EQUACLOSURE OPERATORS ON JOIN SEMIDISTRIBUTIVE LATTICES

This is a working document. All lattices are finite for now.

An *equaclosure operator* on a finite lattice has the following properties.

(1) $x \leq h(x)$.
(2) $x \leq y$ implies $h(x) \leq h(y)$.
(3) $h^2(x) = h(x)$.
(4) $h(0) = 0$.
(5) $h(x) = h(y)$ implies $h(x) = h(xy)$.
(6) $h(x)(y+z) = h(x)y + h(x)z$.

Equaclosure operators were introduced by Adaricheva and Gorbunov [5] to abstract the properties of the equational closure operator on lattices of quasivarieties, which had been put to good use by Dziobiak in [6]. For more, see Section 5.3 of Gorbunov [8]. Note that two quasivarieties have the same equational closure if and only if they have the same free algebras.

The real objective here is to characterize lattices of quasivarieties. These are known to be join semidistributive and biatomic, in addition to admitting an equaclosure operator and being dually algebraic. Moreover, the equational closure operator on quasivarieties satisfies another condition, which is not quite so natural. For atoms a, b we write $a \sim b$ if $|\downarrow(a+b)| = 4$. The last condition is:

(7) If a, b, c, d are atoms of $\mathbf{L}$ with $a \sim d$, $d \nleq h(a)$, $d \leq h(c)$ and $h(c) = h(a+b)$, then $h(c) = h(b+d)$.

A map satisfying (1)–(7) will be called a *strong* equaclop. We note that if h is an equaclop and a, b and c are distinct atoms, then $h(c) = h(a+b)$ if and only if $c \leq a+b$.

1. A general perspective: MD($\mathbf{L}$) and equaclosure operators

Call an element $a \in \mathbf{L}$ *meet distributive* if $a(x+y) = ax + ay$ for all x, $y \in \mathbf{L}$. Let MD($\mathbf{L}$) denote the set of all meet distributive elements of $\mathbf{L}$. Note that this is a meet subsemilattice of $\mathbf{L}$ containing 0 and 1.

With each 0-1-meet subsemilattice $\mathbf{S}$ of MD($\mathbf{L}$) we can associate the closure operator $\sigma(x) = \prod \uparrow x \cap \mathbf{S}$, that is, the least element of $\mathbf{S}$ above x. All these closure operators satisfy conditions (1)–(4) and (6), while none of them need satisfy (5).

We can use (5) directly: $\sigma(x) = \sigma(y)$ implies $\sigma(x) = \sigma(xy)$, or the subsemilattice form

$$(\star) \qquad \uparrow x \cap \mathbf{H} = \uparrow y \cap \mathbf{H} \text{ implies } \uparrow x \cap \mathbf{H} = \uparrow xy \cap \mathbf{H}\,.$$

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Another variation is: *if* $xy \leq h \in \mathbf{H}$*, then either* $x \leq h$*, or there exists* $k \in \mathbf{H}$ *such that* $x \nleq k$ *and* $y \leq k$*, or there exists* $\ell \in \mathbf{H}$ *such that* $x \leq \ell$ *and* $y \nleq \ell$*.*

Theorem 1. *Let* $\mathbf{L}$ *be a finite lattice. There is a one-to-one correspondence between equaclops on* $\mathbf{L}$ *and* 0-1*-meet subsemilattices* $\mathbf{H}$ *of* $\mathrm{MD}(\mathbf{L})$ *satisfying* $(\star)$*. Given an equaclop* η*, let* $\mathbf{H} = \eta(\mathbf{L})$*, the closed elements. Given* $\mathbf{H}$*, let* $\eta(x) = \prod \uparrow x \cap \mathbf{H}$*.*

We order equaclops with the usual closure operator order (dual to the set containment on the corresponding subsemilattices). It is easy to reproduce the result from the untranslated [5] that with this order the equaclops form a meet semilattice.

Theorem 2. *If* $\mathbf{S}$*and* $\mathbf{T}$ *are subsemilattices of* MD *satisfying* $(\star)$*, then* $\mathbf{S} \cdot \mathbf{T} = \mathrm{Sg}(S \cup T)$ *also satisfies* $(\star)$*.*

Proof. Corresponding to $\mathbf{S}\cdot\mathbf{T}$ let $\xi(x) = \sigma(x)\tau(x)$. Note that $\sigma(x) = \sigma(\xi(x))$ and $\tau(x) = \tau(\xi(x))$. So if $\xi(x) = \xi(y)$, then $\sigma(x) = \sigma(y)$ and $\tau(x) = \tau(y)$. But then $\xi(xy) = \sigma(xy)\tau(xy) = \sigma(x)\tau(x) = \xi(x)$, as desired. ☐

Throughout what follows we will let μ denote the map $\mu(x) = \prod \uparrow x \cap \mathrm{MD}(\mathbf{L})$, which may or may not satisfy (5).

