A CLASS OF INFINITE CONVEX GEOMETRIES

KIRA ADARICHEVA AND J. B. NATION

Abstract. Various characterizations of finite convex geometries are well known. This note provides similar characterizations for possibly infinite convex geometries whose lattice of closed sets is strongly coatomic and lower continuous. Some classes of examples of such convex geometries are given.

1. Introduction

There are various ways to characterize finite convex geometries; see Chapter 3 of our [4], which combines results from Dilworth [10], Avann [6], Edelman and Jamison [12], Duquenne [11], and Monjardet [16]. These characterizations can be either combinatorial or lattice theoretical in nature.

Infinite versions of convex geometries occur in several sources, each of which is equivalent to a closure operator with the anti-exchange property, plus some finiteness conditions to provide structure. Crawley and Dilworth [8] consider dually algebraic, strongly coatomic, locally distributive lattices. Adaricheva, Gorbunov and Tumanov [3] discuss closure operators with the anti-exchange property whose closure lattices are weakly atomic, dually spatial and atomistic. Adaricheva and Nation [4] are concerned with algebraic closure operators with the anti-exchange property. See also Sakaki [18], Adaricheva and Pouzet [5], and Adaricheva [2].

Here we consider a fourth type, inspired by Gorbunov’s classic result [13]: Every element of a complete lattice $L$ has a canonical join decomposition if and only if $L$ is strongly coatomic, lower continuous, and join semidistributive. Adding lower semimodularity to the list will give us an infinite class of convex geometries, with unique irredundant join decompositions; see Theorem 7.

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2. The anti-exchange property

Definition 1. A closure system \((X, \gamma)\) satisfies the anti-exchange property if for all \(x \neq y\) and all closed sets \(A \subseteq X\),

\((\text{AEP})\quad x \in \gamma(A \cup \{y\})\) and \(x \not\in A\) imply that \(y \notin \gamma(A \cup \{x\})\).

Equivalently, a closure operator satisfies the anti-exchange property if for all closed sets \(A \subseteq X\) and elements \(x, y \notin A\), if \(\gamma(A \cup \{x\}) = \gamma(A \cup \{y\})\) then \(x = y\).

Examples of closure operators with the anti-exchange property include

- the convex hull operator on Euclidean space \(\mathbb{E}^n\),
- the convex hull operator on an ordered set,
- the subalgebra-generated-by operator on a semilattice,
- the algebraic-subset-generated-by operator on a complete lattice.

For a closure system \((X, \gamma)\), we will let \(\text{Cld}(X, \gamma)\) denote the lattice of \(\gamma\)-closed subsets of \(X\). A closure system is zero-closure if \(\gamma(\emptyset) = \emptyset\).

Definition 2. A zero-closure system that satisfies the anti-exchange property is called a convex geometry.

(This common convention is a bit awkward, as some useful closure operators with the anti-exchange property have a non-empty closure of the empty set. Nonetheless, we shall retain it.)

A lattice is strongly coatomic if \(a < c\) in \(L\) implies that there exists \(b\) such that \(a \leq b < c\). A closure system is strongly coatomic if its lattice of closed sets is so.

Theorem 3. For a strongly coatomic closure system \((X, \gamma)\), the following are equivalent.

1. \((X, \gamma)\) has the anti-exchange property.
2. If \(A < B\) in \(\text{Cld}(X, \gamma)\), then \(|B \setminus A| = 1|\).

Proof. Assume that \((X, \gamma)\) has the AEP. If \(A < B\) in \(\text{Cld}(X, \gamma)\) and \(x, y \in B \setminus A\), then \(\gamma(A \cup \{x\}) = B = \gamma(A \cup \{y\})\), whence \(x = y\) by the AEP.

Suppose that \((X, \gamma)\) satisfies (2). Assume that \(B = \gamma(A \cup \{x\}) = \gamma(A \cup \{y\}) > A = \gamma(A)\). As \(\text{Cld}(X, \gamma)\) is strongly coatomic, there is a closed set \(A'\) such that \(A \leq A' < B\). Then \(B = \gamma(A' \cup \{x\}) = \gamma(A' \cup \{y\})\), so \(x, y \in B \setminus A'\). By (2) we have \(x = y\), as desired. \(\square\)

The equivalence of the preceding theorem is also valid for algebraic closure systems; see [2] and [4].
3. SCLCC Lattices

A subset \( D \) of a lattice \( L \) is **down-directed** if for every pair \( d_1, d_2 \in D \) there exists \( d_3 \in D \) such that \( d_1 \geq d_3 \) and \( d_2 \geq d_3 \) both hold. A complete lattice is **lower continuous** if for every down-directed set \( D \subseteq L, a \lor \bigwedge D = \bigwedge_{d \in D} (a \lor d) \). We denote a strongly coatomic, lower continuous, complete lattice as SCLCC, and a closure system is SCLCC if its lattice of closed sets is so. Note that every finite lattice or closure system is SCLCC. Indeed, our goal in this paper is to find a general class of lattices (not necessarily finite) to which the characterization of finite convex geometries extends naturally.

