A COUNTEREXAMPLE TO THE FINITE HEIGHT CONJECTURE

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By Jónsson's Lemma, the variety $\mathcal{V}(\mathbf{K})$ generated by a finite lattice has only finitely many subvarieties. This led to the conjecture that, conversely, if a lattice variety has only finitely many subvarieties, then it is generated by a finite lattice. A stronger form of the conjecture states that a finitely generated variety $\mathcal{V}(\mathbf{K})$ has only finitely many covers in the lattice Λ of lattice varieties, and that each of these is also generated by a finite lattice. Initial investigations near the bottom of Λ supported these conjectures (see [2]–[15]). However, we will show that they are false by constructing an infinite subdirectly irreducible lattice $\mathbf{L}$ and a finite lattice $\mathbf{F}$ such that $\mathcal{V}(\mathbf{L}) \succ \mathcal{V}(\mathbf{F})$. A modification of this construction yields a finite lattice which generates a variety with infinitely many finitely generated covers.

The idea for this example goes back to a discussion with Ralph Freese for a joint paper with Ralph, Mick Adams and Jürg Schmid [1].

1. The Lattices in Question

Let $\mathbf{L}$ denote the lattice in Figure 1, as labeled. This lattice is most naturally pictured on the surface of a sphere. To simplify the diagram, we have duplicated the elements $\underline{m}$, $\underline{n}$, n_s $(s \in \mathbb{Z})$, and $\overline{n}$. These elements, indicated by a solid circle, appear on both the left and right sides of the diagram, and the two copies of each element should be identified. The same technique has been used for all the figures in this paper.

The lattice $\mathbf{L}$ consists of 14 chains:

$$A = \{\underline{a}\} \cup \{a_s : s \in \mathbb{Z}\} \cup \{\overline{a}\}$$

through

$$N = \{\underline{n}\} \cup \{n_s : s \in \mathbb{Z}\} \cup \{\overline{n}\}$$

with 8 additional elements on top: $\overline{bd}, \overline{df}, \overline{fh}, \overline{hj}, \overline{jl}, \overline{ln}, \overline{nb}, 1$ and 8 on the bottom: $\underline{ac}, \underline{ce}, \underline{eg}, \underline{gi}, \underline{ik}, \underline{km}, \underline{ma}, 0$. The order is as indicated in the diagram; in particular,

$$a_s \leq b_s \geq c_s \leq d_s \geq e_s \leq f_s \geq g_s \leq h_s \geq i_s \leq j_s \geq k_s \leq l_s \geq m_s \leq n_s \geq a_{s+1}.$$

The first thing one must do is check that this is a lattice. (If it were any narrower it wouldn't be.) Note that there is a cyclic automorphism of $\mathbf{L}$ with $a_s \mapsto c_s \mapsto e_s \mapsto \cdots \mapsto m_s \mapsto a_s$ and $b_s \mapsto d_s \mapsto f_s \mapsto \cdots \mapsto n_s \mapsto b_s$.

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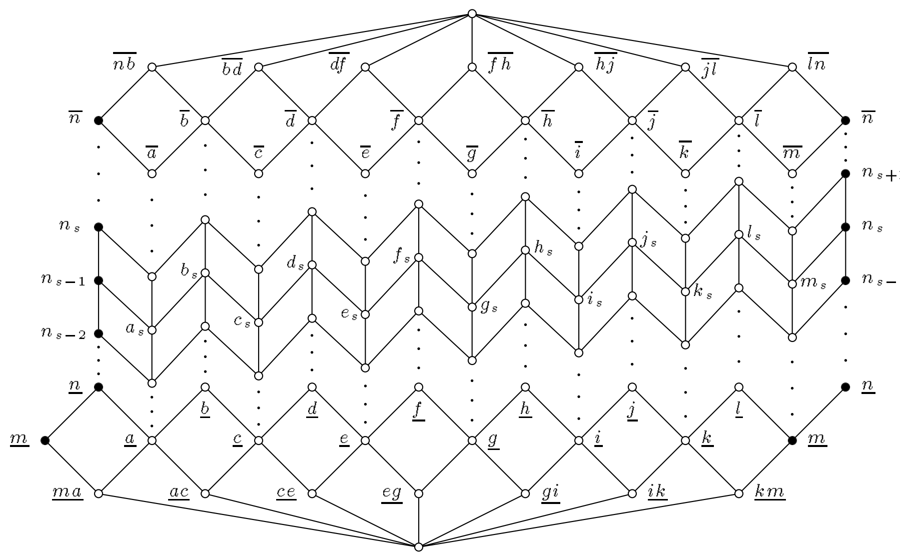

FIGURE 1. $\mathbf{L}$.

