INTERVAL DISMANTLABLE LATTICES

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Abstract. A finite lattice is interval dismantlable if it can be partitioned into an ideal and a filter, each of which can be partitioned into an ideal and a filter, etc., until you reach 1-element lattices. In this note, we find a quasi-equational basis for the pseudoquasivariety of interval dismantlable lattices, and show that there are infinitely many minimal interval non-dismantlable lattices.

Define an interval dismantling of a lattice to be a partition of the lattice into two nonempty, complementary sublattices where one is an ideal and the other a filter. A finite lattice is said to be interval dismantlable if it can be reduced to 1-element lattices by successive interval dismantlings.

In order to work with these lattices, we note that the following are equivalent for a finite lattice $L$:

1. $L = I \cup F$ for some disjoint proper ideal $I$ and filter $F$.
2. $L$ contains a nonzero join prime element.
3. $L$ contains a non-one meet prime element.
4. There is a surjective homomorphism $h : L \to 2$.
5. Some generating set $X$ for $L$ can be split into two disjoint nonempty subsets, $X = Y \cup Z$, such that $\wedge Y \nleq \vee Z$.
6. Every generating set $X$ for $L$ can be split into two disjoint nonempty subsets, $X = Y \cup Z$, such that $\wedge Y \nleq \vee Z$.

So if a lattice $L$ contains no join prime element, then it is interval non-dismantlable. If $L$ contains no join prime element, but every proper sublattice does, then it is minimally interval non-dismantlable. If $L$ contains an interval non-dismantlable sublattice, then $L$ is interval non-dismantlable.

Note that it follows from (2) and (3) that every finite meet semidistributive or join semidistributive lattice is interval dismantlable. The
atoms of a finite meet semidistributive lattice are join prime; dually, the coatoms of a finite join semidistributive lattice are meet prime.

In view of conditions (5) and (6) above, let us say that a subset $X$ of a lattice $L$ is divisible if it can be divided into two nonempty subsets $Y$ and $Z$ such that $\bigwedge Y \nsubseteq \bigvee Z$; else $X$ is indivisible.

It is straightforward to see that interval dismantlable lattices form a pseudoquasivariety, i.e., a class of finite algebraic structures closed under taking substructures and finite direct products. The basic theorem on pseudoquasivarieties is due to C. J. Ash [2]; see also Chapter 2 of V. A. Gorbunov [3].

**Theorem 1.** Let $\mathcal{K}$ be a pseudoquasivariety of structures of finite type. Then $\mathcal{K}$ is the set of all finite structures in the quasivariety $Q = \text{SPU}(\mathcal{K})$, where $U$ denotes the ultraproduct operator.

Thus there is a set of quasi-equations that determines the set of finite interval dismantlable lattices. For each $n \geq 3$, let $X_n = \{x_1, \ldots, x_n\}$ be a set of $n$ variables. Consider the quasi-equations

$$\left( \delta_n \right) \quad \forall \varnothing \subset Y \subset X_n \quad \bigwedge Y \leq \bigvee (X_n \setminus Y) \rightarrow x_1 \approx x_2 .$$

Any indivisible subset $A$ of a lattice $L$ with $|A| \leq n$ satisfies the hypothesis of $\delta_n$. On the other hand, by symmetry the conclusion could be replaced by $x_i \approx x_j$ for any $i \neq j$. Hence the quasi-equation $\delta_n$ expresses that $L$ contains no indivisible subset of size $k$ for $1 < k \leq n$. In particular, $\delta_n$ implies $\delta_{n-1}$.

**Theorem 2.** A finite lattice is interval dismantlable if and only if it satisfies $\delta_n$ for all $n \geq 3$, that is, the lattice contains no indivisible subset of more than one element.

**Proof.** First, assume that $L$ is interval dismantlable. For every $n \geq 3$ and $a \in L^n$, we want to show that $\delta_n$ holds under the substitution $x_i \mapsto a_i$. If $a_1 = a_2$, then the conclusion of $\delta_n$ holds. If $a_1 \neq a_2$, then the sublattice $S = \text{Sg}(a_1, \ldots, a_n)$ is nontrivial and interval dismantlable, and hence $S$ has a decomposition $S = I \cup F$ into a proper ideal and filter. Let $Y = \{a_i : a_i \in F\}$ and $Z = \{a_j : a_j \in I\}$. Then $\bigwedge Y \in F$ and $\bigvee Z \in I$, whence $\bigwedge Y \nsubseteq \bigvee Z$, so that the corresponding inclusion in the hypothesis of $\delta_n$ fails. Thus $\delta_n$ holds for every substitution.