Let $\overline{\mathrm{D}}$ denote the reflexive, transitive closure of D on $J = \mathrm{J}(\mathbf{L})$, regarded as an ordered set.

Lemma 3. *Let* $\mathbf{L}$ *be a finite lattice and* $a \in \mathbf{L}$*. Then* $a \in \mathrm{MD}(\mathbf{L})$ *if and only if* $\downarrow a$ *is* D*-closed. In other words,* $\mu(x)$ *is the* D*-closed ideal generated by* x*. Consequently,* $a \in \mathrm{MD}(\mathbf{L})$ *if and only if* $\downarrow a \cap J$ *is a filter in* $(J, \overline{\mathrm{D}})$*.*

Proof. Suppose $b \leq a$ and $b \leq \sum U$ is a m.n.t.j.c. Then $b \leq a(\sum U) = \sum_{u \in U} au$, whence $u \leq a$ for all $u \in U$. ☐

Corollary 4. *If* $\downarrow a \cap \mathrm{J}(\mathbf{L}) \subseteq \mathrm{D}_0(\mathbf{L})$*, then* $a \in \mathrm{MD}(\mathbf{L})$*.*

The next result provides another (weak) connection between $\mathrm{MD}(\mathbf{L})$ and the distributive reflection of $\mathbf{L}$ in the join semidistributive case.

Theorem 5. *Let* $\mathbf{L}$ *be a finite join semidistributive lattice. If* $a \in \mathrm{MD}(\mathbf{L})$*, then* $\mathrm{CS}(a) \subseteq \mathrm{D}_0(\mathbf{L})$*.*

Proof. Otherwise, let $c \in \mathrm{CS}(a)$ and let $c \leq \sum U$ be a m.n.t.j.c. Then $U \subseteq \downarrow a$ by the lemma, whereas $c \in \mathrm{D}_0(\downarrow a)$, a contradiction. ☐

2. Examples and sufficient conditions

Example. $\mathbf{1} \dot{+} \mathbf{L}$ always has the equaclop

$$h(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

This is hell on the converse of sufficient conditions.

Example. If $\mathbf{L}_1$ and $\mathbf{L}_2$ admit an equaclop, so does $\mathbf{L}_1 \times \mathbf{L}_2$.

Theorem 6. *Let* $\mathbf{L}$ *be a finite lattice. If* $J-\mathrm{D}_0(\mathbf{L})$ *contains a least element* u*, then* $\mathbf{L}$ *admits the equaclop*

$$h(x)=\begin{cases} x & \textit{if } \ x\not\geq u,\\ 1 & \textit{if } \ x\geq u.\end{cases}$$

That includes all subdirectly irreducible lattices in $\mathcal{LB}(1)$, which have exactly one non-join-prime join irreducible.

Proof. We need to know that if $a\not\geq u$, then $a\in\mathrm{MD}(\mathbf{L})$. Since every join irreducible below a will be join prime, this is so. □

The deepest and most satisfying results by far are with respect to the atomistic case, due to Adaricheva, Dziobiak and Gorbunov [3].

Theorem 7. *For a finite atomistic lattice* $\mathbf{L}$*, TFAE.*

1. $\mathbf{L}$ *is the lattice of all quasivarieties contained in some qv* $\mathbf{K}$*.*
2. $\mathbf{L}$ *is isomorphic to the* $\mathrm{Sub}_\wedge(\mathbf{S})$ *for some semilattice* $\mathbf{S}$*.*
3. $\mathbf{L}$ *is biatomic and admits a strong equaclosure operator.*

As a consequence, every finite atomistic Q-lattice is lower bounded, but not conversely.

This result is extended to algebraic atomistic lattices in [4], with appropriate finiteness conditions added.

We include part of the proof of Theorem 7 as being of interest.

Lemma 8. *For any finite semilattice* $\mathbf{P}$ *the lattice* $\mathrm{Sub}_\wedge(\mathbf{P})$ *admits an equaclop.*

Proof. For any $a\in\mathrm{Sub}_\wedge(\mathbf{P})$ we denote by $[a)$ the filter generated by a in $\mathbf{P}$. Then $h_f(a)=[a)$ defines an equaclop on $\mathrm{Sub}_\wedge(\mathbf{P})$. In fact, it was shown in [5] in a more general setting that this is a maximal equaclop on $\mathrm{Sub}_\wedge(\mathbf{P})$.

Also, μ produces an equaclop on every $\mathrm{Sub}_\wedge(\mathbf{P})$, and this is of course the minimal equaclop on $\mathrm{Sub}_\wedge(\mathbf{P})$. For any $p\in\mathbf{P}$, $\mu(p)$ is the minimal subsemilattice X of $\mathbf{P}$ with the following properties:

1. X is a convex subsemilattice with p as the minimal element.
2. If $x\in X$ and $x=y\wedge z$ then $y,z\in X$.

For any $X\in\mathrm{Sub}_\wedge(\mathbf{P})$ we then have $\mu(X)=\bigcup\{\mu(p):p\in X\}$.

