication $w \to y$ precedes $x \to t$. Thus, we can place $y \to x$ in between those groups, following the proper order on all binary implications from Proposition 17. After replacing $A \to x$ by $y \to x$ in proper position of Σ_3, we can still assume that the ordering of remaining part of Σ_2 can be arbitrary. □

Corollary 19. *Suppose Σ_D is the D-basis of some closure system. Consider $\Sigma_D^+ \subseteq \Sigma_D$ obtained from Σ_D by performing the following reductions:*

(a) *Remove $A \to x$, if $A \to y$ and $y \to x$ are also in Σ_D.*
(b) *Remove $z \to x$, if $z \to y$ and $y \to x$ are also in Σ_D.*

Let Σ_1 be a proper ordering of the binary part of Σ_D^+ given in Proposition 17, and let Σ_3 be a subordering of this proper ordering on implications $y \to x$ that appear in triples of $A \to x, A \to y, y \to x$ of (a). Finally, let Σ_2 be some ordering of non-binary implications of Σ_D^+. Then $\sigma = \Sigma_1^\frown \Sigma_2^\frown \Sigma_3$ is an ordered direct sequence for the basis Σ_D^+. In particular, the length of this sequence is no longer than the length of the D-basis.

Proof. Indeed, following the procedure of Lemma 18 we can replace all $A \to x$ from the triples $A \to x, A \to y, y \to x$ in Σ_D by the second copy of $y \to x$ in additional binary part Σ_3 that follows the non-binary part of the D-basis. □

Example 20. *Given the D-basis of the closure system: $\Sigma_D = \langle 3 \to 2, 2 \to 1, 3 \to 1, 45 \to 3, 45 \to 2, 45 \to 1\rangle$, we can produce a shorter basis $\Sigma_D^+ = \{3 \to 2, 2 \to 1, 45 \to 3\}$ with the ordered direct sequence: $\sigma = \langle 3 \to 2, 2 \to 1, 45 \to 3, 3 \to 2, 2 \to 1\rangle$. We note that Σ_D^+ is only half as long as Σ_D,, and its ordered direct sequence σ has the same length as D-basis with optimized binary part but the size of σ is smaller than that of the optimized D-basis.*

9. Closure systems without D-cycles and the E-basis

It turns out that the D-basis can be further reduced, when an additional property holds in a closure system $\langle S, \phi\rangle$. The results of this section follow closely the exposition given in [11], section 2.4.

We will write xDy, for $x, y \in S$, if $y \in Y$ for some minimal cover Y of x. We note that the D-relation is a subset of the dependence relation δ from section **??**.

Definition 21. *A sequence $x_1, x_2, \dots, x_n$, where $n > 1$, is called a D-*cycle*, if $x_1 D x_2 D \dots x_n D x_1$. A finite closure system $\langle S, \phi\rangle$ is said to be* without D-cycles *if it has no D-cycles.*

We note that the lattices of closed sets of closure systems without D-cycles are known in lattice-theoretical literature as *lower bounded*.

For every $x \in S$, let $M(x) = \{Y \subseteq S : Y \text{ is a minimal cover of } x\}$. The family $\phi(M(x)) = \{\phi(Y) : Y \in M(x)\}$ is ordered by set containment, so we can consider its minimal elements. Let $M^*(x) = \{Y \in M(x) : \phi(Y) \text{ is minimal in } \phi(M(x))\}$.

We will write xEy, for $x, y \in S$, if $y \in Y$ for some $Y \in M^*(x)$. According to the definition, if xEy then xDy. On the other hand, the converse is not always true.

Example 22. *Consider the closure system and its D-basis from Example 15. We note that this closure system has no D-cycles. We have three minimal covers of 6: $15, 24$ and 23. Since $\phi(15) = S \setminus 2$, $\phi(24) = S$ and $\phi(23) = S \setminus 45$, we have only two of these covers in $M^*(6)$: 15 and 23. Thus, while $6D4$, we do not have $6E4$.*

We now define two sequences of subsets of S, based on covers from $M(x)$ and $M^*(x)$, correspondingly.

