Bounded finite lattices

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The concept of a bounded lattice was introduced by McKenzie [5] in connection with the study of lattice varieties and sublattices of a free lattice. This note is intended to unify and supplement some of the results concerning bounded finite lattices in Jónsson and the author [4]. In particular, most of the relevant definitions and background for this discussion may be found in that paper, though many originated elsewhere.

The motivation for these results was a desire to find an elementary proof of the following corollary to a theorem of Day ([1], cf. [4] § 9).

**Theorem.** Let \( L \) be a finite semidistributive lattice. Then \( L \) is bounded if and only if \( L \) is lower bounded.

We need to recall several definitions. Following [3], we say that a lattice \( L \) satisfies the condition \( (F) \) if for all \( a \in L \), \( \{ x \in L : x \preceq a \} \) is finite. If \( x \in J(L) \), let \( x^* \) denote the lower cover of \( x \). Dually, \( y^* \) denotes the upper cover of an element \( y \in M(L) \). If \( L \) is semidistributive and satisfies \( (F) \), for \( x \in J(L) \) define \( k(x) = -\sum \{ y \in L : y \equiv x^* \} \) and \( k(x) \in M(L) \). Thus for \( y \in L \), \( y \equiv k(x) \) if and only if \( x^* + y \equiv x \).

We define relations \( A, B, C \) on \( J(L) \) as follows.

(i) \( p A q \) if \( q \preceq p \) and \( p \equiv q + k(q) \).

(ii) \( p B q \) if \( p \equiv q \), \( p \equiv p^* + q \) and \( p \equiv p^* + q^* \) (i.e., \( p \equiv q \), \( q^* \equiv k(p) \) and \( q \equiv k(p) \)).

(iii) \( p C q \) if either \( p A q \) or \( p B q \).

Let \( A^d, B^d, C^d \) denote the duals of these relations on \( M(L) \).

The relevant result from [4], Theorem 4.2 and Corollary 6.4, may be summarized as follows.

**Theorem.** Let \( L \) be a finite semidistributive lattice. Then \( L \) is lower bounded if and only if there does not exist a subset \( \{ p_0, \ldots, p_n \} \subseteq J(L) \) such that \( p_i C p_{i+1} \) \( (0 \equiv i \equiv n-1) \) and \( p_n C p_0 \).
Day's result is then an immediate consequence of the following lemma.

**Lemma.** Let $L$ be semidistributive and satisfy (F), and let $p,q \in J(L)$.

(i) If $pAq$, then $k(p) B^d k(q)$.

(ii) If $pBq$, then $k(p) A^d k(q)$.

**Proof.** (i) Suppose $pAq$ and $k(p) \equiv k(p)^* k(q)$. Then $k(p) + p = k(p)^* = k(p) + k(p)^* k(q)$ whence by semidistributivity $k(p)^* = k(p) + p k(p)^* k(q) \equiv k(p) + p = k(p)$, a contradiction. Thus $k(p) \equiv k(p)^* k(q)$. On the other hand $k(p)^* k(q)^* \equiv p$, so $k(p) \equiv k(p)^* k(q)^*$.

(ii) Let $pBq$. Then $k(p) \equiv q_+$ and $k(p) \equiv q$, so $k(p) \equiv k(q)$. Because $k(q) \equiv q_+ + p \equiv p$, in fact $k(p) \equiv k(q)$. Since $q \equiv k(q)^*$ and $k(q) \cdot q = q_+ < k(p)$, the conclusion follows.

Besides Day's theorem, our lemma yields this result.

**Corollary.** Sharply transferable lattices are projective.

**Proof.** Comparing the characterization of sharply transferable lattices in [3] and the characterization of projective lattices in [2], we can easily reduce the problem to that of showing that a sharply transferable lattice satisfies $D(L) = L$. Let $L$ be sharply transferable and suppose $D(L) \neq L$. Then by the argument of Lemma 6.2 of [4] we may obtain a sequence $\{p_i : i \equiv 0\} \subseteq J(L)$ such that $p_i C p_{i+1}$. By our lemma, $\{k(p_i) : i \equiv 0\} \subseteq M(L)$ and $k(p_i) C^d k(p_{i+1})$. The argument of Theorem 6.3 of [4] then contradicts the fact (from [3]) that $D'(L) = L$.

On the other hand, the lattice figured below is projective and satisfies (F), but is not sharply transferable. Indeed, we have $q_{i+1} A q_i$ (i\equiv 0), from which it follows (again by an argument analogous to that of Theorem 6.3 of [4]) that the condition $(R_v)$ of [3] must fail.

**References**


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Received December 13, 1977

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