LARGEST EXTENSION OF A FINITE CONVEX GEOMETRY

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Abstract. We present a new embedding of a finite join-semidistributive lattice into a finite atomistic join-semidistributive lattice. This embedding turns out to be the largest extension, when applied to a finite convex geometry.

1. Introduction

A convex geometry is a closure space (X, φ) with the anti-exchange property. This property says that if $x \in \varphi(A \cup \{y\})$ for some closed set $A \subseteq X$ and distinct points $x, y \notin A$, then $y \notin \varphi(A \cup \{x\})$. An extension of a convex geometry (X, φ) is another convex geometry (X, ψ), on the same base set X, with the property that every φ-closed set is ψ-closed and the lattice of closed sets of (X, φ) is a sublattice of the lattice of closed sets of (X, ψ). All extensions of a given convex geometry can be naturally ordered, and they form a lattice with respect to this order. The latter fact was formulated implicitly in the proof of [1], Corollary 1.13: when a convex geometry is finite, it always has a largest extension.

The current work is motivated by the fact that an exact formula for this largest extension was not presented in [1]. We generalize the notion of an extension to arbitrary finite closure spaces and we provide a new proof of Theorem 1.11 of [1], that any finite join-semidistributive lattice can be embedded into an atomistic finite join-semidistributive lattice, with the same set of join-irreducible elements. In fact, this new construction gives the largest *join-semidistributive* extension for a given finite join-semidistributive lattice. In particular, when applied to a finite convex geometry, it turns out to be the largest convex geometry extension.

2. Finite closure spaces, convex geometries and lattices

By a *closure space* (X, Φ) we mean a non-void set X with a closure operator Φ on $B(X)$, the set of all subsets of X. The lattice of closed sets of a closure space (X, Φ) will be denoted $\mathrm{Cl}(X, \Phi)$. We will limit our consideration to *finite* closure spaces, as required for the application to convex geometries. Clearly, some results in this section have a straightforward extension to arbitrary closure spaces and complete lattices. See [1] for a discussion of the difficulties encountered with infinite convex geometries.

Given two closure operators Φ and Ψ on the same underlying set X, we say that (X, Ψ) is a *weak extension* of (X, Φ) if $\Psi(Y) \subseteq \Phi(Y)$ for every subset $Y \subseteq X$. In this case we write $(X, \Phi) \sqsubseteq_w (X, \Psi)$.

The following lemma is immediate from the definitions.

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Lemma 2.1. *For the finite closure spaces (X, Φ) and (X, Ψ), the following are equivalent.*

(1) (X, Ψ) *is a weak extension of* (X, Φ).
(2) *Every* Φ*-closed set is* Ψ*-closed.*
(3) $\mathrm{Cl}(X, \Phi)$ *is a meet-subsemilattice of* $\mathrm{Cl}(X, \Psi)$.

We can think of a weak extension of some (X, Φ) as a closure space that has more closed sets than the given one. Note that all closure spaces on a (finite) set X form a lattice $\mathcal{C}(X)$ with respect to $\sqsubseteq_w$. It has a natural join operation, given by

$$(\bigvee \Gamma_i)(Y) = \bigcap \Gamma_i(Y)$$

for any $Y \subseteq X$. See the survey [2] on different aspects of this lattice.

Definition 2.2. A closure space (X, Δ) is a *(strong) extension* of (X, Φ), which will be denoted $(X, \Phi) \sqsubseteq_s (X, \Delta)$, if $\mathrm{Cl}(X, \Phi)$ is a sublattice of $\mathrm{Cl}(X, \Delta)$.

Equivalently, $(X, \Phi) \sqsubseteq_s (X, \Delta)$ whenever $(X, \Phi) \sqsubseteq_w (X, \Delta)$ and, for any closed sets Y, $Z \in \mathrm{Cl}(X, \Phi)$ we have $\Phi(Y \cup Z) = \Delta(Y \cup Z)$.

It is easy to see that the set of all extensions of a closure space (X, Φ) forms an interval in the lattice $\mathcal{C}(X)$.