Let $D_0 = E_0 = \{p \in S : p \in \phi(p_1, \dots, p_k) \text{ implies } p \in \phi(p_i) \text{ for some } i \leqslant k\}$. If D_k and E_k are defined, then $D_{k+1} = D_k \cup \{s \in S : \text{ if } s \lhd Y \text{ then } s \lhd Z \text{ for some } Z \subseteq D_k, Z \ll Y \text{ and } Z \in M(s)\}$. Similarly, $E_{k+1} = E_k \cup \{s \in S : \text{ if } s \lhd Y \text{ then } s \lhd Z \text{ for some } Z \subseteq E_k, Z \ll Y \text{ and } Z \in M^*(s)\}$. Apparently, $E_k \subseteq D_k$, for any k. The following result is proved in [11], Theorem 2.51.

Lemma 23. *If $\langle S, \phi\rangle$ is a reduced closure system without D-cycles, then, for some k, $S = E_k = D_k$.*

As a consequence, we can often shorten the D-basis for a closure system without D-cycles. We will say that $s \in S$ has D-rank $k = 0$, if $s \in D_0$, and $k > 0$, if $s \in D_k \setminus D_{k-1}$. According to Lemma 23, every $s \in S$ in a closure system without D-cycles has a D-rank.

Recall that a basis is called *aggregated* when all its premises are different. Every basis can be brought to the aggregated form by combining conclusions of all implications with the same premises.

Theorem 24. *Let $\langle S, \phi \rangle$ be a reduced closure system without D-cycles. Consider a subset Σ_E of the D-basis that is the union of two sets of implications:*

(1) $\{y \to x : x \in \phi(y)\}$,
(2) $\{X \to x : X \in M^*(x)\}$.

Then

(a) Σ_E *is a basis for* $\langle S, \phi \rangle$.
(b) Σ_E *is ordered direct.*
(c) *The aggregated form of* Σ_E *is ordered direct.*

Proof. To begin with, it is not true that every cover of an element $x \in S$ refines to a cover in $M^*(x)$, so Σ_E must be ordered more carefully than Σ_D. Nonetheless, mimicking the proof of Theorem 2.50 of [11], we can construct an order on Σ_E that makes it an ordered direct basis. This will be done for the aggregated E-basis, proving parts (a) and (c) simultaneously; part (b) then follows.

Consider the aggregated form of Σ_E. Given an implication $X \to Y$ in this basis, let $D^*(X \to Y)$ be the maximal D-rank of elements in X, and $D_*(X \to Y)$ be the minimal D-rank of elements in Y. Then $D^*(X \to Y) < D_*(X \to Y)$.

Order the implications following the rule: put the implications $x \to Y$ first (aggregated form of binary part of Σ_E), and for the rest, if $D^*(X_1 \to Y_1) < D^*(X_2 \to Y_2)$ then $X_1 \to Y_1$ precedes $X_2 \to Y_2$ in the order.

Claim. *If $X_1 \to Y_1$ and $X_2 \to Y_2$ are in the aggregated E-basis, and $x \in Y_1 \cap X_2$, then $X_1 \to Y_1$ precedes $X_2 \to Y_2$.*

Indeed, if the D-rank of x is k, then $D^*(X_2 \to Y_2) \geq k \geq D_*(X_1 \to Y_1) > D^*(X_1 \to Y_1)$. Hence, $X_1 \to Y_1$ will appear in the order before $X_2 \to Y_2$.

Now take any input set Z. We want to show that $\phi(Z)$ can be obtained when applying the aggregated basis in the described order. We argue by induction on the rank of an element $z \in \phi(Z) \setminus Z$.

If $z \in D_0$, then it only can be obtained via some implication $x \to Y$, for some $x \in Z$, and $z \in Y$, and implications $x \to Y$ form an initial segment in the ordered sequence of the basis. Now assume that it is already proved that all elements of $\phi(Z) \setminus Z$ of rank at most k can be obtained in some initial segment of the sequence for the basis. If we have now element z of rank $k+1$, then it can be obtained via an implication $X \to Y$ with $X \subseteq \phi(Z)$, $z \in Y$, and $D^*(X) < k+1$. By the induction hypothesis, all elements in $X \subseteq \phi(Z) \setminus Z$ can be obtained via implications located in some initial segment of the sequence, and by the Claim above, all those implications precede $X \to Y$. Thus, all implications producing elements of rank $k+1$ from $\phi(Z)$ will be located after the segment of the sequence producing all rank k elements. □

To illustrate the ordering of an E-basis, consider again the closure system given in Example 15. As we know from Example 22, Σ_E exists and includes all implications of the D-basis, except $24 \to 6$. Elements $1, 2, 4$ have D-rank 0; elements $3, 5$ have D-rank 1, and D-rank of 6 is 2. This allows to impose a proper ordering on implications of Σ_E that turns it into ordered direct:
(1) $5 \to 4$, (2) $6 \to 3$, (3) $3 \to 1$, (4) $2 \to 1$, (5) $14 \to 3$, (6) $24 \to 5$, (7) $23 \to 6$, (8) $15 \to 6$. This basis is also aggregated.

