Closure operators and lattice extensions

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Abstract. For closure operators Γ and Δ on the same set X, we say that Δ is a weak (resp. strong) extension of Γ if $\mathrm{Cl}(X, \Gamma)$ is a complete meet-subsemilattice (resp. complete sublattice) of $\mathrm{Cl}(X, \Delta)$. This context is used to describe describe the extensions of a finite lattice that preserve various properties.

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The purpose of these notes is to make a systematic treatment of lattice extensions. Most of the results are not new, in the sense that the ideas have been understood and used as needed for some time. The author would like to thank Kira Adaricheva for discussions on lattice extensions, which led to the writing of this note.

1. Generalities

Let $\mathcal{C}(X)$ denote the collection of closure operators on a nonempty set X. For $\Gamma \in \mathcal{C}(X)$, let $\mathrm{Cl}(X, \Gamma)$ denote the lattice of Γ-closed sets.

$\mathcal{C}(X)$ is a complete lattice. The order on $\mathcal{C}(X)$ is denoted by $\sqsubseteq_w$, and $\Gamma \sqsubseteq \Delta$ holds when one of the following three equivalent properties obtains.

1. For all $S \subseteq X$, $\Delta(S) \subseteq \Gamma(S)$.

2. For all $S \subseteq X$, $\Gamma(S) = S$ implies $\Delta(S) = S$.

3. $\mathrm{Cl}(X, \Gamma)$ is a complete meet-subsemilattice of $\mathrm{Cl}(X, \Delta)$.

In that case Δ is called a *weak extension* of Γ. The join operation on $\mathcal{C}(X)$ is given by $(\bigvee \Gamma_i)(S) = \bigcap \Gamma_i(S)$.

We say that Δ is a *strong extension* of Γ, written $\Gamma \sqsubseteq_s \Delta$, if one of these three equivalent properties holds.

1. If $\Gamma(S_j) = S_j$ for all j, then $\Delta(\bigcup S_j) = \Gamma(\bigcup S_j)$.

2. $\Gamma \sqsubseteq_w \Delta$, and if $\Gamma(S_j) = S_j$ for all j, then $\Gamma(\bigcup S_j) \subseteq \Delta(\bigcup S_j)$.

3. $\mathrm{Cl}(X, \Gamma)$ is a complete sublattice of $\mathrm{Cl}(X, \Delta)$.

Note that $\sqsubseteq_s$ is a transitive relation, and that

$$(*) \qquad \Gamma \sqsubseteq_w \Delta \sqsubseteq_w E \text{ and } \Gamma \sqsubseteq_s E \text{ implies } \Gamma \sqsubseteq_s \Delta.$$

Not much else can be said in general.

THEOREM 1. *Every $\Gamma \in \mathcal{C}(X)$ has a maximum strong extension M_Γ, and $\Gamma \sqsubseteq_s \Delta$ if and only if $\Gamma \sqsubseteq_w \Delta \sqsubseteq_w M_\Gamma$.*

We sketch two proofs.

(1) If $\Gamma \sqsubseteq_s \Delta_j$ for all j, then $\Gamma \sqsubseteq_s \bigvee \Delta_j$. Then use (*).

(2) Define the closure system M_Γ by the rule that $S \subseteq X$ is M_Γ-closed iff $\Gamma(\bigcup T_i) \subseteq S$ whenever $\Gamma(T_i) = T_i$ for all i and $\bigcup T_i \subseteq S$.

The basic properties of maximum strong extensions can be summarized as follows.

1. $\Gamma \sqsubseteq_s \Delta$ implies $M_\Delta \sqsubseteq M_\Gamma$.
2. $M_\Gamma = \Gamma$ exactly when $\Gamma(x) = \{x\} \cup \Gamma(\emptyset)$ for every $x \in X$. This implies that $\mathrm{Cl}(X, \Gamma)$ is atomistic.
3. $M_{M_\Gamma} = M_\Gamma$.
4. Every subset of X is M_Γ-closed, i.e., $\mathrm{Cl}(X, M_\Gamma) = \mathcal{P}(X)$, if and only if the Γ-closed sets are closed under unions.