Next one checks that it is subdirectly irreducible: the monolith μ of $\mathbf{L}$ collapses each of the sets $\{a_s : s \in \mathbb{Z}\}, \dots, \{n_s : s \in \mathbb{Z}\}$ to a single point. Thus $\mathbf{L}/\mu$ is given in Figure 2, and is easily seen to be a subdirect product of two copies of the lattice $\mathbf{F}$ in Figure 3.

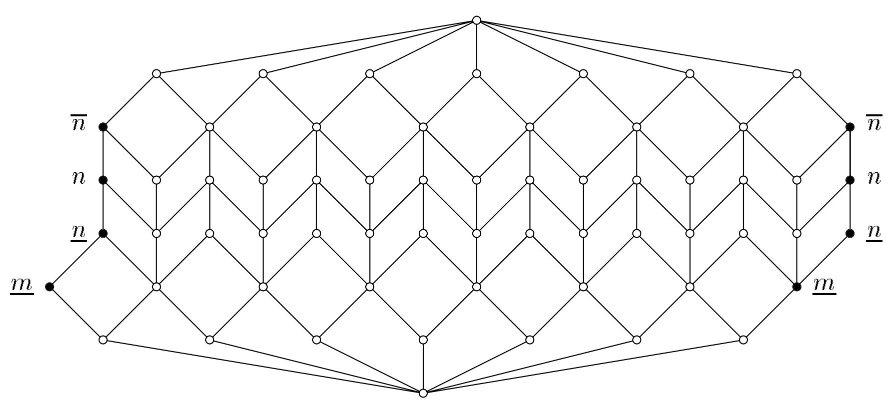

FIGURE 2. $\mathbf{L}/\mu$.

Note that $\mathbf{L}$ has a maximal sublattice $\mathbf{T}$ consisting of

$$T = \{0, \underline{ma}, \underline{ac}, \underline{ce}, \underline{eg}, \underline{gi}, \underline{ik}, \underline{km}, \underline{a}, \underline{b}, \underline{c}, \underline{d}, \underline{e}, \underline{f}, \underline{g}, \underline{h}, \underline{i}, \underline{j}, \underline{k}, \underline{l}, \underline{m}, \underline{n}, \overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, \overline{g}, \overline{h}, \overline{i}, \overline{j}, \overline{k}, \overline{l}, \overline{m}, \overline{n}, \overline{bd}, \overline{df}, \overline{fh}, \overline{hj}, \overline{jl}, \overline{ln}, \overline{nb}, 1\}.$$

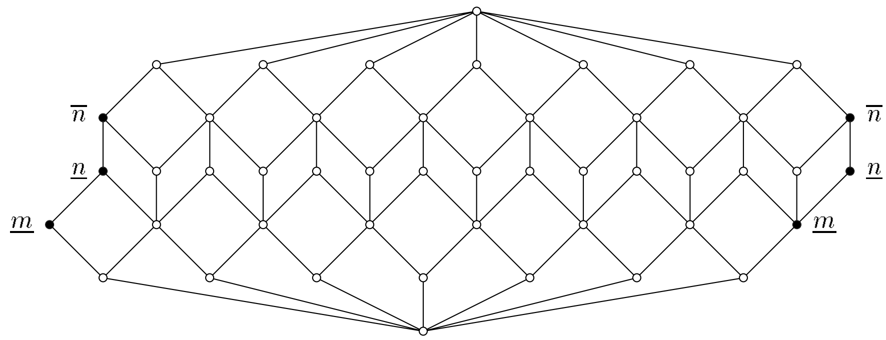

FIGURE 3. **F**.

2. GETTING AROUND ULTRAPRODUCTS

First we reduce to finitely generated lattices.