Theorem 2.3. *Every finite closure space* (X, Φ) *has a largest extension* (X, M_Φ) *in* $\mathcal{C}(X)$*, which is determined by the rule that a subset* $C \subseteq X$ *is* M_Φ*-closed if and only if it satisfies the following property.*

$$\textit{For any } A \subseteq X, \textit{ if } \Phi(a) \subseteq C \textit{ for all } a \in A \textit{ then } \Phi(A) \subseteq C. \tag{M}$$

Moreover, $(X, \Phi) \sqsubseteq_s (X, \Psi)$ *if and only if* $(X, \Phi) \sqsubseteq_w (X, \Psi) \sqsubseteq_w (X, M_\Phi)$.

Every finite lattice L can be viewed as the lattice of closed sets of a closure space on the set $\mathrm{J}(L)$ of its (nonzero) join-irreducible elements. That is, if we define the closure operator σ on $\mathrm{J}(L)$ by $\sigma(Y) = [0, \bigvee Y] \cap \mathrm{J}(L)$ for any $Y \subseteq \mathrm{J}(L)$, then L is isomorphic to the lattice of closed sets $\mathrm{Cl}(\mathrm{J}(L), \sigma)$. Moreover, the join-irreducibility of the elements in $\mathrm{J}(L)$ is reflected in the property that

$$\sigma(x) \setminus \{x\} \text{ is closed for every } x \in \mathrm{J}(L). \tag{†}$$

Conversely, let X be a finite set, and let ϕ be a closure operator on X with the property that $\phi(x) \setminus \{x\}$ is closed for every $x \in X$. Let $L = \mathrm{Cl}(X, \phi)$. Then the rule $x \to \phi(x)$ defines an isomorphism from (X, ϕ) onto $(\mathrm{J}(L), \sigma)$.

This correspondence allows us to treat finite lattices and closure operators satisfying (†) interchangeably. We intend to exploit this equivalence. In particular, we note that in lattice terms, K is an extension of L if and only if $\mathrm{J}(L) = \mathrm{J}(K)$ and there is a lattice embedding $\varphi : L \to K$ such that

$$\{x \in \mathrm{J}(L) : x \leq a\} = \{x \in \mathrm{J}(K) : x \leq \varphi(a)\}$$

for all $a \in L$. The above set will be denoted by $\mathrm{J}(a)$. In a later paper [11], we will give a systematic treatment of the extensions of a finite lattice.

Recall the definition of the anti-exchange property for a closure space (see, e.g., [9]).

Definition 2.4. A finite closure space (X, Φ) satisfies the *anti-exchange property* if the following statement holds.

$$\begin{gathered} p \in \Phi(C \cup \{q\}) \text{ and } p \notin C \text{ imply that } q \notin \Phi(C \cup \{p\}) \\ \text{for all } p \neq q \text{ in } X \text{ and all closed } C \subseteq X. \end{gathered} \tag{AEP}$$

If the AEP holds and $\phi(\emptyset) = \emptyset$, then we say that (X, Φ) is a *convex geometry.*

It is a straightforward exercise that convex geometries satisfy the property (†).

Lemma 2.5. *Let (X, Φ) be a finite convex geometry. Then $\Phi(x) \setminus \{x\}$ is closed for every $x \in X$. Hence $\Phi(x)$ is join-irreducible for every $x \in X$.*

In order to interpret convex geometries in lattice-theoretic terms, we recall some standard definitions.

A *join decomposition* of an element x in a finite lattice L is a subset $C \subseteq \mathrm{J}(L)$ such that $x = \bigvee C$.

A lattice $(L, \wedge, \vee)$ is called *join-semidistributive* if

$$x \vee y = x \vee z \text{ implies that } x \vee y = x \vee (y \wedge z)$$

for all x, y, $z \in L$.

By a *canonical join representation* of an element $x \in L$ we mean a finite subset $C \subseteq L$ such that

(1) $x = \bigvee C$ irredundantly;
(2) if $x = \bigvee B$ then $C \ll B$, i.e., for any $c \in C$ there is $b \in B$ such that $c \leq b$.

Thus a canonical join representation of x is a join decomposition that refines all other join decompositions, in the sense of (2).

A critical fact, due to Jónsson and Kiefer [8], is that a finite lattice L is join-semidistributive if and only if every element of L has a canonical join representation. In a finite join-semidistributive lattice, we will use $\mathrm{CS}(x)$ to denote the set of elements of L that give the canonical join representation of x. Other characterizations of finite join-semidistributive lattices are gathered in [7].