2. Application to finite lattices

Every finite lattice $\mathbf{L}$ can be viewed as the lattice of closed sets of a closure system on the set of its (nonzero) join irreducible elements. That is, if we define the closure operator $\Gamma_{\mathbf{L}}$ on $\mathrm{J}(\mathbf{L})$ by

$$\Gamma_{\mathbf{L}}(Q) = \{p \in \mathrm{J}(\mathbf{L}) : p \leq_{\mathbf{L}} \bigvee_{\mathbf{L}} Q\}$$

for any $Q \subseteq \mathrm{J}(\mathbf{L})$, then $\mathbf{L}$ is isomorphic to the lattice of closed sets $\mathrm{Cl}(\mathrm{J}(\mathbf{L}), \Gamma_{\mathbf{L}})$.

Given a finite set J, there are a couple of useful ways to describe the finite lattices with $\mathrm{J}(\mathbf{L}) = J$.

1. The ordered set $\mathbf{J} = (J, \leq)$ is given and $\mathbf{L}$ is the $(0, \vee)$-semilattice generated by $\mathbf{J}$ subject to relations of the form $p \leq \bigvee Q$ with $q \nleq p$ for some $q \in Q$.

2. $\mathbf{L} \cong \mathrm{Cl}(J, \Gamma)$ for a closure operator with the property that $\Gamma(\{x\}) - \{x\}$ is closed for every $x \in J$.

In the description (1), the set of relations used can be any set of joins that (i) are valid in $\mathbf{L}$, and (ii) include all the minimal nontrivial join covers (abbreviated m.n.t.j.c.) of $\mathbf{L}$. In the description (2), $\Gamma = \Gamma_{\mathbf{L}}$ as given above.

Since we will be considering different lattices with the same set of join irreducibles, it is convenient to introduce the notation that for $x \in \mathbf{L}$,

$$J_{x\mathbf{L}} = \{p \in J : p \leq_{\mathbf{L}} x\}.$$

Let $\mathbf{K}$ and $\mathbf{L}$ be finite lattices with the same set of join irreducible elements J. Extending the notation, write $\mathbf{K} \sqsubseteq_w \mathbf{L}$ when $\Gamma_{\mathbf{K}} \sqsubseteq_w \Gamma_{\mathbf{L}}$, and $\mathbf{K} \sqsubseteq_s \mathbf{L}$ when $\Gamma_{\mathbf{K}} \sqsubseteq_s \Gamma_{\mathbf{L}}$. In these situations, we say that $\mathbf{L}$ is a *weak* (resp. *strong*) *extension* of $\mathbf{K}$. These relations can be described in various terms.

THEOREM 2. *The following are equivalent for finite lattices* $\mathbf{K}$ *and* $\mathbf{L}$.

1. $\mathbf{K} \sqsubseteq_w \mathbf{L}$.
2. $J_{x\mathbf{K}}$ *is* $\Gamma_{\mathbf{L}}$*-closed for every* $x \in \mathbf{K}$.
3. $p \leq_{\mathbf{L}} \bigvee_{\mathbf{L}} Q$ *implies* $p \leq_{\mathbf{K}} \bigvee_{\mathbf{K}} Q$ *for all* $Q \subseteq J$.
4. *There is a meet semilattice embedding* $\varphi : \mathbf{K} \leq \mathbf{L}$ *with* $J_{x\mathbf{K}} = J_{\varphi(x)\mathbf{L}}$ *for every* $x \in \mathbf{K}$.

THEOREM 3. *The following are equivalent for finite lattices* $\mathbf{K}$ *and* $\mathbf{L}$.