Lemma 1. *If $\mathcal{U}$ and $\mathcal{V}$ are lattice varieties with $\mathcal{U} < \mathcal{V}$, then there is a finitely generated, subdirectly irreducible lattice in $\mathcal{V} - \mathcal{U}$.*

Proof. Let $p \approx q$ be an equation which holds in $\mathcal{U}$ and fails in $\mathcal{V}$. Then $p \approx q$ fails in some finitely generated lattice $\mathbf{M} \in \mathcal{V}$, and hence in some subdirectly irreducible factor of $\mathbf{M}$. □

Next we avoid ultraproducts.

Lemma 2. *Let $\mathbf{L}$ be the lattice in Figure 1. Then every finitely generated, subdirectly irreducible lattice in $\mathcal{V}(\mathbf{L})$ is in $\mathrm{HS}(\mathbf{L})$.*

In fact, every subdirectly irreducible lattice in $\mathcal{V}(\mathbf{L})$ is in $\mathrm{HS}(\mathbf{L})$. The proof is a bit more technical, and Lemma 1 renders it unnecessary.

Proof. Assume $\mathbf{K}$ is a finitely generated subdirectly irreducible lattice in $\mathcal{V}(\mathbf{L})$, whence $\mathbf{K} \in \mathrm{HSP_u}(\mathbf{L})$, say

$$\mathbf{Q} = \prod \mathbf{L}/U$$
$$\mathbf{S} \leq \mathbf{Q}$$
$$\mathbf{K} \cong \mathbf{S}/\kappa.$$

Now the ultrapower $\mathbf{Q}$ can be described as follows. At the top and bottom it has a sublattice isomorphic to $\mathbf{T}$. In between, it is the union of 14 chains $\mathcal{A}$ through $\mathcal{N}$, each of which is a union of copies of $\mathbb{Z}$. For some totally ordered index set Σ we can write

$$\mathcal{A} = \{a_s^\sigma : \sigma \in \Sigma,\ s \in \mathbb{Z}\}$$

and similarly for $\mathcal{B}$ through $\mathcal{N}$. These are ordered by $a_s^\sigma \leq a_t^\tau$ if $\sigma < \tau$, or if $\sigma = \tau$ and $s \leq t$. Of course we also have

$$a_s^\sigma \leq b_s^\sigma \geq c_s^\sigma \leq d_s^\sigma \geq e_s^\sigma \leq f_s^\sigma \geq g_s^\sigma \leq h_s^\sigma$$
$$\geq i_s^\sigma \leq j_s^\sigma \geq k_s^\sigma \leq l_s^\sigma \geq m_s^\sigma \leq n_s^\sigma \geq a_{s+1}^\sigma.$$

The top: $\{\overline{a}, \dots, \overline{n}, \overline{bd}, \dots, \overline{nb}, 1\}$, the bottom: $\{0, \underline{ac}, \dots, \underline{ma}, \underline{a}, \dots, \underline{n}\}$, and the sets $Q^\sigma = \{x_s^\sigma : x = a, \dots, n \text{ and } s \in \mathbb{Z}\}$ form the connected components of $\mathbf{Q}$.

Observe that for any $\sigma_1, \dots, \sigma_p \in \Sigma$, the set $\mathbf{T} \cup Q^{\sigma_1} \cup \dots \cup Q^{\sigma_p}$ is a sublattice of $\mathbf{Q}$. For if $x \in Q^\sigma$ and $y \in Q^\tau$, then $x \vee y \in Q^{\sigma \vee \tau} \cup \mathbf{T}$ and $x \wedge y \in Q^{\sigma \wedge \tau} \cup \mathbf{T}$. Every finitely generated sublattice of $\mathbf{Q}$ is contained in a sublattice of this type.

For each $\sigma \in \Sigma$, define a congruence relation ρ^σ on $\mathbf{Q}$ as follows. This ρ^σ collapses a_s^τ to $\overline{a}$ if $\tau > \sigma$, a_s^τ to $\underline{a}$ if $\tau < \sigma$, and similarly for the chains $\mathcal{B}$ through $\mathcal{N}$. It is the identity of Q^σ and the rest of $\mathbf{T}$ (the elements $0, \underline{ma}, \dots, \underline{km}, \overline{bd}, \dots, \overline{nb}, 1$). One needs to check that ρ^σ is a congruence and that $\mathbf{Q}/\rho^\sigma \cong \mathbf{L}$.