It is easy to design a semilattice $\mathbf{P}$ for which μ does not coincide with any maximal equaclop that $\mathrm{Sub}_\wedge(\mathbf{P})$ might have. □

Example 9.

We note that μ, when it is not a maximal equaclop on some lattices of the form $\mathrm{Sub}_\wedge(\mathbf{P})$, might be not a *strong* equaclop.

As an example, consider a semilattice $P^*=\mathbf{2}\times\mathbf{3}$ with the minimal element $c=a\wedge b$, and $b=d\wedge e$, where $d>a$. Evidently, μ defined on $\mathrm{Sub}_\wedge(\mathbf{P}^*)$ satisfies $\mu(c)=P^*,\mu(a)=a$, thus the premises of (7) are satisfied, but $\mu(c)\neq\mu(b+d)$.

The following result from Adaricheva and Gorbunov [5] is actually weaker, but a good source of examples. Let $\mathbf{P}_5$ be the ordered set with $a < b < c > d > e$.

Theorem 10. *The (atomistic, lower bounded) lattice of convex subsets of a finite ordered set* $\mathbf{P}$ *is a Q-lattice if and only if* $\mathbf{P}$ *contains none of* $\mathbf{4}$, $\mathbf{2}^2$, $\mathbf{P}_5$, *and* $\mathbf{P}_5^d$.

3. Necessary conditions

Roughly speaking, most lattices don't support an equaclop. So we want to catalogue some of the necessary conditions for one to do so.

(A) Dziobiak's condition from [6] is still one of the most useful.

Lemma 11. *Let* $\mathbf{L}$ *be a finite lattice that admits an equalclop. If an element* u *is the join of* n *atoms of* $\mathbf{L}$, *then* $\downarrow u$ *contains at most* $2^n - 1$ *atoms.*

(B) The next condition is an easy consequence of the fact that if $\mathbf{L}$ admits an equaclop h, then $h \geq \mu$.

Theorem 12. *Let* $\mathbf{L}$ *be a finite lattice. If* $\mathbf{L}$ *contains elements* $x, y > 0$ *such that* $xy = 0$ *and* $\mu(x) = \mu(y)$, *then* $\mathbf{L}$ *does not admit an equaclop.*

Simple though it is, it covers a lot of examples.

(C) We can also work on the top end of the lattice.

Lemma 13. *Let* $\mathbf{L}$ *be a finite lattice that admits an equaclop* h, *and let* $a = m_h(1)$ *be the least element such that* $h(x) = 1$. *If* $x \in \mathbf{L}$ *and* $x \leq \sum U$ *is a m.n.t.j.c. and* $a \leq \sum U$, *then* $a \leq x$.

Proof. The condition implies that $h(x) \geq \mu(x) \geq a$, whence $x \geq a$. □

Corollary 14. *Let* $\mathbf{L}$ *be a finite lattice that admits an equaclop* h, *and let* $a = m_h(1)$. *If* $x \, D \, u \geq a$, *then* $x \geq a$.

Corollary 15. *Let* $\mathbf{L}$ *be a finite lattice that admits an equaclop* h, *and let* $a = m_h(1)$ *and* $x \in \mathbf{L}$. *If* $1 = \sum U$ *is a m.n.t.j.c. of* x, *then* $x \geq a$. *If also there exists* y *with* $\mu(y) \geq x$, *then* $xy \geq a$.

There are lots of examples; one should go here, say the standard lattice illustrating $y \, B \, x \, B \, r$.

(D) This condition is a bit esoteric, and as currently formulated applies only to finite atomistic lattices. It is offered as a new type of result to pursue.

Let $\mathbf{B}$ be the *bowtie* with a_1, $a_2 < b_1$, b_2. For $k \geq 2$ let $\mathbf{Y}_k$ be the ordered set with $a < b < c_j$ for $1 \leq j \leq k$.

Lemma 16. *Let* $\mathbf{L}$ *be a finite atomistic lattice. If either* $\mathbf{B}$ *or some* $\mathbf{Y}_k$ *(*$k \geq 2$*) is an order filter in* $(\mathrm{J}(\mathbf{L}), \overline{\mathrm{D}})$, *then* $\mathbf{L}$ *does not admit an equalclop.*

Proof. If $\mathbf{B}$ is a filter, then a_1, $a_2 \leq b_1 + b_2$ violates Dziobiak.

If $\mathbf{Y}_k$ is a filter, then $a \leq b + c_1$ say. Then $\mu(a) = \mu(b)$, and we can apply Theorem 12. □

4. The algorithm

In this section we build an algorithm that determines whether a given lattice has an equaclop. If it does the algorithm produces the minimal equaclop on the lattice. We start from the description of equaclops in terms of equapartitions suggested in [5].