1. $\mathbf{K} \sqsubseteq_s \mathbf{L}$.
2. *If* $Q \subseteq J$ *then*
$$\Gamma_{\mathbf{L}}(\bigcup_{q \in Q} J_{q\mathbf{K}}) = J_{b\mathbf{K}}$$
where $b = \bigvee_{\mathbf{K}} Q$.
3. *For every* $Q \subseteq J$ *we have*
$$p \leq_{\mathbf{L}} \bigvee_{\mathbf{L}} \bigcup_{q \in Q} J_{q\mathbf{K}} \quad \text{if and only if} \quad p \leq_{\mathbf{K}} \bigvee_{\mathbf{K}} Q.$$

4. *For every* $Q \subseteq J$ *we have*
$$p \leq_{\mathbf{L}} \bigvee_{\mathbf{L}} Q \quad \textit{implies} \quad p \leq_{\mathbf{K}} \bigvee_{\mathbf{K}} Q$$
and
$$p \leq_{\mathbf{K}} \bigvee_{\mathbf{K}} Q \quad \textit{implies} \quad p \leq_{\mathbf{L}} \bigvee_{\mathbf{L}} \bigcup_{q \in Q} J_{q\mathbf{K}}.$$

5. *There is a lattice embedding* $\varphi : \mathbf{K} \leq \mathbf{L}$ *with* $J_{x\mathbf{K}} = J_{\varphi(x)\mathbf{L}}$ *for every* $x \in \mathbf{K}$.

THEOREM 4. *If* $\mathbf{K} \sqsubseteq_s \mathbf{L}$, *then* $\operatorname{Sub}\mathbf{K} \leq \operatorname{Sub}\mathbf{L}$ *and* $\operatorname{Con}\mathbf{K} \geq \operatorname{Con}\mathbf{L}$.

The first part of the theorem is obvious, and the second part requires a little analysis. We recall the representation of congruence relations on a finite lattice in terms of the dependency relation; see, e.g., [4], pages 40–41.

For p, $q \in \mathrm{J}(\mathbf{K})$ we define $p\, D_{\mathbf{K}}\, q$ whenever q is a member of a minimal nontrivial join cover of p in $\mathbf{K}$. The reflexive, transitive closure $\overline{D}_{\mathbf{K}}$ of D is a quasi-order on $\mathrm{J}(\mathbf{K})$, which induces an equivalence relation $\equiv$. Let $\mathbf{Q_K}$ be the ordered set $\langle \mathrm{J}(\mathbf{K})/\equiv, \overline{D}_{\mathbf{K}}\rangle$. Then $\operatorname{Con}\mathbf{K}$ is isomorphic to the lattice of order ideals $\mathcal{O}(\mathbf{Q_K})$.

LEMMA 5. *If* $\mathbf{K} \sqsubseteq_s \mathbf{L}$ *and* $p\, D_{\mathbf{K}}\, q$, *then* $p\, D_{\mathbf{L}}\, q$.

Proof. Let $p \leq_{\mathbf{K}} \bigvee_{\mathbf{K}} Q$ be a m.n.t.j.c. in $\mathbf{K}$. Then $p \leq_{\mathbf{L}} \bigvee_{\mathbf{L}} \bigcup_{q \in Q} J_{q\mathbf{K}}$. Refine this to a m.n.t.j.c. $p \leq_{\mathbf{L}} \bigvee_{\mathbf{L}} S$ in $\mathbf{L}$, and note that $S \subseteq \bigcup_{q \in Q} J_{q\mathbf{K}}$ as $J_{r\mathbf{L}} \subseteq J_{r\mathbf{K}}$ for each $r \in J$. By part (5) of the preceding theorem we have $p \leq_{\mathbf{K}} \bigvee_{\mathbf{K}} S$. Since $S \ll Q$, by the minimality of Q we have $Q \subseteq S$.

LEMMA 6. *If* $\mathbf{P}$ *is an ordered set, and* $\mathbf{P}'$ *is obtained from* $\mathbf{P}$ *by adding relations* $x \leq y$ *and collapsing, then* $\mathcal{O}(\mathbf{P}') \subseteq \mathcal{O}(\mathbf{P})$.

These two lemmas combine to prove the second part of Theorem 4.

3. A Catalog of Extensions

The closure operator $\Gamma_{\mathbf{K}}$ is determined by the order on $J = \mathrm{J}(\mathbf{K})$ and by the minimal nontrivial join covers of $\mathbf{K}$. Extensions are determined by weakening the order on J and requiring more joinands in the minimal nontrivial join covers. With this heuristic, we list some basic types of extensions.

