PRIME INTERVALS AND MAXIMAL CHAINS IN FINITE DIMENSIONAL SEMIMODULAR LATTICES

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ABSTRACT. We show that in a finite dimensional semimodular lattice L, from any prime interval we can reach any maximal chain C by an up- and a down-perspectivity. Therefore, L is a congruence-preserving extension of C.

A classical theorem of Dedekind states that the factors of any chief series (maximal chain of normal subgroups) of a finite group are invariant. C. Jordan, Hölder and Wielandt generalized this result to the factors of any composition series (maximal chain of subnormal subgroups). The lattice theoretic version goes thusly. Recall that intervals $[a, b]$ and $[c, d]$ are *perspective* if $b \vee c = d$ and $b \wedge c = a$ or *vice versa*. *Projectivity* is the transitive closure of perspectivity.

Theorem 1. *Let C and D be two maximal chains in a finite length semimodular lattice, say*

$$0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1$$
$$0 = d_0 \prec d_1 \prec \cdots \prec d_n = 1.$$

Then there is a permutation π of the set $\{1, \ldots, n\}$ such that $[c_{i-1}, c_i]$ is projective in two steps (up-down) to $[d_{\pi(i)-1}, d_{\pi(i)}]$ for all i.

Recall that the lattice of subnormal subgroups of a finite group is lower semimodular, so the theorem applies dually there to yield the Jordan-Hölder theorem. Here we give a very natural proof, that as far as we know is new.

Proof. By induction on length(L). The statement is obvious for length$(L) \leq 2$, so let length$(L) > 2$.

Let k be the largest integer with $c_1 \nleq d_k$, noting $k < n$. If $k = 0$, then $c_1 = d_1$ and the statement follows by the induction hypothesis. So we can assume that $k > 0$.

For $0 \leq j \leq n$, let $e_j = c_1 \vee d_j$. Note that $e_0 = c_1$ and $e_k = e_{k+1} = d_{k+1}$, and indeed $e_j = d_j$ for $j \geq k + 1$. Now

$$c_1 = e_0 \prec e_1 \prec \cdots \prec e_k = e_{k+1} \prec e_{k+2} \prec \cdots \prec e_n = 1$$

is a maximal chain in the interval $[c_1, 1]$. By induction, there is an injective map $\sigma : \{2, \ldots, n\} \to \{1, \ldots, k, k + 2, \ldots, n\}$ such that, for $i > 1$, each interval $[c_{i-1}, c_i]$ is projective up to some prime interval $\mathfrak{p}_i$ in L, which in turn projects down to $[e_{\sigma(i)-1}, e_{\sigma(i)}]$. For $j \leq k$, $[e_{j-1}, e_j]$ projects down to $[d_{j-1}, d_j]$, while for $j > k + 1$

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we have $[e_{j-1}, e_j] = [d_{j-1}, d_j]$. Meanwhile, $[0, c_1]$ projects up to $[d_{k-1}, d_k]$. So we may take π to be the permutation with $\pi(i) = \sigma(i)$ for $i \neq 1$, and $\pi(1) = k$. □

Here we note another nice consequence of this theorem, that could prove useful and seems to have escaped attention.

Corollary 2. *Let* $\mathbf{L}$ *be a finite length semimodular lattice, and let* C *be any maximal chain in* $\mathbf{L}$*. Then any congruence relation on* $\mathbf{L}$ *is uniquely determined by its restriction to* C*.*

Proof. Indeed, a congruence is determined by the covers that it collapses, and every covering pair is contained in a maximal chain. Let θ be a congruence on $\mathbf{L}$, and let ψ be the congruence generated by θ restricted to a maximal chain C. Then $\psi \leq \theta$. Meanwhile, if a/b is a prime interval, then by the theorem, a/b is projective to some prime interval c/d in C. So if $a\theta b$ then $c\theta d$, whence $a\psi b$. Thus $\theta \leq \psi$. □

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