Theorem 17. *Let* $\mathbf{L}$ *be a finite lattice, and let* $\mathrm{MD}(\mathbf{L})$ *be its semilattice of meet distributive elements. Then* $\mathbf{L}$ *admits an equaclop iff there exists a subset* $\mathbf{H} \subseteq \mathrm{MD}(\mathbf{L})$ *and elements* $b_h \in \mathbf{L}$ *for* $h \in \mathbf{H}$*, such that*

1. $0 \in \mathbf{H}$,
2. $b_h \leq h$ *for all* $h \in \mathbf{H}$,
3. $\mathbf{L} = \bigcup_{h \in \mathbf{H}} [b_h, h]$,
4. *if* $b_h \leq k$ *for some* $h, k \in \mathbf{H}$, *then* $h \leq k$.

If the conditions hold, the map $\eta(x) = h$ *if* $x \in [b_h, h]$ *is an equaclop.*

First note that condition (4) and symmetry imply that the intervals $[b_h, h]$ are pairwise disjoint, so the definition of η makes sense. The rest of the proof is entirely straightforward, just checking that (1)–(6) of the definition of an equaclop hold.

Here is the algorithm for determining whether a finite lattice admits an equalclop.

1. Find $\mathrm{MD}(\mathbf{L}) = \{m \in \mathbf{L} : \downarrow m \text{ is D-closed}\}$.
2. Order $\mathrm{MD}(\mathbf{L})$ with a decreasing linear extension
$$1 = m_0, m_1, \ldots, m_t = 0$$
so that $m_i \geq m_j$ implies $i \leq j$.
3. For each i, let $b_i = \prod\{x \in \mathbf{L} : \mu(x) = m_i\}$. (If $b_i = 0$ for any $i < t$, we could stop now as $\mathbf{L}$ does not admit by Lemma 12.)
4. Call the procedure Refine$(0, t, (m_0, b_0), \ldots, (m_t, b_t), admit)$.
5. If $admit = 0$ then $\mathbf{L}$ does not admit an equaclop. If $admit = 1$, then define $\eta(x) = m_i$ for $b_i \leq x \leq m_i$.

The main recursive procedure refines the original approximation $\mathbf{H} = \mathrm{MD}(\mathbf{L})$ and $\eta = \mu$ by removing elements and combining intervals to obtain a subset $\mathbf{H}'$ that satisfies the conditions of Theorem 17, if possible.

Procedure: Refine$(j, t, (m_0, b_0), \ldots, (m_t, b_t), admit)$.

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(1) If $j = t$ then $admit = 1$; return.
(2) If for all $s < t$, $b_j \leq m_s$ implies $m_j \leq m_s$, then call Refine$(j+1, t, (m_0, b_0), \ldots, (m_t, b_t), admit)$.
(3) Else choose $s$ minimal such that $b_j \leq m_s$ and $m_j \nleq m_s$.
(4) At this point we have $m_s \oplus m_j = m_r$ say, where $r \leq j$.
     (i) $b_r \leftarrow b_r b_s$.
    (ii) If $b_r = 0$ then return.
   (iii) Remove $(m_s, b_s)$, collapse the indexing, and $t \leftarrow (t-1)$.
(5) Call Refine$(r, t, (m_0, b_0), \ldots, (m_t, b_t), admit)$.
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We claim that the following conditions hold at the beginning of each pass ("call" statement) of the procedure.

1. $m_0 = 1$ and $m_t = 0$ and $b_i \leq m_i \in \mathrm{MD}(\mathbf{L})$ for all i.
2. $\mathbf{L} = \bigcup_{i \leq t}[b_i, m_i]$.
3. If $k < j$ and $b_k \leq m_\ell$, then $m_k \leq m_\ell$.
4. For any $h \in \mathrm{EC}(\mathbf{L})$, $h(b_i) \geq m_i$ for all i.

Moreover,

5. $j \leq t$, and after each pass, either j increases or t decreases.

These properties combine to show that the algorithm works.

Theorem 18. *Apply the preceding algorithm to a finite lattice* $\mathbf{L}$*.*

6. *If the procedure terminates with admit* $= 0$*, then* $\mathbf{L}$ *admits no equaclop.*
7. *If the procedure terminates with admit* $= 1$*, then* η *as defined in step 5 of the algorithm is the minimum equaclop on* $\mathbf{L}$*.*

We start the proof. Let us specify that a pass starts with the parameters

$$(j, t, (m_0, b_0), \ldots, (m_t, b_t), \mathit{admit})$$

which, after checking the first case we will assume satisfy (1)–(4).

(1) We start with $m_0 = 1$ and $m_t = 0$. Removals are made in step 5. It is straightforward to check that 1 and 0 are never removed (because we return whenever $b_r = 0$).

(2) This holds at the beginning, and we only remove intervals that are contained in retained ones.