Proposition 25. *Suppose $\Sigma_D = \{s_1, s_2, \dots, s_n\}$ is a D-basis of some reduced closure system $\langle S, \phi \rangle$ and $|S| = m$. It requires time $O(mn^2)$ to determine whether* *the closure system is without D-cycles, and if it is, to build its ordered direct basis Σ_E.*

Proof. Since the D-relation is a subset of S^2, it will contain at most m^2 pairs. On the other hand, it is built from implications $X \to x$, so the other upper bound for pairs in D-relation is mn. Evidently, the closure system is without D-cycles iff its D-relation can be extended to a linear order. There exists an algorithm that can decide whether $\langle S, D\rangle$ can be extended to a partial order on S in time $O(m + |D|)$, see Theorem 11.1 in [11]. We will see below that the rest of the algorithm will take time $O(mn^2)$, which makes the total time also $O(mn^2)$.

Assuming the first part of algorithm provides a positive answer and there are no D-cycles, we proceed by finding the ranks of all elements. It will take at most n operations to find set D_0: include p into D_0, if it does not appear as a conclusion in any (non-binary) implication $X \to x$ of the D-basis, where $x \lhd X$. If the system is without D-cycles, then $\pi_\Sigma(D_0) \setminus D_0$ gives elements of rank 1, $\pi^2_\Sigma(D_0) \setminus \pi(D_0)$ elements of rank 2, etc. Note that $\pi_\Sigma(X)$ is defined in the beginning of section 6. Computation of $\pi_\Sigma(X)$ requires n steps, since $\Sigma = \Sigma_D$ in our case has n implications. After at most m iterations of π, one would obtain the whole S, whence, $O(mn)$ operations are needed to obtain the ranks of all elements from S.

It remains to decide which implications from the D-basis should remain in the E-basis. To that end, for each element $x \in S$ we need to compare the closures $\phi(X)$ of subsets X, for which $X \to x$ is in the D-basis, and choose for the E-basis those that are minimal. There is at most n implications $X \to x$, for a given $x \in S$, and the closure $\phi(X)$, for each such X, can be found in $O(s(\Sigma_D))$ steps. It will take time $O(n^2)$ to determine all minimal subsets among $O(n)$ given subsets $\phi(X)$, associated with fixed $x \in S$. Hence, it will require time $O(mn^2)$ for all $x \in S$.

The size of the E-basis will be at most n, and it will take time $O(n^2)$ to order it with respect to the rank of elements, per Corollary 24. □

When a closure system has D-cycles, the subset Σ_E of Σ_D, defined in Corollary 24, may not form a basis.

Example 26. *Consider $S = \{1, 2, 3, 4\}$ and a closure operator defined by the D-basis*

$$13 \to 2, 24 \to 3, 14 \to 2, 14 \to 3.$$

This closure system has the cycle $2D3D2$. It is easy to verify that Σ_E has only $13 \to 2$ and $24 \to 3$, so the last two implications from the D-basis cannot be recovered from Σ_E.

Further results about closure systems without D-cycles, and more generally systems whose closure lattice is join semidistributive, will be presented in [3].

10. D-basis versus Duquenne-Guigues canonical basis

We recall the definition of the canonical basis introduced by V. Duquenne and J.L. Guigues in [9], see also [5]. This applies to arbitrary closure systems, not just reduced ones.