Conversely, let us show that every finite lattice that satisfies all $\delta_n$ is interval dismantlable. We do so by induction on $|L|$. To begin, the 1-element lattice satisfies every $\delta_n$ and is trivially interval dismantlable. So consider a finite lattice $L$ with $|L| > 1$. Choose a generating set $X = \{a_1, a_2, \ldots, a_k\}$ for $L$ with $a_1 \neq a_2$. Since $L$ satisfies $\delta_k$ and
the conclusion fails, there is a nontrivial splitting \( X = Y \cup Z \) with \( \bigwedge Y \notin \bigvee Z \). This splits \( L \) into a proper ideal and filter, \( L = I \cup F \), and each of these is a smaller lattice that satisfies \( \delta_n \) for all \( n \). By induction, both \( I \) and \( F \) are interval dismantlable, and so \( L \) is as well. □

Any class of finite lattices closed under sublattices can be characterized by the exclusion of its minimal non-members. Examples of minimal interval non-dismantlable lattices include \( M_3 \) and the lattices in Figure 1, which fail \( \delta_4 \). We would like to show that the pseudoquasivariety of finite interval dismantlable lattices is not finitely based, for which we need an infinite sequence of minimal interval non-dismantlable lattices, such that any finite collection of the quasi-equations \( \delta_j \) is satisfied by at least one of them. The next theorem provides this by generalizing the top right example of Figure 1.

**Figure 1.** Three minimal interval non-dismantlable lattices.

**Theorem 3.** There is a sequence of minimal interval non-dismantlable lattices \( K_n \) \((n \geq 4)\) such that each \( K_n \) satisfies \( \delta_j \) for \( 3 \leq j < n \), but fails \( \delta_n \).

**Proof.** For \( n \geq 4 \), we construct a lattice \( K_n \) as follows. The carrier set is \( n \times (n - 2) = \{(i, j) : 0 \leq i < n \text{ and } 0 \leq j < n - 2\} \), with the order given by \( (i, j) \leq (k, \ell) \) if \( j \leq \ell \) and either \( 0 \leq k - i \leq \ell - j \) or
n + k - i ≤ ℓ - j, plus a top element T and bottom element B. Thus we are thinking of the first coordinates modulo n, as if wrapped around a cylinder. The covers in the middle portion of the lattice are given by (i, j) < (i, j + 1) and (i, j) < (i + 1 mod n, j + 1) where 0 ≤ i < n and 0 ≤ j < n - 3. The middle portion of the lattice \( K_5 \) is illustrated in Figure 2.

For a generating set, we can take \( X = \{(i, 0) : i < n\} \). This has the property that any pair of distinct elements of \( X \) meets to \( B \), while the join of any \( n - 1 \) is \( T \). Thus \( K_n \) fails \( \delta_n \) and is interval non-dismantlable.

In view of the circular symmetry, we may consider the maximal sub-lattices not containing the generator \((0, 0)\). These are easily seen to be \( S_0 = K_n \setminus \{(0, j) : j < n - 2\} \) and \( T_0 = K_n \setminus \{(j, j) : j < n - 2\} \). Both these are interval dismantlable. For \( S_0 = \uparrow (1, 0) \cup \downarrow (n - 1, n - 3) \), with the filter being dually isomorphic to the lattice Co(n - 2) of convex subsets of an \( n - 2 \) element chain, and hence meet semidistributive, and the ideal being isomorphic to Co(n - 2) and hence join semidistributive. Likewise \( T_0 = \uparrow (n - 1, 0) \cup \downarrow (n - 2, n - 3) \), with the filter being meet semidistributive and the ideal being join semidistributive.

To see that \( K_n \) satisfies \( \delta_j \) for \( 3 ≤ j < n \), consider an arbitrary generating set \( X \) for \( K_n \). For each \( k \) with \( 0 ≤ k < n \), the set \( S_k = K_n \setminus \{(k, \ell) : \ell < n - 2\} \) is a proper sublattice of \( K_n \). Hence \( X \not\subseteq S_k \), i.e., \( X \) contains an element of the form \( (k, \ell) \) for each \( k < n \). Thus \( |X| ≥ n \). So every subset of \( K_n \) with fewer than \( n \) elements generates a proper sublattice, which is interval dismantlable. Therefore \( K_n \) satisfies \( \delta_j \) for \( j < n \).

Discussion. The original notion of dismantlability is that a finite lattice is dismantlable if it can be reduced to a 1-element lattice by successively removing doubly irreducible elements. These lattices were
characterized independently by Ajtai [1] and Kelly and Rival [4], as those lattices not containing an \( n \)-crown for \( n \geq 3 \). Dismantlable lattices do not form a pseudoquasivariety, as they are not closed under finite direct products.

More generally, we can define a sublattice dismantling of a lattice to be a partition of the lattice into two nonempty, complementary sublattices. A finite lattice is said to be sublattice dismantlable if it can be reduced to 1-element lattices by successive sublattice dismantlings. Clearly both the original dismantlable lattices and interval dismantlable lattices are sublattice dismantlable, and this class does form a pseudoquasivariety. It would be interesting to characterize sublattice dismantlable lattices.

**References**