(3) This holds vacuously at the first step. The procedure is called from steps 2 and 5. If the call is from step (2), it holds by induction and the condition of step (2). If the call is from (5), it holds by induction since $r \leq j$.

(4) The original definition of b_i ensures that this holds at the first step. Note that b_r is altered in step 4. Assume that $h \in \mathrm{EC}(\mathbf{L})$, $b_j \leq m_s$ and $m_s \oplus m_j = m_r$. Let b_r denote the old value, and $b_r' = b_r b_s$ the new value. Now $h(b_r) = h(m_r)$ by induction. Also $h(b_s) = h(m_s) \geq m_s + h(b_j) \geq m_s + m_j$, whence $h(b_s) \geq m_s \oplus m_j = m_r$ and $h(b_s) = h(m_r)$. Consequently $h(b_r') = h(b_r b_s) = h(m_r)$, as desired.

(5) is clear, and shows that the algorithm terminates.

The returns are from step 1 and 4. If it returns with $\mathit{admit} = 0$, then there is no equaclop by property (4). If it returns with $\mathit{admit} = 1$, then η is an equaclop by Theorem 17, and it is the minimum one by property (4).

5. Structural approach

It was shown in [1] that any finite lower bounded lattice is embedded into $\mathrm{Sub}_\wedge(\mathbf{P})$ for some finite (meet-) semilattice P. This embedding can always be assumed to be $(0,1)$-embedding. On the other hand, it is not atom-preserving.

Example 19.

The lb (and biatomic) lattice $\mathbf{Co}(\mathbf{P}_5)$ of convex subsets of partially ordered set $\mathbf{P}_5 = \{a, b, c, d, e\}$ with $a < b < c > d > e$ cannot be embedded into $\mathrm{Sub}_\wedge(\mathbf{P})$ atom-preservingly for any finite semilattice $\mathbf{P}$. It is also known from [5] that $\mathbf{Co}(\mathbf{P}_5)$ does not have an equaclop. These two observations about $\mathbf{Co}(\mathbf{P}_5)$ might be connected, but it is still vague.

Question 20. *Is it true that if a finite lb lattice can be embedded into* $\mathrm{Sub}_\wedge(\mathbf{P})$ *atom-preservingly then it has an equaclop? (the answer might be easy, but no counter-example comes to mind yet)*

We collect some easy facts about equaclops on sublattices of $\mathrm{Sub}_\wedge(\mathbf{P})$.

In the sequel it would be convenient to consider the semilattices with 1, so that the bottom element of $\mathrm{Sub}_\wedge(\mathbf{P})$ is $\{1\}$.

Let ϵ be a quasiorder on $\mathbf{P}$. We will call this quasiorder *distributive*, if $(a \wedge b)\epsilon c$ implies there are $a', b' \in \mathbf{P}$ such that $a\epsilon a', b\epsilon b'$ and $c = a' \wedge b'$. Denote by $\mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$ the lattice of ϵ-closed subsemilattices of P, i.e. such subsemilattices $X \in \mathrm{Sub}_\wedge(\mathbf{P})$ that $a\epsilon x$ and $a \in X$ implies $x \in X$. We will always assume that $\{1\}$ is ϵ-closed.

Lemma 21. *For any distributive quasiorder ϵ on $\mathbf{P}$,* $\mathrm{Sub}_\wedge(\mathbf{P}, \epsilon) \leq \mathrm{Sub}_\wedge(\mathbf{P})$.

Let us call an element $x \in P$ an ϵ-element, if $x\epsilon y$ implies $y \geq x$.

The following property of a quasiorder was introduced in [9].

We say that the quasiorder ϵ is *filterable*, if for any ϵ-element $x \in P$ it follows from $y > x$ and $y\epsilon z$ that $z \geq x$.

An example of filterable quasiorder is any quasiorder ϵ such that $a\epsilon b$ implies $b \geq a$.

Lemma 22. *Let ϵ be a distributive filterable quasiorder on $\mathbf{P}$. Then there is an equaclop on* $\mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$.

Proof. Let $X \in \mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$, then $x = \bigwedge X$ is an ϵ-element. It follows from the filterability of ϵ that $[X) \in \mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$, hence, for any $X \in \mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$ and for equaclop h_f on $\mathrm{Sub}_\wedge(\mathbf{P})$, we have $h_f(X) \in \mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$. Therefore, the restriction of h_f on $\mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$ provides an equaclop on $\mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$. □

Problem 23. *Can every finite lower bounded lattice with an equaclop be presented as* $\mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$ *for suitable $\mathbf{P}$ and distributive and filterable quasiorder ϵ on $\mathbf{P}$?*

As the first step toward such a description we present a construction of every lower bounded lattice in the form $\mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$, for some distributive quiasiorder ϵ on $\mathbf{P}$. This construction is due in part to construction of every lower bounded lattice given in [1] and in part to construction employing colored trees in [11]. As a matter a fact, our current construction borrows the definition of relation ϵ from [11], but uses it on original construction of [1], which seems to be simplier than colored trees.