Definition 27. *The canonical basis of a closure system (S, ϕ) consists of implications $X \to Y$ for $X, Y \subseteq S$, that satisfy the following properties:*

(1) $X \subset \phi(X) = Y$;

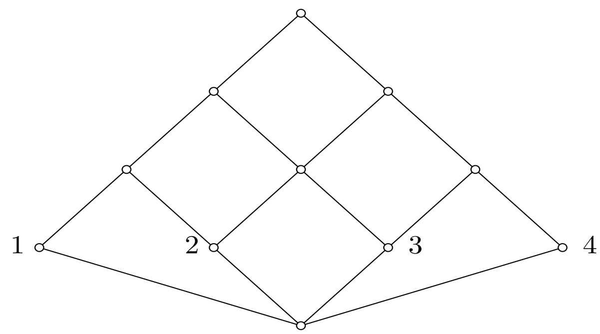

FIGURE 4. Example 26

(2) *for any ϕ-closed set Z, either $X \subseteq Z$ or $Z \cap X$ is ϕ-closed;*
(3) *if $W \subseteq X$, $\phi(W) = Y$ and W satisfies (2) in place of X, then $W = X$.*

The subsets $X \subset S$ with properties (1) and (2) are usually called *quasi-closed*, see [4]. The meaning of (2) is that adding X to the family of closed sets of ϕ produces the family of closed sets of another closure operator. Property (3) indicates that among all quasi-closed subsets with the same closure one needs to choose the minimal ones. This basis is called *canonical*, since it is minimal, in that no implication can be removed from it without altering ϕ, and every other minimal implicational basis for ϕ can be obtained from it. In particular, no other basis can have a smaller number of implications. Note that here the implications are of the form $X \to Y$, where Y is not necessarily a one-element set. We will also call it the *D-G basis*, to distinguish from canonical direct basis.

To bring this basis in comparison with other bases discussed in this paper, each implication $X \to Y$ may be replaced by set of implications $X \to y$, $y \in Y \setminus X$. We will call this modification of the canonical basis the *unit D-G basis*.

In many cases the canonical basis may be turned into an ordered direct basis.

Example 28. *Consider again the closure system from Example 15. The canonical basis is*

$$2 \to 1, 3 \to 1, 5 \to 4, 6 \to 3, 6 \to 1, 14 \to 3, 123 \to 6, 1345 \to 6, 12346 \to 5.$$

Besides, it is ordered direct in the given order.

In general, though, the canonical basis cannot be ordered so that it becomes direct. Thus, it is not ordered direct. The following two examples were uncovered by running a computer program and checking about a million of various closure systems on 5- and 6-element sets. The first example demonstrates a closure system, where the canonical basis cannot be ordered, while the unit expansion of this basis does admit an ordering to make it direct. The second example shows that some canonical bases cannot be ordered in either form.

Example 29.

Let (S,ϕ) be a closure system on $S=\{1,2,3,4,5,6\}$, given by the family of closed sets: $\{\emptyset,1,2,3,4,6,36,26,13,24,14,35,23,16,135,136,236,1246,2345,S\}$. The lattice representation of this system is given in Figure 5.

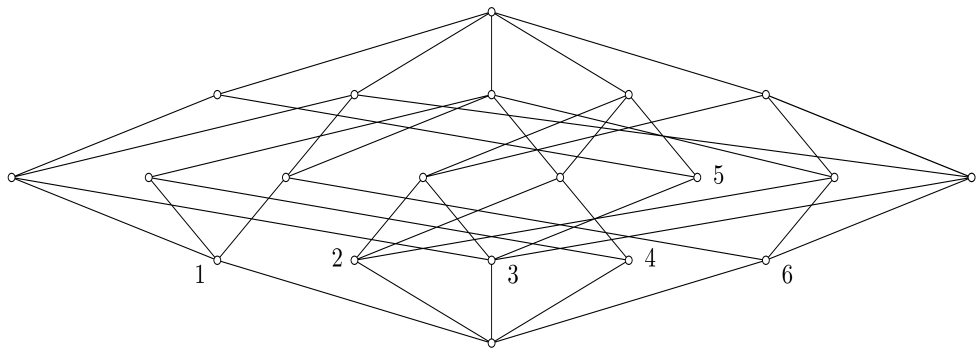

FIGURE 5. Example 29

Then the canonical basis is $5\to 3, 34\to 25, 12\to 46, 46\to 12, 235\to 4, 356\to 124$. It is easy to show that this basis cannot be ordered. Indeed, in order to obtain $\phi(145)=S$ in one application of canonical basis, one would need to put $5\to 3$ first, then $34\to 25$, followed by $12\to 46$. On the other hand, $\phi(123)=S$, too, and the only implication applicable to 123 is $12\to 46$, but it comes after $34\to 25$, and one cannot obtain 5 in the closure otherwise.