The maximum extension of a finite lattice

The maximum extension $\mathbf{M_K}$ has the property that $\mathbf{K} \sqsubseteq_s \mathbf{L}$ if and only if $\mathbf{K} \sqsubseteq_w \mathbf{L} \sqsubseteq_w \mathbf{M_K}$. Here are two characterizations of $\mathbf{M} = \mathbf{M_K}$.

1. A set $C \subseteq J$ is $\Gamma_{\mathbf{M}}$-closed iff for every $S \subseteq J$, if $J_{s\mathbf{K}} \subseteq C$ for every $s \in S$ then $J_{b\mathbf{K}} \subseteq C$, where $b = \bigvee_{\mathbf{K}} S$.

2. For all $S \subseteq J$ and every $p \in J$, $p \leq_{\mathbf{K}} \bigvee_{\mathbf{K}} S$ if and only if $p \leq_{\mathbf{M}} \bigvee_{\mathbf{M}} \bigcup_{s \in S} J_{s\mathbf{K}}$.

We note that $\mathbf{M}_{\mathbf{K}}$ is atomistic, and that $\mathbf{M}_{\mathbf{K}}$ is Boolean if and only if $\mathbf{K}$ is distributive.

The Tischendorf extension of a finite lattice

The extension $\mathbf{T} = \mathbf{T}_{\mathbf{K}}$ of Michael Tischendorf [7] is the smallest atomistic extension such that $\overline{D}_{\mathbf{T}} = \overline{D}_{\mathbf{K}}$, and hence $\mathbf{Q}_{\mathbf{T}} = \mathbf{Q}_{\mathbf{K}}$. (Thus congruence relations are preserved in a natural way.) It is determined by the closure rules that $p \leq_{\mathbf{T}} \bigvee_{\mathbf{T}} S$ whenever $p \leq_{\mathbf{K}} \bigvee_{\mathbf{K}} S$ is a m.n.t.j.c. in $\mathbf{K}$. In other words, the extension is obtained by removing the order on J and retaining the minimal nontrivial join covers. This natural extension has proved useful, and inspired some of the other extensions considered herein.

The maximum congruence-preserving extension of a finite lattice

There is a largest extension $\mathbf{B} = \mathbf{B}_{\mathbf{K}}$ such that $\overline{D}_{\mathbf{B}} = \overline{D}_{\mathbf{K}}$, with the property that $\mathbf{K} \sqsubseteq_s \mathbf{L}$ and $\overline{D}_{\mathbf{L}} = \overline{D}_{\mathbf{K}}$ if and only if $\mathbf{K} \sqsubseteq_w \mathbf{L} \sqsubseteq_w \mathbf{B}$. In particular, $\mathbf{K} \sqsubseteq_w \mathbf{T} \sqsubseteq_w \mathbf{B}$, so $\mathbf{B}$ is atomistic. This extension is determined by the rules that $p \leq_{\mathbf{B}} \bigvee_{\mathbf{B}} \bigcup_{s \in S} (J_{s\mathbf{K}} \cap \{t \in J : p \overline{D}_{\mathbf{K}} t\})$ whenever $p \leq_{\mathbf{K}} \bigvee_{\mathbf{K}} S$ is a m.n.t.j.c. in $\mathbf{K}$.

Recall that $\mathbf{L}$ is *lower bounded* of rank k if and only if $\overline{D}_{\mathbf{L}}$ is acyclic and $\mathbf{Q}_{\mathbf{L}}$ has height at most k. Clearly this extension preserves lower boundedness and rank: $\mathbf{B}_{\mathbf{K}} \in \mathcal{LB}_k$ if and only if $\mathbf{K} \in \mathcal{LB}_k$.

The maximum join semidistributive extension of a finite join semidistributive lattice

A finite join semidistributive lattice $\mathbf{K}$ has a largest join semidistributive extension $\mathbf{S} = \mathbf{S}_{\mathbf{K}}$, with the property that $\mathbf{K} \sqsubseteq_s \mathbf{S}$, $\mathbf{S}$ is join semidistributive, and $\mathbf{L} \sqsubseteq_w \mathbf{S}$ whenever $\mathbf{K} \sqsubseteq_s \mathbf{L}$ and $\mathbf{L}$ satisfies $\mathrm{SD}_{\vee}$. This extension was investigated in [1]. To describe it, we define $\mathrm{D}^0_{x\mathbf{K}}$ to be the set of all elements that are join prime in the interval $[0, x]$ of $\mathbf{K}$. We give two descriptions of $\mathbf{S}$.