Theorem 24. *For every finite lower bounded lattice L one can find a finite semilattice $\mathbf{P}$ and a distributive quasiorder ϵ on P such that $L = \mathrm{Sub}_\wedge(\mathbf{P}, \epsilon)$.*

Proof. We first remind the definition of P given in [1].

The semilattice $\mathbf{P} = \langle P, * \rangle$ is build on the set P of all sequences $\overline{d} = \langle d_1, \dots, d_n \rangle$ of join-irreducible elements $d_1, \dots, d_n \in J(L)$, $n \geq 1$, such that $d_i D d_{i+1}$ and d_1 is the minimal element with respect to relation D (i.e. there is no such $c \in J(L)$ that cDd_1). We also allow the empty sequence ι as a an element of P. The semilattice $\mathbf{P}$ is then defined as freely generated by its elements modulo the relations

$$\overline{d} = \iota * \overline{d}, \overline{d} = \overline{d} * \overline{f},$$

when $\overline{d}$ is an initial segment of $\overline{f}$, for non-empty sequences $\overline{d}$ and $\overline{f}$, and

$$\langle \overline{d} a \rangle = \Pi\{\langle \overline{d} a b_i \rangle : 1 \leq i \leq n\},$$

where $a \leq b_1 \vee \ldots \vee b_n$ is a nontrivial minimal join cover of a in L.

In particular, $\mathbf{P}$ has a top element $1_\mathbf{P}$ represented by the empty sequence ι.

It was proved in [1] that every element of $\mathbf{P}$ can uniquely be presented as "reduced" $*$-product of elements of P, meaning that in that form none of the relations above can be applied to the product.

For $a \in \mathbf{L}$, we let P_a be the set of all $\overline{d} \in P$ with $l(\overline{d}) \leq a$. Here $l(\overline{d})$ denotes the last term of $\overline{d}$. For the empty sequence ι we define $l(\iota) = 0_\mathbf{L}$. In particular, $1_\mathbf{P} \in P_a$ for every $a \in L$. It was proved in [1] that the mapping Φ from $\mathbf{L}$ that sends any element $a \in L$ to the $*$-subsemilattice of $\mathbf{P}$ generated by P_a, is an embedding of $\mathbf{L}$ into the lattice of subsemilattices (with $1_\mathbf{P}$) of $\mathbf{P}$.

Now we want to introduce the quiasiorder ϵ on $\mathbf{P}$.

Let $p = \overline{p_1} * \cdots * \overline{p_n}$ and $q = \overline{q_1} * \cdots * \overline{q_s}$ are elements of $\mathbf{P}$ in their canonical reduced form. Then we define $p \epsilon q$, if $l(q_1) \vee \ldots \vee l(q_s) \leq_\mathbf{L} l(p_1) \vee \ldots \vee l(p_n)$. In particular, $p \epsilon \iota$ for every $p \in \mathbf{P}$.

First, we prove that this quasiorder is distributive.

Suppose that $p * r \epsilon q$, where $p = \overline{p_1} * \cdots * \overline{p_n}$, $r = \overline{r_1} * \cdots * \overline{r_m}$ and $q = \overline{q_1} * \cdots * \overline{q_s}$ are canonical forms of elements p, r and q. We bring $p * r$ to the canonical form $\overline{u_1} * \cdots * \overline{u_k}$. It is straightforward to check that $l(\overline{u_1}) \vee \ldots \vee l(\overline{u_k}) \leq_\mathbf{L} S = l(\overline{p_1}) \vee \ldots \vee l(\overline{p_n}) \vee l(\overline{r_1}) \vee \ldots \vee l(\overline{r_m})$. Hence, $l(\overline{q_1}) \vee \ldots \vee l(\overline{q_s}) \leq_\mathbf{L} l(\overline{p_1}) \vee \ldots \vee l(\overline{p_n}) \vee l(\overline{r_1}) \vee \ldots \vee l(\overline{r_m})$.

Every $l(\overline{q_i})$ is either below some $l(\overline{p_j})$ or $l(\overline{r_k})$, or has a non-trivial minimal join cover $Q_i \ll S$. In the first case, $\overline{p_j} \epsilon \overline{q_i}$ or $\overline{r_k} \epsilon \overline{q_i}$. In the second case, we consider elements $\langle \overline{q_i} q \rangle$, for all $q \in Q_i$. By the definition, $\langle \overline{q_i} \rangle = \Pi\{\langle \overline{q_i} q_{i_j} \rangle : q_{i_j} \in Q_i\}$, and $p_i \epsilon \langle \overline{q_i} q \rangle$ or $r_k \epsilon \langle \overline{q_i} q \rangle$, for each $q \in Q_i$.