As was mentioned, the unit expansion of this canonical basis is still ordered direct: one would need to place implications $12\to 4$ and $12\to 6$ around $34\to 2$ and $34\to 5$, thusly: $5\to 3, 12\to 4, 34\to 2, 34\to 5, 12\to 6, 46\to 2, 46\to 1, 235\to 4, 356\to 1, 356\to 2, 356\to 4$.

As always, the D-basis is ordered direct in both forms: in its original unit form, and in the aggregated form. For example, the aggregated form of D-basis in this example is $5\to 3, 34\to 25, 12\to 46, 46\to 12, 25\to 4, 56\to 124, 123\to 5, 134\to 6$.

One needs to run the canonical basis two times to ensure the closure of arbitrary subset, i.e., apply $6\cdot 2=12$ implications, versus only 8 implications of the aggregated D-basis. In the unit form, the canonical basis has 11 implications and the D-basis has 13, but the ordering of the canonical basis requires special care.

Remark 30.

Example 29 also shows that the D-basis, unlike the canonical basis, can be *redundant* (even in its aggregated form): this means that some implications can be removed, and the remaining ones still define the same closure system. In the D-basis of our example, both implications $123\to 5, 134\to 6$ can be removed, since they follow from $34\to 25, 12\to 46$. On the other hand, the basis without these two implications is no longer ordered direct.

The following example shows that the canonical basis might be un-orderable in either form.

Example 31.

Let (S, ϕ) be a closure system on $S = \{1,2,3,4,5,6\}$, given by the family of closed sets: $\{\emptyset, 1, 2, 3, 5, 6, 12, 13, 14, 16, 23, 123, 124, 135, 256, 1346, S\}$. The lattice representation of this system is given in Figure 6.

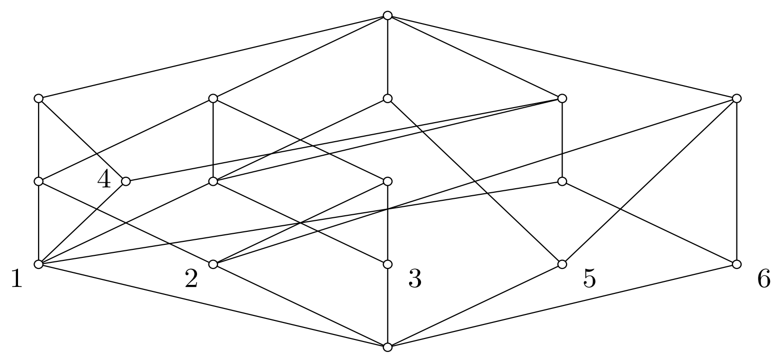

FIGURE 6. Example 31

The canonical basis has 9 implications:
$4 \to 1, 15 \to 3, 35 \to 1, 25 \to 6, 56 \to 2, 26 \to 5, 36 \to 14, 134 \to 6, 146 \to 3$.
There is a single implication $36 \to 14$ that can be expanded to two unit implications $36 \to 1$ and $36 \to 4$.

The proof that the unit expansion of canonical basis cannot be ordered to make it direct, follows from consideration of the next three closures:

- $45 \to 145 \to 1345 \to 13456 \to S$, hence, $134 \to 6$ should be placed later than $15 \to 3$.
- $1234 \to 12346 \to S$, hence $26 \to 5$ should be placed later than $134 \to 6$.
- $126 \to 1256 \to 12356 \to S$, hence $15 \to 3$ should be placed later than $26 \to 5$, which contradicts the combination of the previous two items.

For comparison, the aggregated D-basis has 15 implications:
$4 \to 1, 45 \to 26, 36 \to 14, 34 \to 6, 15 \to 3, 46 \to 3, 35 \to 1, 25 \to 6, 26 \to 5, 56 \to 2, 126 \to 34, 235 \to 4, 156 \to 4, 234 \to 5, 125 \to 4$.

Thus, one run of the aggregated D-basis (15 implications) wins over two runs (18 implications) of the canonical basis. In unit expansions: D-basis (18 implications) still wins over two runs (20) of canonical basis.

In this example, the D-basis is 4 implications shorter than the canonical unit direct basis, which has 22 implications.

Our earlier analysis of the binomial part of the D-basis in Proposition 17carries over to a partial optimization of the canonical basis.