1. A set $C \subseteq J$ is $\Gamma_{\mathbf{S}}$-closed iff for every $a \in \mathbf{K}$, if $\mathrm{D}^0_{a\mathbf{K}} \subseteq C$ then $J_{a\mathbf{K}} \subseteq C$.

2. For all $T \subseteq J$ and every $p \in J$, $p \leq_{\mathbf{K}} \bigvee_{\mathbf{K}} T$ if and only if $p \leq_{\mathbf{S}} \bigvee_{\mathbf{S}} \bigcup_{t \in T} \mathrm{D}^0_{t\mathbf{K}}$.

However, there may be lattices $\mathbf{L}$ with $\mathbf{K} \sqsubseteq_s \mathbf{L} \sqsubseteq_w \mathbf{S_K}$ and $\mathbf{L}$ not join semidistributive. Also, the join of two join semidistributive extensions of $\mathbf{K}$ need not be join semidistributive. The lattice $\mathbf{S_K}$ is always atomistic.

For a finite lower bounded lattice $\mathbf{K}$ we have

$$\mathbf{K} \sqsubseteq_w \mathbf{T_K} \sqsubseteq_w \mathbf{B_K} \sqsubseteq_w \mathbf{S_K} \sqsubseteq_w \mathbf{M_K}$$

and all are strong extensions of $\mathbf{K}$. These can all be distinct.

4. Convex geometries and lower local distributivity

Finite join semidistributive lattices are those in which every element has a *canonical* join representation. A classical result of R. P. Dilworth [2] is that every element of a finite lattice $\mathbf{L}$ has a *unique irredundant* join representation if and only if $\mathbf{L}$ is lower locally distributive, i.e., if the interval $[m_x, x]$ is distributive for every $x \in \mathbf{L}$, where $m_x = \bigwedge\{y \in L : x \succ y\}$. Since a unique irredundant join decomposition is *a fortiori* canonical, every finite lower locally distributive lattice is join semidistributive. On the other hand, in a finite atomistic join semidistributive lattice, the canonical join representation of an element is its unique irredundant decomposition. Hence *if* $\mathbf{K}$ *satisfies* $\mathrm{SD}_\vee$*, then* $\mathbf{S_K}$ *is lower locally distributive.*

These lattices have been studied recently under the guise of convex geometries. A *convex geometry* is a closure system (X, Γ) on a finite set X where Γ satisfies the anti-exchange property: if $p \in \Gamma(C \cup \{q\})$ and $p \notin \Gamma(C)$, then $q \notin \Gamma(C \cup \{p\})$ for all $p \neq q$ in X and all $C \subseteq X$. The basic equivalence is found in Edelman and Jamison [5] (see also Monjardet [6]).

THEOREM 7. *The following are equivalent for a finite lattice L.*

1. $\mathbf{L} \cong \mathrm{Cl}(X, \Gamma)$ *for some convex geometry* (X, Γ).
2. $(\mathrm{J}(\mathbf{L}), \Gamma_{\mathbf{L}})$ *is a convex geometry.*
3. $\mathbf{L}$ *is lower locally distributive.*

P. H. Edelman [3] has shown that the join of two convex geometries in $\mathcal{C}(X)$ is again a convex geometry. Indeed, suppose that Γ, $\Delta \in \mathcal{C}(X)$ are such that both $\mathrm{Cl}(X, \Gamma)$ and $\mathrm{Cl}(X, \Delta)$ are lower locally distributive. It is easy to show that if $A \subseteq X$, C is the unique irredundant decomposition of $\Gamma(A)$, and D is the unique irredundant decomposition of $\Delta(A)$, then $C \cup D$ is the unique irredundant decomposition of $(\Gamma \vee \Delta)(A) = \Gamma(A) \cap \Delta(A)$ in $\mathrm{Cl}(X, \Gamma \vee \Delta)$.

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