It follows that q can be presented as a product $\overline{v_1} * \cdots * \overline{v_d}$, where, for each $\overline{v_k}$, either $\overline{p_i} \epsilon \overline{v_k}$, for some $\overline{p_i}$, or $\overline{r_j} \epsilon \overline{v_k}$, for some $\overline{r_j}$. If for every k there is i such that $\overline{p_i} \epsilon \overline{v_k}$, then $p \epsilon q$ and $q = q * 1_\mathbf{P}$. Similarily, when for every k there is j such that $\overline{r_j} \epsilon \overline{v_k}$. Otherwise, we form the product of all $\overline{v_k}$ for which $\overline{p_i} \epsilon \overline{v_k}$, for some i, and call that element q_1, we form the product of all $\overline{v_k}$ for which $\overline{r_j}\epsilon\overline{v_k}$, for some j, and call that element q_2. We notice that $p\epsilon q_1$ and $r\epsilon q_2$, and also $q = q_1 * q_2$, what is required for distributivity.

We now want to show that $\mathbf{L}$ is isomorphic to $\mathrm{Sub}_\wedge(\mathbf{P},\epsilon)$. Evidently, $\Phi(a)$ is ϵ-closed subsemilattice, hence, we have an embedding of $\mathbf{L}$ into $\mathrm{Sub}_\wedge(\mathbf{P},\epsilon)$.

It is enough to show that Φ is onto. Take any $A \in \mathrm{Sub}_\wedge(\mathbf{P},\epsilon)$, and let $a = \bigvee\{l(\overline{p}) : \overline{p} \in A\}$. We notice that if $p = \overline{p_1} * \cdots * \overline{p_n}$ is in A, then $p\epsilon\overline{p_i}$, thus $\overline{p_i}$ are also in A, since A is ϵ-closed. It follows that, for every such element $p \in A$, we have $\bigvee l(\overline{p_i}) \leq_L a$. It remains to show that $A = \Phi(a)$. Evidently, $A \subseteq \Phi(a)$. Now, take any $q = \overline{q_1} * \cdots * \overline{q_n} \in \Phi(a)$ in its reduced form. For every $\overline{q_i}$ we have either $l(\overline{q_i}) \leq_{\mathbf{L}} l(\overline{p})$ for some $\overline{p} \in A$, or we can find for $l(\overline{q_i})$ a non-trivial minimal cover $Q \ll \bigvee\{l(\overline{p}) : \overline{p} \in A\}$. In the first case, $\overline{p}\epsilon q_i$, hence $\overline{q_i}$ must be in A. In the second case, $\overline{q_i} = \Pi\{\langle\overline{q_i}q\rangle : q \in Q\}$. Every $\langle\overline{q_i}q\rangle$ must be in A, since $\overline{p}\epsilon\langle\overline{q_i}q\rangle$ for some $\overline{p} \in A$, and $\overline{q_i}$ must be in A, since A is $*$-subsemilattice. Thus, every $\overline{q_i}$ is in A, hence $q \in A$, and we are done.

□

We want to look closer at the sublattices of $\mathrm{Sub}_\wedge(\mathbf{P})$ that might have an equaclop.

Lemma 25. *Let* $\mathbf{L} \leq \mathrm{Sub}_\wedge(\mathbf{P})$ *be a* $0,1$*-sublattice of* $\mathrm{Sub}_\wedge(\mathbf{P})$*. For any* $a \in \mathbf{L}$*, let* $h(a) = \Sigma\{x \in \mathbf{L} : x \leq [a)\}$*. Then* h *on* $\mathbf{L}$ *satisfies properties* $1-5$ *of equaclop.*

Proof. Need only to check 5.

If $h(a) = h(b)$, for some $a, b \in \mathbf{L}$, then, in particular, $\bigwedge_{\mathbf{P}} b = \bigwedge_{\mathbf{P}} h(b) = \bigwedge_{\mathbf{P}} h(a) = \bigwedge_{\mathbf{P}} a$. It follows that $\bigwedge_{\mathbf{P}} a = \bigwedge_{\mathbf{P}}(a \wedge b)$, thus $h(a \wedge_{\mathbf{L}} b) = h(a)$. □

Fix some $0,1$-embedding ϕ of lb lattice $\mathbf{L}$ into $\mathrm{Sub}_\wedge(\mathbf{P})$. For any $a \in \mathbf{L}$, we define $u_a^\phi = \bigwedge_{\mathbf{P}} \phi(a)$. For any $p \in \mathbf{P}$, let also define an element $m_p = \Pi_{\mathbf{L}}\{x \in L : p \in \phi(x)\}$. Note that m_p is defined for every $p \in \mathbf{P}$, when $\mathbf{L}$ is 1-sublattice of $\mathrm{Sub}_\wedge(\mathbf{P})$.