Proposition 32. *The binary part of the unit expansion of the D-G canonical basis of any reduced closure system coincides with the binary part of the D-basis (or, E-basis, if it exists) of the same system.*

Proof. We recall that the binary part of the D-basis of closure system $\langle S, \phi\rangle$ consists of implications $y \to x$, where $x \in \phi(y) \setminus y$. This implies that $\{y\}$ is not a ϕ-closed set. Besides, it is a quasi-closed set, since the intersection of $\{y\}$ with any ϕ-closed set is either $\{y\}$ or $\emptyset$. Evidently, $\{y\}$ will be the minimum quasi-closed set with the closure $\phi(y)$. Hence, $\{y\} \to \phi(y) \setminus y$ should be an implication in the canonical basis. Evidently, the unit expansion of $\{y\} \to \phi(y) \setminus y$ gives all the implications in the D-basis with the premise y. Vice versa, every implication in the canonical basis of the form $y \to Y$ implies that $Y = \phi(y) \setminus y$. Hence, $y \to y'$ for $y' \in Y$ should appear in the D-basis. □

The following statement is an immediate consequence of Proposition 17 and Proposition 32. We recall that L stands for the lattice of closed sets of $\langle S, \phi\rangle$, and $(J(L), \leqslant)$ is a partially ordered set of join-irreducible elements of L.

Corollary 33. *Let Σ_c be the canonical basis of $\langle S, \phi\rangle$, where $|S| = m$. Let $\Sigma_c' \subseteq \Sigma_c$ have all implications $y \to Y$ from Σ_c, and let n be the number of implications in the unit expansion of Σ_c'. Then an algorithm that requires $O(mn + n^2)$ time will replace each implication $y \to Y$ in Σ_c' by $y \to Y'$, $Y' \subseteq Y$, where $\phi(y)$ covers $\phi(y')$ in $(J(L), \leqslant)$, for each $y' \in Y'$. If Σ_c'' is this new set of implications, then the algorithm will also put an appropriate order on Σ_c'' in such a way that $\rho_{\Sigma_c'} = \rho_{\Sigma_c''}$.*

Thus, the optimization of the canonical basis inspired by Proposition 17 is in the form of a possible size reduction of some implications.

We finish this section with a comparison of the canonical D-G basis with the D-basis on some illustrative examples. We consider one particular type of closure systems for which the description of the canonical basis is easy. The closure system (S, ϕ) is called *a convex geometry*, if ϕ satisfies the anti-exchange axiom: if $x \in \phi(C + y)$ and $x \notin C$, then $y \notin \phi(C + x)$, for all $x \neq y$ in S and all closed $C \subseteq S$.

For any closed set X in a convex geometry, the set of extreme points of X is defined as $Ex(X) = \{x \in X : x \notin \phi(X \setminus x)\}$. It is well-known that, in every convex geometry, $X = \phi(Ex(X))$. The equivalent statement in the framework of lattice theory is that the closure lattice of a finite convex geometry has unique irredundant join decompositions; see, for example, [2].

An important example of convex geometry is $Co(R^n, A)$, where A is a finite set of points in R^n, and $Co(R^n, A)$ stands for geometry of convex sets relative to A. In other words, the base set of such closure system is A, and closed sets are subsets X of A with the property that whenever point $a \in A$ is in convex hall of some points from X, then a must be in X (see more details of the definition, for example, in [2]).

Lemma 34. *If Y is the premise of an implication from the canonical basis of some convex geometry $Co(R^n, A)$, then Y is the set of extreme points of a closed set $\phi(Y)$ such that every subset of Y is closed.*

Proof. Evidently, the premise Y of every implication of the canonical basis contains $ex(\phi(Y))$. Moreover, $Co(R^n, A)$ satisfies the n-Carathéodory property, see [1], which means that $|Ex(\phi(Y))| \leqslant n + 1$. Suppose there exists an implication $Y \to z$ in canonical basis with $Ex(Y) = \{y_1, y_2, \dots, y_{n+1}\}$, $z \in \phi(Y)$ and $z \notin \phi(Y')$ for every $Y' \subset ex(Y)$. We claim that every $Y_i = Y \setminus \{y_i\}$ is closed. Indeed, suppose w.l.o.g. that $x \in \phi(Y_{n+1}) = \phi(y_1, \dots, y_n)$, $x \notin \{y_1, \dots, y_n\}$. Then simplex generated by $y_1, \dots, y_{n+1}$ is split into simplices generated by $X_1 = \{x, y_2, \dots, y_{n+1}\}$, $X_2 = \{y_1, x, y_3, \dots, y_{n+1}\}, \dots$, $X_n = \{y_1, y_2, \dots, y_{n-1}, x, y_{n+1}\}$. Then z must be in one of those simplices, say, $z \in \phi(X_i)$. Since Y is quasi-closed, $\phi(Y_{n+1}) \subset \phi(Y)$ implies $x \in Y$, and $\phi(X_i) \subset \phi(Y)$ implies $z \in Y$, a contradiction with the assumption $z \notin Y$.