Now let ξ^σ be the restriction of ρ^σ to $\mathbf{S}$. Since $\mathbf{S}$ is finitely generated, by the observation above it is contained in a sublattice $\mathbf{T} \cup Q^{\sigma_1} \cup \dots \cup Q^{\sigma_p}$, and for these (finitely many) ξ^σ's we have $\xi^{\sigma_1} \cap \dots \cap \xi^{\sigma_p} = 0$ in Con $\mathbf{S}$. But κ is (finitely) meet prime in Con $\mathbf{S}$, and hence $\kappa \geq \xi^{\sigma_r}$ for some r. Thus $\mathbf{K} \cong \mathbf{S}/\kappa \in \mathrm{HS}(\mathbf{L})$. □

So we are reduced to considering subdirectly irreducible lattices $\mathbf{K} \in \mathrm{HS}(\mathbf{L})$. Because $\mathbf{L}$ is finitely generated, every proper sublattice is contained in a maximal one. As $\mathbf{L}/\mu \in \mathcal{V}(\mathbf{F})$, if we can also show that every maximal sublattice of $\mathbf{L}$ is in $\mathcal{V}(\mathbf{F})$, then we will have $\mathcal{V}(\mathbf{L}) \succ \mathcal{V}(F)$, as desired.

3. Maximal Sublattices

We have seen that $\mathbf{T}$ is a maximal sublattice of $\mathbf{L}$. Let us show that there are exactly 14 more:

$$\mathbf{S}_1 = \mathbf{L} - (A \cup B \cup \{\overline{bd}, \underline{ma}\})$$
$$\mathbf{S}_2 = \mathbf{L} - (D \cup E \cup \{\overline{bd}, \underline{eg}\})$$
$$\mathbf{S}_3 = \mathbf{L} - (C \cup D \cup \{\overline{df}, \underline{ac}\})$$
$$\mathbf{S}_4 = \mathbf{L} - (F \cup G \cup \{\overline{df}, \underline{gi}\})$$
$$\mathbf{S}_5 = \mathbf{L} - (E \cup F \cup \{\overline{fh}, \underline{ce}\})$$
$$\mathbf{S}_6 = \mathbf{L} - (H \cup I \cup \{\overline{fh}, \underline{ik}\})$$
$$\mathbf{S}_7 = \mathbf{L} - (G \cup H \cup \{\overline{hj}, \underline{eg}\})$$
$$\mathbf{S}_8 = \mathbf{L} - (J \cup K \cup \{\overline{hj}, \underline{km}\})$$
$$\mathbf{S}_9 = \mathbf{L} - (I \cup J \cup \{\overline{jl}, \underline{gi}\})$$
$$\mathbf{S}_{10} = \mathbf{L} - (L \cup M \cup \{\overline{jl}, \underline{ma}\})$$
$$\mathbf{S}_{11} = \mathbf{L} - (K \cup L \cup \{\overline{ln}, \underline{ik}\})$$
$$\mathbf{S}_{12} = \mathbf{L} - (N \cup A \cup \{\overline{ln}, \underline{ac}\})$$

$$\mathbf{S}_{13} = \mathbf{L} - (M \cup N \cup \{\overline{nb}, \underline{km}\})$$
$$\mathbf{S}_{14} = \mathbf{L} - (B \cup C \cup \{\overline{nb}, \underline{ce}\}).$$

Each of these sublattices is of the form $\mathbf{S} = \mathbf{L} - [p, q]$ where q is a coatom of $\mathbf{L}$ and p is one of the pair of atoms joining to q. One must check that these are indeed sublattices of $\mathbf{L}$, and that each one is a subdirect product of infinitely many copies of the corresponding sublattice of $\mathbf{F}$. There are up to automorphism two cases to consider, represented by say $\mathbf{S}_1$ and $\mathbf{S}_2$.

Corresponding to $\mathbf{S}_1$ we have the sublattice $\mathbf{R}_1 = \mathbf{F} - \{\overline{a}, \underline{a}, \overline{b}, \underline{b}, \overline{bd}, \underline{ma}\}$ of $\mathbf{F}$. For any $s \in \mathbb{Z}$, there is a congruence ζ_s on $\mathbf{S}_1$ which for $z \in \{c, \dots, n\}$ collapses z_t to $\overline{z}$ if $t > s$, collapses z_t to $\underline{z}$ if $t \leq s$, and otherwise is the identity on $\mathbf{S}_1$. Then $\mathbf{S}_1/\zeta_s \cong \mathbf{R}_1$ for every s, and $\bigcap_{s\in\mathbb{Z}} \zeta_s = 0$ in Con $\mathbf{S}_1$.