For each element x in a finite lattice L, let $m_x = \bigwedge\{y \in L : x \succ y\}$. We say that L is *lower locally distributive* if the interval $[m_x, x]$ is distributive for every $x \in L$. A classic result of R. P. Dilworth [4] is that every element of a finite lattice L has a unique irredundant join decomposition if and only if L is lower locally distributive. Since a unique irredundant join decomposition is *a fortiori* canonical, every finite lower locally distributive lattice is join-semidistributive.

With these definitions in hand, we can state the main equivalence, based on Edelman and Jamison [6] (see also Monjardet [10]).

Theorem 2.6. *The following are equivalent for a finite lattice L.*

(1) $L \cong \mathrm{Cl}(X, \Gamma)$ *for some convex geometry* (X, Γ).
(2) $(\mathrm{J}(L), \sigma)$ *is a convex geometry.*
(3) L *is lower locally distributive.*

Recall that a lattice is *atomistic* if every element is a join of atoms. In the finite case, this means that the join-irreducible elements are atoms. In a finite join-semidistributive atomistic lattice, the canonical join representation of an element will be its unique irredundant join decomposition. Hence we have the following result [1].

Corollary 2.7. *Any finite atomistic lattice is the closure lattice of some finite convex geometry if and only if it is join-semidistributive.*

3. Embedding finite join-semidistributive lattices

When a finite lattice L satisfies some additional property (*), we can ask whether there exists a largest extension of L (w.r.t. the order $\sqsubseteq_w$ on $\mathcal{C}(\mathrm{J}(L))$) that also satisfies the property (*). In this section we will answer this question in the affirmative for the case when (*) is the join-semidistributive law.

Theorem 3.1. *Any finite join-semidistributive lattice has a largest join-semidistributive extension. This largest join-semidistributive extension is atomistic, and hence the lattice of closed sets of a convex geometry.*

It was shown in [1] that every finite join-semidistributive lattice can be embedded into an atomistic finite join-semidistributive lattice, preserving the bounds (i.e., 0 and 1 of the lattice) as well as the number of join-irreducible elements. Theorem 3.1 improves this result by using a natural construction which yields the largest such extension. The proof is contained in Lemmas 3.2 to 3.7. Thus if K is the extension of L given by Theorem 3.1, and the lattice M is also a join-semidistributive extension of L, then $M \sqsubseteq_w K$ in $\mathcal{C}(\mathrm{J}(L))$. Example 4.7 of the next section will show that L may have lattice extensions that are larger than K, and thus not join-semidistributive. L may also have extensions that are smaller than K and not join-semidistributive; see Examples 3.8 and 4.4.

For any $a \in L$, let $\mathrm{D}_0(a)$ denote the set of elements that are join-prime in the interval $[0, a]$, i.e., the elements $w \in [0, a]$ such that $w \leq u \vee v \leq a$ implies $w \leq u$ or $w \leq v$. Evidently, $\mathrm{CS}(a) \subseteq \mathrm{D}_0(a) \subseteq \mathrm{J}(a) = \mathrm{J}(L) \cap [0, a]$.

Define a closure operator ϵ on $\mathrm{J}(L)$ of which the closed sets are *all* subsets C in $\mathrm{J}(L)$ satisfying the following property:

$$\text{For any } a \in L, \text{ if } \mathrm{D}_0(a) \subseteq C \text{ then } \mathrm{J}(a) \subseteq C. \tag{E}$$

Lemma 3.2. *For any $j \in \mathrm{J}(L)$, $C = \{j\}$ satisfies* (E). *In particular, the lattice of closed sets of ϵ is atomistic.*

Proof. Indeed, if $\mathrm{D}_0(a) \subseteq C = \{j\}$, then $a = j$ is join-irreducible. For $a \in \mathrm{J}(L)$ we have $\mathrm{D}_0(a) = \mathrm{D}_0(a_*) \cup \{a\}$, where a_* is the unique element covered by a. We conclude that $0 \prec a$, and hence $\mathrm{J}(a) = \{a\} \subseteq C$. □