Consider the following property of ϕ for any $a \in \mathbf{L}$:

$$(\diamond) \qquad \text{If } p > u = u_a^\phi \text{ in } \mathbf{P} \text{ then } \phi(m_p) \leq \uparrow u (\in \mathrm{Sub}_\wedge(\mathbf{P})).$$

Lemma 26. *If* $\phi : L \to \mathrm{Sub}_\wedge(\mathbf{P})$ *is an embedding of a lower bounded lattice* $\mathbf{L}$ *into* $\mathrm{Sub}_\wedge(\mathbf{P})$ *that satisfies* $(\diamond)$ *for each* $a \in \mathbf{L}$*, then* h *on* $\mathbf{L}$ *defined in Lemma 25 is an equaclop.*

Proof. We need to check prooperty 6 of equaclop.

Let $A, V, W \in \phi(L)$ and $u = \bigwedge_P A$. If $t \in h(A)(W + V)$, then $t \in h(A)$ and $t = w \wedge v$, for some $w \in W, v \in V$. Since $u = u_A^\phi$ and $w, v \geq u$, we have according to $(\diamond)$ that $\phi(m_w), \phi(m_v) \leq [u)$. It follows from the definition of h that $\phi(m_w), \phi(m_v) \leq h(A)$. Hence, $w, v \in h(A)$ and $t \in h(A)W + h(A)V$. □

Example 27.

An embedding of $\mathbf{L} = \mathbf{Co}(\mathbf{P}_5)$ into $\mathrm{Sub}_\wedge(\mathbf{P})$ that uses algorithm from [1] does not satisfy ($\diamond$). Every atom of $\mathbf{Co}(\mathbf{P}_5)$, except c, is mapped to one-element sublattice of $\mathbf{P}$, and c is mapped to 3-element sublattice $\{c_1, c_2, c_3\}$, with $c_3 = c_1 \wedge c_2$. We also have $b = c_1 \wedge a$ and $d = c_2 \wedge d$ in $\mathbf{P}$.

Thus, we have b and d that are $\mathbf{L}$-elements, for which ($\diamond$) fails: $c_1 \geq b$ and $m_{c_1} = \{c_1, c_2, c_3\}$, thus $m_{c_1} \not\leq [b)$ (analogously, for d).

Question 28.

We would like to investigate the following: let $\mathbf{L}$ be a finite lb lattice with equaclop, and ϕ be some $0,1$-embedding into $\mathrm{Sub}_\wedge(\mathbf{P})$, where ($\diamond$) fails for some $a \in \mathbf{L}$. Can this embedding be fixed at a?

Namely, can we build an over-semilattice $\mathbf{P}_1 \geq \mathbf{P}$ such that $\mathbf{L}$ is still $0,1$-embedded into $\mathrm{Sub}_\wedge(\mathbf{P_1})$, and ($\diamond$) holds at $a \in \mathbf{L}$ under this new embedding?

The idea of Lemma 25 and consecutive question 28 is that any equaclop on lb lattice $\mathbf{L}$ could be presented in the overlattice $\mathrm{Sub}_\wedge(\mathbf{P_1})$ as a *filterable* equaclop, i.e. such ϕ that $\phi(x) \subseteq [x)$, for any $x \in \mathbf{L}$.

The natural first step would be to check the equaclops on $\mathrm{Sub}_\wedge(\mathbf{P})$. There was a long standing hypothesis that for every (strong) equaclop h on $\mathrm{Sub}_\wedge(\mathbf{P})$ one can find a semilattice P^* such that $\mathrm{Sub}_\wedge(\mathbf{P})$ is isomorphic to $\mathrm{Sub}_\wedge(\mathbf{P}^*)$ and h is filterable on $\mathrm{Sub}_\wedge(\mathbf{P}^*)$.

The following example shows that this is not true.

Example 29.

Let S be a semilattice on Fig.2. According to results of [2], if $\mathrm{Sub}_\wedge(\mathbf{P})$ isomorphic to $\mathrm{Sub}_\wedge(\mathbf{S})$ for some semilattice P, then P and S are isomorphic.

Define an equaclop on $\mathrm{Sub}_\wedge(\mathbf{S})$ as a minimal equaclop ϕ satisfying $b \in \phi(\{c_1, c_2\})$. It is straightforward to check that such an equaclop exists and, because of the required condition, it doesn't satisfy $\phi(X) \subseteq [X)$.

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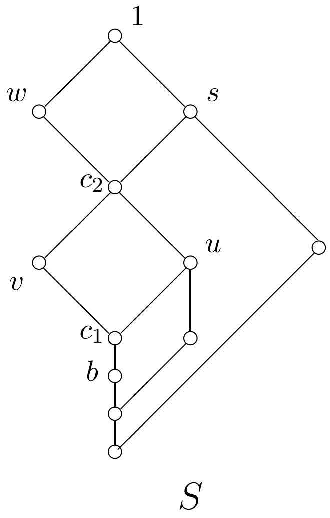

FIGURE 1

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