Likewise, $\mathbf{F}$ has a sublattice $\mathbf{R}_2 = \mathbf{F} - \{\overline{d}, \underline{d}, \overline{e}, \underline{e}, \overline{bd}, \underline{eg}\}$ corresponding to $\mathbf{S}_2$. For any $s \in \mathbb{Z}$, there is a congruence η_s on $\mathbf{S}_2$ which is maximal such that it separates the prime quotients projective to $[f_s, f_{s+1}]$ in $\mathbf{S}_2$. For $z \in \{f, \dots, n\}$, η_s collapses z_t to $\overline{z}$ if $t > s$, collapses z_t to $\underline{z}$ if $t \leq s$, while an element x_t with $x \in \{a, b, c\}$ is collapsed to $\overline{x}$ if $t > s+1$, and to $\underline{x}$ if $t \leq s+1$. Otherwise η_s is the identity on $\mathbf{S}_2$. As before, $\mathbf{S}_2/\eta_s \cong \mathbf{R}_2$ for every s, and $\bigcap_{s\in\mathbb{Z}} \eta_s = 0$ in Con $\mathbf{S}_2$.

The next lemma shows that each sublattice $\mathbf{S}_r$ is maximal, and that $\mathbf{T}$ is the only other maximal sublattice of $\mathbf{L}$.

Lemma 3. *If $\mathbf{U}$ is a proper sublattice of $\mathbf{L}$ and $\mathbf{U} \not\subseteq \mathbf{T}$, then $\mathbf{U} \subseteq \mathbf{S}_r$ for some r.*

Proof. As $\mathbf{U} \not\subseteq \mathbf{T}$, we have $x_s \in \mathbf{I}$ for some $x \in \{a, \dots, n\}$. Now x_s and $\mathbf{T}$ generate $\mathbf{L}$, and the coatoms of $\mathbf{L}$ generate $\mathbf{T}$. Hence $q \notin \mathbf{U}$ for some coatom q. There is a unique pair of atoms such that $p \vee p' = q$, and we must have $\mathbf{U} \cap [p, q] = \emptyset$ or $\mathbf{U} \cap [p', q] = \emptyset$. Thus $\mathbf{U} \subseteq \mathbf{L} - [p, q]$ or $\mathbf{U} \subseteq \mathbf{L} - [p', q]$. □

Therefore $\mathbf{L}$ has exactly 15 maximal sublattices, all of which are in $\mathcal{V}(\mathbf{F})$, which is what we set out to prove.

Theorem 4. $\mathcal{V}(\mathbf{L}) \succ \mathcal{V}(\mathbf{F})$.

4. Infinitely Many Finitely Generated Covers

In this section we will describe a collection of finite, subdirectly irreducible lattices $\mathbf{L}_z$ $(1 \leq z < \omega)$ such that every variety $\mathcal{V}(\mathbf{L}_z)$ covers the same finitely generated variety $\mathcal{V}(\mathbf{G})$. Recall that the variety generated by a finite, subdirectly irreducible lattice is join irreducible in Λ. Rather than describe the lattice $\mathbf{G}$ directly, we will prove that for each z, $\mathrm{HS}(\mathbf{L}_z) - \{\mathbf{L}_z\}$ contains the same subdirectly irreducible lattices. This implicitly yields $\mathbf{G}$ as a direct product of 37 lattices.

Each $\mathbf{L}_z$ $(z \geq 1)$ consists of 14 chains:

$$A = \{\underline{a}\} \cup \{a_s : 0 \leq s < z\} \cup \{\overline{a}\}$$

through

$$N = \{\underline{n}\} \cup \{n_s : 0 \le s < z\} \cup \{\overline{n}\}$$

and the 16 additional elements $\overline{bd}$, $\overline{df}$, $\overline{fh}$, $\overline{hj}$, $\overline{jl}$, $\overline{ln}$, $\overline{nb}$, 1, $\underline{ac}$, $\underline{ce}$, $\underline{eg}$, $\underline{gi}$, $\underline{ik}$, $\underline{km}$, $\underline{ma}$, 0. The top and bottom are ordered as before, and in the middle we have