Lemma 3.3. *Let L be a finite lattice, and a, $b \in L$.*

1. $\mathrm{D}_0(a \vee b) \subseteq \mathrm{D}_0(a) \cup \mathrm{D}_0(b)$.
2. *If $a \leq b$, then* $\mathrm{D}_0(b) \cap [0, a] \subseteq \mathrm{D}_0(a)$.
3. *If $K = \mathrm{Cl}(\mathrm{J}(L), \psi)$ is an extension of L and $C = \psi(c)$ is join-prime in the interval $[0, \mathrm{J}(a)]$ of K, then $c \in \mathrm{D}_0(a)$ in L.*

Proof. The first two parts follow directly from the definition of D_0. For the third part, we note that for any $c \in \mathrm{J}(L)$ and $x \in L$ we have $c \in \mathrm{J}(x)$ if and only if $\psi(c) \subseteq \mathrm{J}(x)$ in K. Hence a nontrivial join-cover of c in L translates into a nontrivial join-cover of $\psi(c)$ in K. □

Lemma 3.4. *For any $S \subseteq \mathrm{J}(L)$, $\epsilon(S) = S \cup \bigcup\{\mathrm{J}(a) : \mathrm{D}_0(a) \subseteq S, a \in L\}$.*

Proof. Let Y denote the set on the right hand side of the equality. Evidently $S \subseteq Y \subseteq \epsilon(S)$, so we need to show that Y is an ϵ-closed set.

Assume $\mathrm{D}_0(c) \subseteq Y$ for some $c \in L$. We shall prove that there is an element b with $c \leq b$ and $\mathrm{D}_0(b) \subseteq S$, so that $\mathrm{J}(c) \subseteq \mathrm{J}(b) \subseteq Y$.

Indeed, since $\mathrm{D}_0(c) \subseteq Y$, for any $p \in \mathrm{D}_0(c) \setminus S$ there exists $c_p \in L$ such that $p \in \mathrm{J}(c_p)$ and $\mathrm{D}_0(c_p) \subseteq S$. Put $b = c \vee \bigvee\{c_p : p \in \mathrm{D}_0(c) \setminus S\}$. Then $c \leq b$, and we want to show that $\mathrm{D}_0(b) \subseteq S$.

Suppose $z \in \mathrm{D}_0(b) \setminus S$. Then, as $z \leq b$ and z is join prime in $[0, b]$, we have either $z \leq c$ or $z \leq c_p$ for some $p \in \mathrm{D}_0(c) \setminus S$. In the first case, we have $\mathrm{D}_0(b) \cap [0, c] \subseteq \mathrm{D}_0(c)$, whence $z \in \mathrm{D}_0(c) \setminus S$, which by the above remark reduces to the second case. In the second case, since $z \in \mathrm{D}_0(b)$ and $c_p \leq b$ we have $z \in \mathrm{D}_0(c_p) \subseteq S$, a contradiction. □

In view of Theorem 2.6, in order to show that the lattice of ϵ-closed sets is join-semidistributive, it suffices to prove that it has the AEP.

Lemma 3.5. $(\mathrm{J}(L), \epsilon)$ *satisfies the anti-exchange property.*

Proof. Suppose C is an ϵ-closed set, $p \neq q$, $p, q \notin C$, $p \in \epsilon(C \cup \{q\})$ and $q \in \epsilon(C \cup \{p\})$. It follows from Lemma 3.4 that there exist $a \geq p$ and $b \geq q$ such that $\mathrm{D}_0(a) \subseteq C \cup \{q\}$ and $\mathrm{D}_0(b) \subseteq C \cup \{p\}$.

Let $d = a \vee b$. Then $p, q \leq d$, and by Lemma 3.3 we have $\mathrm{D}_0(d) \subseteq C \cup \{p, q\}$. Since $p \notin \mathrm{D}_0(a)$ and $q \notin \mathrm{D}_0(b)$, we have $p, q \notin \mathrm{D}_0(d)$. Hence $\mathrm{D}_0(d) \subseteq C$. Since C is an ϵ-closed set, p and q must be in C, a contradiction. □

Lemma 3.6. *L is a* $(0, 1)$*-sublattice of the lattice of closed sets of* $(\mathrm{J}(L), \epsilon)$.