Similar argument applies for any other Y with $Ex(\phi(Y)) < n+1$. □

In the next two examples, we consider convex geometries of the form $Co(R^2, A)$ and compare the canonical bases and D-bases.

Example 35. *If A is a set of points in general position, i.e., no three points are on a line, then the D-basis and canonical basis of convex geometry $Co(R^2, A)$ are the same.*

Due to the 3-Carathéodory property, all covers can be reduced to covers by three elements. So the D-basis consists of implications $abc \to x$, for all triangles abc that have x inside.

Now, for any relatively convex subset $X \subseteq A$, $Ex(X)$ consists of the vertices of a convex n-gon that holds all the points of X inside. If there exists $y \in X \setminus Ex(X)$, then there are $a, b, c \in X$ with $y \in \phi(a, b, c)$. Hence, $\{a, b, c\} \subseteq X$ is not closed. This will not contradict Lemma 34, only if $X = \{a, b, c\}$. Thus, the only implications in the canonical basis are $abc \to x$, where x is inside triangle abc. It follows that the D-basis and canonical basis are the same.

Example 36. *If A is a set of points that is not in general position, then the canonical basis of $Co(R^2, A)$ is a proper subset of the D-basis.*

Indeed, consider a point configuration of 5 points: a, b, c form a triangle, x is inside the triangle, and d is on the side ab, so that x is also inside triangle dbc. The D-basis is $ab \to d$, $abc \to x$, $bcd \to x$, while the canonical basis is $ab \to d$, $bcd \to x$.

Note that abc cannot be a premise of an implication in the canonical basis due to Lemma 27, since the subset ab is not closed.

We note that Lemma 34 is not true for arbitrary convex geometries.

Example 37. *Take the convex geometry $(\{a, b, c, d, x\}, \phi)$ of Example 36. Adding another closed set $\{b, c, d\}$ will result in a new convex geometry $(\{a, b, c, d, x\}, \psi)$ with the canonical basis $ab \to d, abcd \to x$. Note that in implication $abcd \to x$, $d \notin Ex(abcd)$, and subset ab is not closed.*

11. Testing the performance of D-basis

The performance of D-basis in comparison with the D-G unit basis and canonical unit direct basis was tested on $300,000$ randomly generated closure systems on base sets of 6 and 7 elements.

The primary advantage of the D-basis is its ordered directness. By contrast, computing the closure of a non-closed set using the canonical basis will always take at least two passes: the final pass produces nothing and exists solely to determine that the ability of the basis to expand the given set has been exhausted.

In the testing on domain length 6, with inputs sets of length 3, the D-G unit basis cycled through, on average, 22.9 implications before returning the closure. By comparison, the direct canonical (optimal) basis took 15.8 such steps and the D-basis took only 12.7 checks on average. Due to their ordered directness, the number of implications checked in the direct optimal and D-basis was equivalent to the number of implications they contained.

Domain: {1,2,3,4,5,6}

Closed Sets	D-G Unit Basis	Direct Optimal Basis	D-Basis
5	32.37	18.71	16.45
10	22.59	16.24	12.23
15	21.22	15.54	12.47
20	18.13	13.06	11.45
25	15.54	11.09	10.34
30	11.70	7.96	7.65

Average implications checked to expand an arbitrary 3-element set.

FIGURE 7. Bases comparison on domain set 6

It was observed that the efficiency gap between the direct and indirect bases was greatest when there were fewer closed sets, meaning that more subsets could be expanded through the bases' implications. This relation is shown in the Figure 8.

