

Prime intervals and maximal chains in semimodular lattices of finite length

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ABSTRACT. We show that in a semimodular lattice L of finite length, from any prime interval we can reach any maximal chain C by an up- and a down-perspectivity. Therefore, C is a congruence-determining sublattice of L .

A classical theorem of R. Dedekind [2]) (see also R. Dedekind [3]) states that the factors of any chief series (maximal chain of normal subgroups) of a finite group are invariant. C. Jordan, O. Hölder, and H. Wielandt generalized this result to the factors of any composition series (maximal chain of subnormal subgroups).

Formulations of this result for modular lattices are well-known; they are called the *Jordan-Hölder Theorem*. For semimodular lattices of finite length, most books (see, for instance [1, page 40], [4, page 226]) contain only a much weaker result: any two maximal chains in a semimodular lattice have the same length.

There is a stronger version, however, for semimodular lattices that requires only a slight extension of Dedekind's original argument. This can be found in [6, page 444] or [7, Chapter 9]. (Recall that the intervals $[a, b]$ and $[c, d]$ are *perspective* if $b \vee c = d$ and $b \wedge c = a$ or *vice versa*. *Projectivity* is the transitive closure of perspectivity.)

Theorem 1. *Let C and D be two maximal chains in a finite length semimodular lattice,*

$$\begin{aligned} C: \quad & 0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1 \\ D: \quad & 0 = d_0 \prec d_1 \prec \cdots \prec d_n = 1. \end{aligned}$$

Then there is a permutation π of the set $\{1, \dots, n\}$ such that $[c_{i-1}, c_i]$ is projective to $[d_{\pi(i)-1}, d_{\pi(i)}]$, for all i .

In this note, we improve on Theorem 1 by replacing projectivity with *two* perspectivities.

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Theorem 2. *Let C and D be two maximal chains in a finite length semimodular lattice, say*

$$\begin{aligned} C: \quad & 0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1 \\ D: \quad & 0 = d_0 \prec d_1 \prec \cdots \prec d_n = 1. \end{aligned}$$

Then there is a permutation π of the set $\{1, \dots, n\}$ with the following property: there exists a (prime) interval \mathfrak{p}_i such that $[c_{i-1}, c_i]$ is up-perspective to \mathfrak{p}_i and \mathfrak{p}_i is down-perspective to $[d_{\pi(i)-1}, d_{\pi(i)}]$, for all i .

Remark. Recall that the lattice of subnormal subgroups of a finite group is lower semimodular, so the dual of this theorem applies to yield the full Jordan-Hölder theorem.

Proof. By induction on $\text{length}(L)$. The statement is obvious for $\text{length}(L) \leq 2$, so let $\text{length}(L) > 2$.

Let k be the largest integer with $c_1 \not\leq d_k$; note that $k < n$. If $k = 0$, then $c_1 = d_1$ and the statement follows by the induction hypothesis. So we can assume that $k > 0$.

For $0 \leq j \leq n$, let $e_j = c_1 \vee d_j$. Note that $e_0 = c_1$ and $e_k = e_{k+1} = d_{k+1}$, and indeed $e_j = d_j$ for $j \geq k + 1$. Now

$$c_1 = e_0 \prec e_1 \prec \cdots \prec e_k = e_{k+1} \prec e_{k+2} \prec \cdots \prec e_n = 1$$

is a maximal chain in the interval $[c_1, 1]$. By induction, there is an injective map $\sigma : \{2, \dots, n\} \rightarrow \{1, \dots, k, k+2, \dots, n\}$ such that, for $i > 1$, each interval $[c_{i-1}, c_i]$ is projective up to some prime interval \mathfrak{p}_i in L , which in turn projects down to $[e_{\sigma(i)-1}, e_{\sigma(i)}]$. For $j \leq k$, $[e_{j-1}, e_j]$ projects down to $[d_{j-1}, d_j]$, while for $j > k + 1$ we have $[e_{j-1}, e_j] = [d_{j-1}, d_j]$. Meanwhile, $[0, c_1]$ projects up to $[d_{k-1}, d_k]$. So we may take π to be the permutation with $\pi(i) = \sigma(i)$ for $i \neq 1$, and $\pi(1) = k$. \square

Given a sublattice S of a lattice L , we say that S is a *congruence-determining sublattice* of L if any congruence relation on L is uniquely determined by its restriction to S ; see [5].

Corollary 3. *Let L be a finite dimensional semimodular lattice and C be a maximal chain in L . Then C is a congruence-determining sublattice of L .*

Proof. Indeed, a congruence is determined by the prime intervals it collapses, and every prime interval is contained in a maximal chain. Let α be a congruence on L , and let β be the congruence generated by α restricted to a maximal chain C . Then $\beta \leq \alpha$. Meanwhile, if $[a, b]$ is a prime interval, then by the theorem, $[a, b]$ is projective to some prime interval $[c, d]$ in C . So if $a \equiv b \pmod{\alpha}$, then $c \equiv d \pmod{\alpha}$, whence $a \equiv b \pmod{\beta}$. Thus $\alpha \leq \beta$. We conclude that $\alpha = \beta$, verifying that C is a congruence-determining sublattice of L . \square

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