Proof. Define a map $\alpha : L \to \mathrm{Cl}(\mathrm{J}(L), \epsilon)$ by the rule $\alpha(x) = \mathrm{J}(x) = \{j \in \mathrm{J}(L) : j \leq x\}$ for $x \in L$. It is straightforward to check that $\mathrm{J}(x)$ is ϵ-closed for any x, and that α preserves meets and the bounds. To show that α preserves joins, we need to prove that $\mathrm{J}(x \vee y)$ is the least ϵ-closed set that contains $\mathrm{J}(x)$ and $\mathrm{J}(y)$. If C is ϵ-closed and $\mathrm{J}(x) \cup \mathrm{J}(y) \subseteq C$ then $\mathrm{D}_0(x) \cup \mathrm{D}_0(y) \subseteq C$, and, in view of Lemma 3.3, $\mathrm{D}_0(x \vee y) \subseteq C$. This implies $\mathrm{J}(x \vee y) \subseteq C$. □

Lemma 3.7. $\mathrm{Cl}(\mathrm{J}(L), \epsilon)$ *is the largest join-semidistributive extension of L.*

Proof. It is enough to show that for every join-semidistributive extension $K = \mathrm{Cl}(\mathrm{J}(L), \psi)$ of L, any ψ-closed set satisfies (E).

Let $B \in K$ be a ψ-closed set. Let $a \in L$, and assume that $\mathrm{D}_0(a) \subseteq B$. As K is an extension of L, the set $\mathrm{J}(a)$ is ψ-closed. Since K is join-semidistributive, one can consider its canonical join representation $\mathrm{J}(a) = \bigvee_K C_i$. Each C_i is of the form $\psi(c_i)$ for some $c_i \in \mathrm{J}(L)$, and is join-prime in the interval $[0, \mathrm{J}(a)]$ of K. By Lemma 3.3(3), each c_i is in $\mathrm{D}_0(a)$, whence $c_i \in B$. Thus $\mathrm{J}(a) \subseteq B$. This proves that B satisfies (E), as desired. □

Remark 3.8. Despite Lemma 3.7, we can still have a join-semidistributive lattice L, its largest join-semidistributive extension K, and a non-join-semidistributive lattice M with $L \sqsubseteq_s M \sqsubseteq_w K$. Figure 1 gives an example of this phenomenon.

Remark 3.9. We note that the new construction of an embedding of a finite join-semidistributive lattice into an atomistic join-semidistributive lattice differs from the one given in Theorem 1.11 of [1]. This can be seen on the following example.

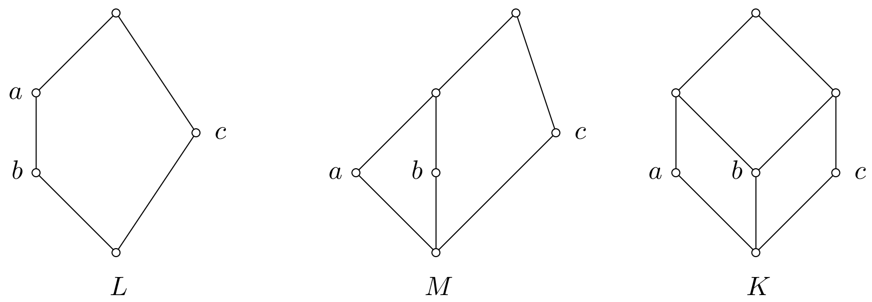

FIGURE 1

For a poset P, let $\mathbf{Co}(P)$ denote the lattice of convex subsets of P. Let $\mathbf{4}$ denote the four-element chain. Then $\mathbf{Co}(\mathbf{4})$ is the lattice defined by the generators (atoms) a, b, c, d and the relations $b \leq a \vee c$, $c \leq b \vee d$, and $b, c \leq a \vee d$.

Let $L = \mathbf{Co}(\mathbf{4}) \setminus \{\{a\}, \{d\}\}$. Then $L \cong \mathbf{3} \times \mathbf{3}$, where $\mathbf{3}$ denotes a three-element chain, and the construction of Theorem 1.11 in [1] embeds L into $\mathbf{Co}(\mathbf{4})$.

On the other hand, it is easy to verify that the construction of the current paper embeds L into Boolean lattice $B(4)$. Indeed, it is easy to see that $D \sqsubseteq_s B(\mathrm{J}(D))$ for any finite distributive lattice D, so that the largest extension of a finite distributive lattice is Boolean.