$$\begin{aligned} \underline{n} \ge\, & a_0 \le b_0 \ge c_0 \le d_0 \ge e_0 \le f_0 \ge g_0 \le h_0 \ge i_0 \le j_0 \ge k_0 \le l_0 \ge m_0 \le n_0 \\ \ge\, & a_1 \cdots \le h_{z-1} \ge i_{z-1} \le j_{z-1} \ge k_{z-1} \le l_{z-1} \ge m_{z-1} \le n_{z-1} \ge \overline{a}. \end{aligned}$$

The lattice $\mathbf{L}_2$ is drawn in Figure 4. Note that $\mathbf{L}_z$ is not cyclically symmetric: the end chains A and N play a special role.

Again one must check that $\mathbf{L}_z$ is a subdirectly irreducible lattice. If μ_z denotes the monolith of $\mathbf{L}_z$, then $\mathbf{L}_z/\mu_z$ is isomorphic to the (simple) lattice $\mathbf{G}_1$ of Figure 5.

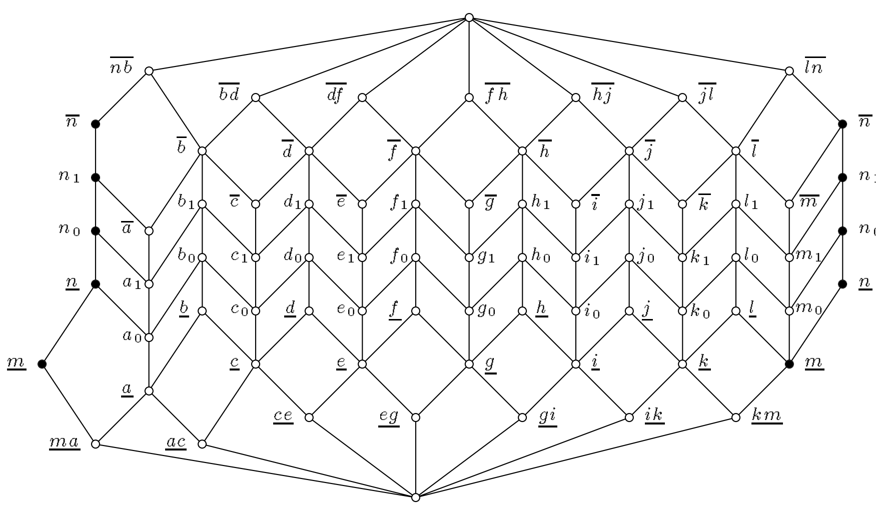

FIGURE 4. **L**2.

We again identify the subset T_z of $\mathbf{L}_z$ consisting of

$$\begin{aligned} T_z = \{0, & \underline{ma}, \underline{ac}, \underline{ce}, \underline{eg}, \underline{gi}, \underline{ik}, \underline{km}, \underline{a}, \underline{b}, \underline{c}, \underline{d}, \underline{e}, \underline{f}, \underline{g}, \underline{h}, \underline{i}, \underline{j}, \underline{k}, \underline{l}, \underline{m}, \underline{n}, \\ & \overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, \overline{g}, \overline{h}, \overline{i}, \overline{j}, \overline{k}, \overline{l}, \overline{m}, \overline{n}, \overline{bd}, \overline{df}, \overline{fh}, \overline{hj}, \overline{jl}, \overline{ln}, \overline{nb}, 1\}. \end{aligned}$$

However, T_z is not a sublattice; indeed, each $\mathbf{L}_z$ is generated by its coatoms.

Every $\mathbf{L}_z$ has 14 maximal sublattices:

$$\mathbf{S}_{1z} = \mathbf{L}_z - (A \cup B \cup \{\overline{bd}, \underline{ma}\})$$

through

$$\mathbf{S}_{14z} = \mathbf{L}_z - (B \cup C \cup \{\overline{nb}, \underline{ce}\})$$

exactly as before. Moreover, the next lemma shows that these are all the maximal sublattices of $\mathbf{L}_z$.