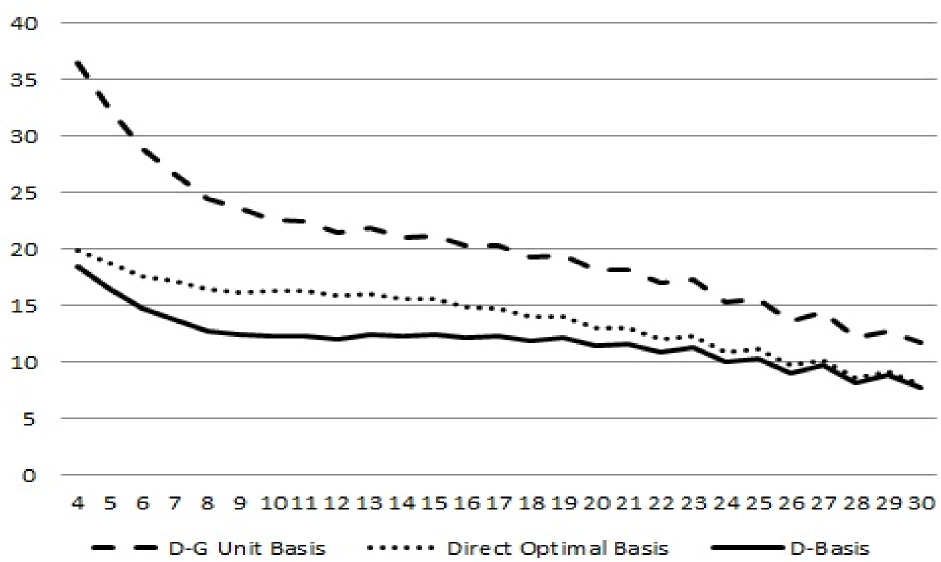

FIGURE 8. Bases comparison on domain set 6

We saw similar results on bases of domain length 7. There, we once again saw the convergence of the D-Basis and direct optimal as the number of closed sets approached either extreme, with a more pronounced gap in between.

There were 33.8 checks on average for the D-G unit basis, and 26.0 and 19.0 for the direct optimal and D-basis, respectively, see Figure 9.

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Domain: {1,2,3,4,5,6,7}

Closed Sets	D-G Unit Basis	Direct Optimal Basis	D-Basis
5	46.73	27.70	23.57
10	33.74	26.26	17.92
15	32.11	26.80	18.59
20	31.01	25.68	19.43
25	29.77	23.99	19.66
30	26.71	20.64	17.73

Average implications checked to expand an arbitrary 3-element set.

Figure 9. Bases comparison on domain set 7

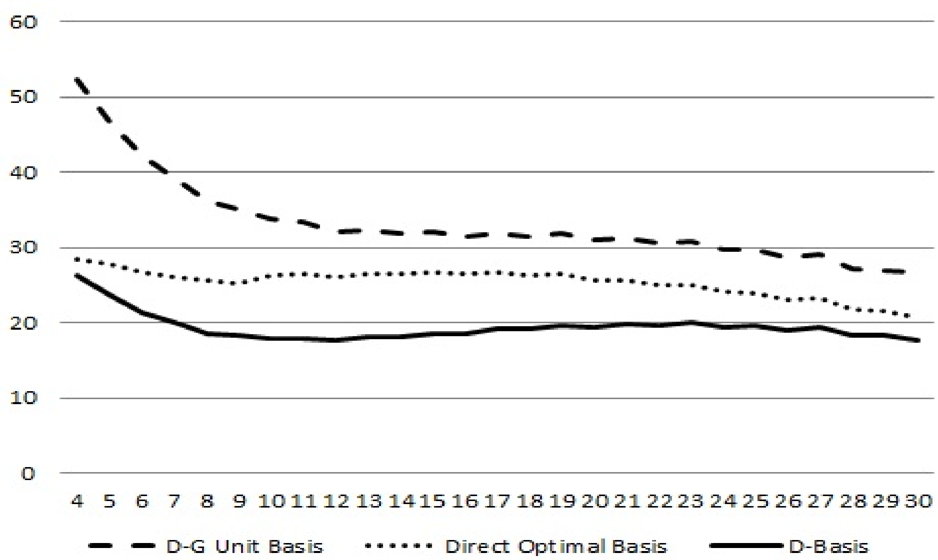

Figure 10. Bases comparison on domain set 7

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Department of Mathematical Sciences, Yeshiva University, New York, NY 10016, USA

E-mail address: `adariche@yu.edu`

Department of Mathematics, University of Hawaii, Honolulu, HI 96822, USA

E-mail address: `jb@math.hawaii.edu`

Yeshiva University, New York, NY 10033, USA

E-mail address: `rrand@yu.edu`