Remark 3.10. A natural question to consider is the following: will the current embedding construction preserve lower boundedness? Recall that a lattice L is *lower bounded* if there exists a homomorphism $g : \mathrm{FL}(n) \twoheadrightarrow L$ from a finitely generated free lattice onto L such that every element $x \in L$ has a least pre-image. It is known that lower bounded lattices form a proper subclass of the class of join-semidistributive lattices.

Following A. Day [3], for $a, b \in \mathrm{J}(L)$ we define aDb if $a \neq b$ and there exists $c \in L$ such that $a \leq b \vee c$ and $a \nleq b_* \vee c$. By a *D-cycle* we mean a finite sequence $a_0, \dots, a_n = a_0$ of join irreducible elements of L such that $a_i D a_{i+1}$ for all $i < n$. Day gave a test to determine lower boundedness:

A finite lattice L is lower bounded iff it contains no D-cycle.

M. Tischendorf's well-known procedure for embedding a finite lattice into an atomistic finite lattice preserves lower boundedness [12]. On the other hand, the construction given in [1] does not have this property. We want to show that the embedding construction given in the current paper also fails to preserve lower boundedness. (A finite lower bounded lattice has a largest lower bounded extension, which may be larger than Tischendorf's extension. This will be dealt with in a later paper.)

Consider the $(\vee, 0)$-semilattice generated by the set of six elements $J = \{a_1, a_2, b_1, b_2, p, q\}$ subjected to the following relations: $p \leq a_1 \vee a_2$, $p \leq b_2$, $q \leq a_2$, $q \leq b_1 \vee b_2$. Evidently, we get a finite lattice L with $J = \mathrm{J}(L)$. Since the join dependency relation D on L consists of pDa_1, pDa_2, qDb_1, qDb_2 only, L is lower bounded.

It is not hard to check that the sets $A = \{a_1, a_2\}, B = \{b_1, b_2\} \subseteq \mathrm{J}(L)$ are ϵ-closed. Also, for $a = a_1 \vee a_2$ and $b = b_1 \vee b_2$ we have $\mathrm{D}_0(a/0) = \{a_1, a_2, q\}$, $\mathrm{D}_0(b/0) = \{b_1, b_2, p\}$. In view of Lemma 3.4 this implies $p \in \epsilon(A \cup \{q\})$ and $q \in \epsilon(B \cup \{p\})$. Thus pDq and qDp in $\mathrm{Cl}(\mathrm{J}(L), \epsilon)$. Hence this atomistic extension is not lower bounded.

4. Application to finite convex geometries

The aim of this section is to show that the construction given in the previous section provides the largest extension for a finite convex geometry. That is, we are interested in the case when (X, ψ) and (X, φ) are both convex geometries, and $(X, \psi) \sqsubseteq_s (X, \varphi)$.

One can consider the collection $\mathcal{G}(X)$ of all convex geometries defined on a finite set X. The following result is due to P. Edelman.

Theorem 4.1. [5] *$\mathcal{G}(X)$ forms a join-subsemilattice in $\mathcal{C}(X)$, the lattice of all closure spaces on X.*

Combining this with the fact that $\sqsubseteq_s$ is transitive, we obtain Corollary 1.13 of [1].

Corollary 4.2. *Every finite convex geometry $C \in \mathcal{C}(X)$ has a largest extension among convex geometries on X.*

Since the lattice of closed subsets of a convex geometry is join-semidistributive, the proof of Theorem 3.1 provides us with a description of this maximum convex geometry extension.

Theorem 4.3. *Let (X, φ) be a finite convex geometry and $L = \mathrm{Cl}(X, \varphi)$. Then the collection of all subsets of X satisfying* (E) *represents the closed sets of the largest extension of this convex geometry in $\mathcal{G}(X)$.*

Remark 4.4. The crux of Theorem 4.1 is that lower local distributivity (i.e., unique join representations, or equivalently the AEP) is preserved under joins in $\mathcal{C}(X)$. The corresponding statement for join-semidistributivity (i.e., canonical join representations) is not true. Figure 2 illustrates a join-semidistributive lattice L, and two join-semidistributive extensions K_1 and K_2 of L, whose join in $\mathcal{C}(X)$ is not join-semidistributive.