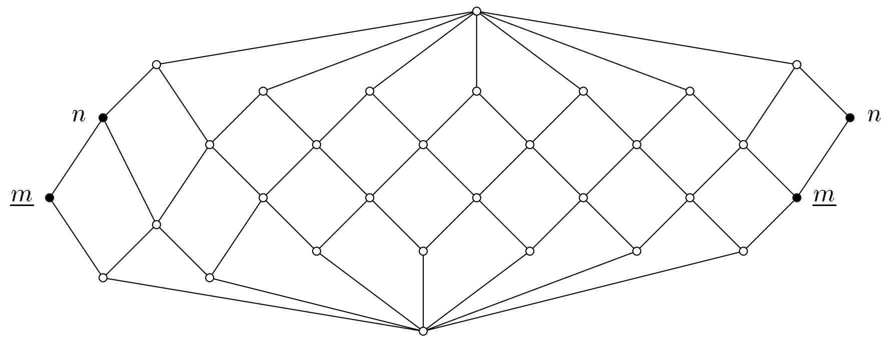

FIGURE 5. $\mathbf{G}_1$.

Lemma 5. *If* $\mathbf{U}$ *is a proper sublattice of* $\mathbf{L}_z$*, then* $\mathbf{U} \subseteq \mathbf{S}_{tz}$ *for some* t *with* $1 \le t \le 14$.

Proof. Since $\mathbf{L}_z$ is generated by its coatoms, there must be a coatom q with $q \notin \mathbf{U}$. Again q is the join of two atoms, $q = p \vee p'$, so we must have $\mathbf{U} \cap [p, q] = \emptyset$ or $\mathbf{U} \cap [p', q] = \emptyset$. Hence $\mathbf{U} \subseteq \mathbf{S}_{tz}$ for some t. □

This brings us to the major difference between this case and the infinite one, *viz.*, describing the subdirect decomposition of the maximal sublattices of $\mathbf{L}_z$.

First, let us describe a decomposition of $\mathbf{S}_{tz} = \mathbf{L}_z - (X \cup Y \cup \{p, q\})$ with $XY \ne AB$, MN, NA (so $t \ne 1$, 12, 13). We claim that each such sublattice is a subdirect product of 3 different lattices $\mathbf{G}_r$, $\mathbf{G}_s$, $\mathbf{G}_u$ depending only on the choice of X, Y, p, q (and not z). As there are 11 choices of t subject to the above restrictions, this gives 33 more lattices $\mathbf{G}_2, \dots, \mathbf{G}_{34}$ in $\mathcal{V}(\mathbf{L}_z)$, besides $\mathbf{G}_1$. (There will be more from the other cases, and we do not need to know whether they are pairwise nonisomorphic.)

Notice that the prime quotients of the chains A through N (omitting some consecutive pair XY) of any $\mathbf{S}_{tz}$ ($t \ne 1$, 12, 13) fall into $z + 2$ projectivity classes of three different types: one beginning with $[\underline{a}, a_0]$, one ending with $[n_{z-1}, \overline{n}]$, and z in the middle which cycle around. We will define α so that it is the largest congruence on $\mathbf{S}_{tz}$ separating $\underline{a}$ and a_0. Similarly, β_w for $0 \le w < z$ will be the largest congruence not collapsing $[a_w, a_{w+1}]$ (for $w = z - 1$, $[a_{z-1}, \overline{a}]$), and γ will be the largest congruence separating n_{z-1} and $\overline{n}$. These congruences will give a subdirectly decomposition of $\mathbf{S}_{tz}$ into subdirectly irreducible lattices of three different isomorphism types $\mathbf{S}/\alpha$, $\mathbf{S}/\beta$ and $\mathbf{S}/\gamma$ (not depending on z).

A typical example illustrates this perhaps better than a general explanation. Consider $\mathbf{S} = \mathbf{S}_{7z} = \mathbf{L}_z - (G \cup H \cup \{\overline{hj}, \underline{eg}\})$. The 3 different types of congruence relations on $\mathbf{S}_{7z}$ are as follows. The congruence α collapses the intervals $[a_0, \overline{a}], \dots, [f_0, \overline{f}]$ and $[\underline{i}, \overline{i}], \dots, [\underline{n}, \overline{n}]$, and is the identity otherwise. For $0 \le w < z$, the congruence β_w collapses the intervals $[\underline{a}, a_w], \dots, [\underline{f}, f_w]$ and and $[i_w, \overline{i}], \dots, [n_w, \overline{n}]$ and, if $w + 1 < z$, $[a_{w+1}, \overline{a}], \dots, [f_{w+1}, \overline{f}]$ and, if $w-1 \geq 0$, $[\underline{i}, i_{w-1}], \ldots, [\underline{n}, n_{w-1}]$. The congruence γ collapses the intervals $[\underline{a}, \overline{a}], \ldots, [\underline{f}, \overline{f}]$ and $[\underline{i}, i_{z-1}], \ldots, [\underline{n}, n_{z-1}]$. One must check the following facts:

1. α, β_w, γ are congruence relations on $\mathbf{S}$,
2. $\alpha \wedge \gamma \wedge \bigwedge_{w<z} \beta_w = 0$ in Con $\mathbf{S}$,
3. $\mathbf{S}/\beta_w \cong \mathbf{S}/\beta_u$ for $1 \leq w \leq u < z$.

Thus $\mathbf{S}$ is a subdirect product of $\mathbf{G}_r = \mathbf{S}/\alpha$, $\mathbf{G}_s = \mathbf{S}/\beta_w$ (for any w), and $\mathbf{G}_u = \mathbf{S}/\gamma$.

Each of the remaining sublattices $\mathbf{S}_{1z}$, $\mathbf{S}_{12z}$ and $\mathbf{S}_{13z}$ is a subdirect power of a single lattice. There are $z+1$ projectivity classes of prime quotients in the chains of these sublattices: the one containing $[\underline{c}, c_0]$, the ones with $[c_w, c_{w+1}]$ for $0 \leq w < z-1$, and that containing $[c_{z-1}, \overline{c}]$. Again α, β_w and γ will be the maximal congruence relations on $\mathbf{S}$ separating these, and again they will provide a subdirect decomposition. However, this time all the quotient lattices will be isomorphic.

For example, on $\mathbf{S}_{1z}$ with $z > 1$ we have the congruence α which collapses $[c_0, \overline{c}], \ldots, [n_0, \overline{n}]$, the congruences β_w for $0 \leq w < z-1$ collapsing $[\underline{c}, c_w], \ldots, [\underline{n}, n_w]$ and $[c_{w+1}, \overline{c}], \ldots, [n_{w+1}, \overline{n}]$, and the congruence γ collapsing $[\underline{c}, c_{z-1}], \ldots, [\underline{n}, n_{z-1}]$. If $z = 1$, then only α and γ are defined. Again one must check that these are indeed congruence relations, and that $\alpha \wedge \gamma \wedge \bigwedge_{w<z-1} \beta_w = 0$ in Con $\mathbf{S}_{1z}$. Moreover, $\mathbf{S}_{1z}/\alpha \cong \mathbf{S}_{1z}/\beta_w \cong \mathbf{S}_{1z}/\gamma \cong \mathbf{G}_{35}$ say, independently of z and w. The case of $\mathbf{S}_{13z}$ is dual, and that of $\mathbf{S}_{12z}$ is similar. These cases account for three additional lattices $\mathbf{G}_{35}, \mathbf{G}_{36}, \mathbf{G}_{37}$ in $\mathcal{V}(\mathbf{L}_z)$.

Now let $\mathbf{G} = \prod_{1 \leq r \leq 37} \mathbf{G}_r$. We have shown that, for every $z \geq 1$,

1. $\mathbf{L}_z$ is subdirectly irreducible, so $\mathbf{L}_z \notin \mathcal{V}(\mathbf{G})$ by Jónsson's Lemma,
2. $\mathbf{G}_1, \ldots, \mathbf{G}_{37} \in \mathcal{V}(\mathbf{L}_z)$ and hence $\mathcal{V}(\mathbf{G}) < \mathcal{V}(\mathbf{L}_z)$,
3. $\mathbf{L}_z/\mu_z \cong \mathbf{G}_1 \in \mathcal{V}(\mathbf{G})$ and every maximal sublattice $\mathbf{S}_{tz}$ of $\mathbf{L}_z$ is in $\mathcal{V}(\mathbf{G})$.

This establishes the desired result.

Theorem 6. $\mathcal{V}(\mathbf{L}_z) \succ \mathcal{V}(\mathbf{G})$ *whenever* $1 \leq z < \omega$.

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