The unique irredundant join representation of a closed subset A of a convex geometry (X, ϕ) can be described in the terms of a closure space. For any $A \subseteq X$, we denote by $ex_\phi(A)$ the set of *extreme points* of A, i.e., the set of all $y \in A$ such that $y \notin \phi(A \setminus y)$. By the result of P. Edelman and R. Jamison [6], a closure space (X, ϕ) is a convex geometry if and only if every ϕ-closed set is the closure of its extreme points.

Theorem 4.5. *Let $A = \phi(A)$, where $A \neq \emptyset$, be a closed set of a convex geometry (X, ϕ). Let $B = ex_\phi(A) = \{y \in A : y \notin \phi(A \setminus y)\}$. Then $A = \bigvee_{y \in B} \phi(y)$ is the unique irredundant decomposition of A in* $\mathrm{Cl}(X, \phi)$.

Proof. The definition of an extreme point makes it evident that each $\phi(y)$ for $y \in B$ is a maximal join-prime element in $[0, A]$. The result of Edelman and Jamison guarantees that there are enough of these to join to A. $\square$

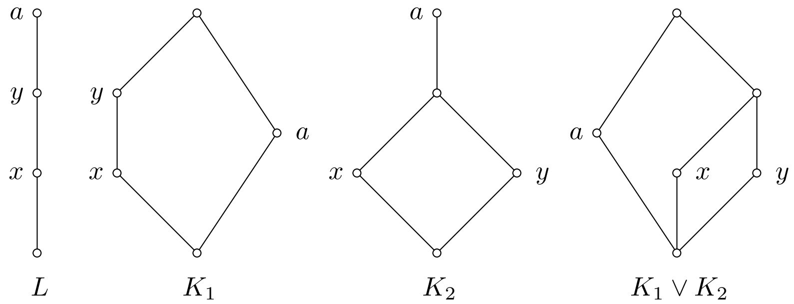

FIGURE 2

Substituting the result of Theorem 4.5 into the condition of Theorem 2.3 yields a description of the closure spaces that are extensions of a convex geometry (X, ϕ).

Corollary 4.6. *Let (X, ϕ) be a convex geometry on a finite set X. Then a closure space (X, ψ) is an extension of (X, ϕ) if and only if every ψ-closed set $C = \psi(C)$ satisfies the following property:*

$$\textit{For any } A \subseteq X, \textit{ if } \phi(y) \subseteq C \textit{ for all } y \in ex_\phi(A) \textit{ then } \phi(A) \subseteq C. \tag{M$'$}$$

However, the largest closure space extending (X, ϕ), which is determined by the closure rule (M′), need not be a convex geometry, as shown by the following example.

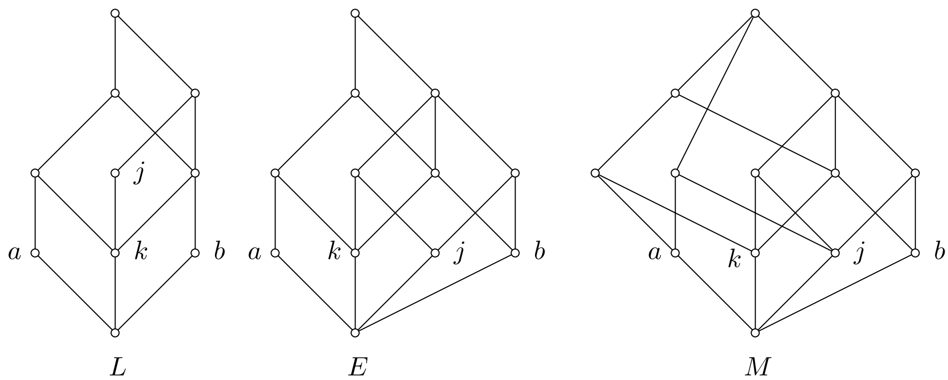

FIGURE 3

Example 4.7. Let L be the first lattice in Figure 3. It is easily checked that L is the lattice of closed sets of a convex geometry (X, ϕ) with $X = \mathrm{J}(L) = \{a, b, j, k\}$.

The second lattice, labelled E, is the largest join-semidistributive extension of L, and is calculated using the closure rule (E). The third lattice, labelled M, is in fact the maximal extension of L, and is determined by the closure rule (M).

The lattice M is not join-semidistributive, as is seen by the fact that $1_M = a \vee j \vee k = a \vee j \vee b > a \vee j \vee (k \wedge b) = a \vee j$.

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