Some basic properties of SCLCC lattices follow.

**Lemma 4.** Let \( L \) be an SCLCC lattice.

1. Every nonzero join irreducible element is completely join irreducible.
2. If \( w \succ c \) in \( L \), then there exists an element \( k \) that is minimal w.r.t. \( k \leq w, k \not\prec c \), and each such element is join irreducible.
3. Every element of \( L \) is a (perhaps infinite) join of join irreducible elements.

**Proof.**

1. We prove the contrapositive. Assume that \( w = \bigvee X \) with \( x < w \) for all \( x \in X \). Choose any \( x_0 \in X \). Then \( x_0 < w \), so by strong coatomicity there exists \( c \in L \) such that \( x_0 \leq c \prec w \). Since \( \bigvee X = w > c \), there exists an element \( x_1 \in X \) such that \( x_1 \not\preceq c \). But then \( w = x_1 \lor c \) is finitely join reducible. (Note that this part uses only strong coatomicity.)

2. Given a covering pair \( w \succ c \) in \( L \), let \( P = \{ x \in L : x \leq w, x \not\preceq c \} \). Then \( P \) is a nonempty ordered set, and \( c \lor x = w \) for every \( x \in P \). For any chain \( D \subseteq P \), we have by lower continuity that \( c \lor \bigwedge D = \bigwedge_{d \in D} (c \lor d) = w \), and thus \( \bigwedge D \in P \). By Zorn’s Lemma, \( P \) contains a minimal element \( k \), as claimed. Moreover, any such element \( k \) is completely join irreducible, as every element strictly below \( k \) is below \( c \land k \).

3. The statement is true for the least element \( 0 \) of \( L \), since by convention \( \bigvee \emptyset = 0 \). So consider \( w > 0 \) in \( L \), and let \( m \) be the join of all join irreducible elements \( x \) with \( x \leq w \). The claim is that \( m = w \). Suppose to the contrary that \( m < w \). Then there exists an element \( c \in L \) such that \( m \leq c \prec w \). By part (2), there is a completely join irreducible element \( k \) with \( k \leq w \) and \( k \not\preceq c \). But \( k \leq w \) implies \( k \leq m \) by construction, and \( m \leq c \), so this is a contradiction. Hence \( m = w \). \( \square \)
A complete lattice in which every element is a join of completely join irreducible elements is said to be spatial. Clearly, this is a desirable property for any sort of “geometry.” In [5], it was shown that every weakly atomic convex geometry is spatial. The preceding lemma says that SCLCC lattices have an even better property: a complete lattice is strongly spatial if it is spatial and every nonzero join irreducible element is completely join irreducible. The set of nonzero join irreducible elements of a lattice \( L \) will be denoted by \( \text{Ji}(L) \).

Note that every complete lattice can be represented via a closure system in various ways. If the lattice is spatial, as SCLCC lattices are, then a standard representation would use the completely join irreducible elements of \( L \) as the set \( X \). The next observation relates this to condition (2) of Theorem 3.

**Lemma 5.** Let \( (X, \gamma) \) be a closure system such that the lattice of closed subsets \( \text{Cld}(X, \gamma) \) is strongly coatomic and satisfies the property that \( A \prec B \) implies \( |B \setminus A| = 1 \). Then there is a one-to-one correspondence between \( X \setminus \gamma(\emptyset) \) and the nonzero completely join irreducible closed sets of \( (X, \gamma) \).

**Proof.** Always in a closure system, if \( B \) is a completely join irreducible closed set, then \( B = \gamma(x) \) for any \( x \in B \setminus B_* \), where \( B_* \) denotes the unique lower cover of \( B \). On the other hand, if \( \text{Cld}(X, \gamma) \) is strongly coatomic and \( x \notin \gamma(\emptyset) \), then there is a closed set \( A \) such that \( A \prec \gamma(x) \). If \( (X, \gamma) \) also satisfies the condition, then the only choice for \( A \) is \( \gamma(x) \setminus \{x\} \). That makes \( \gamma(x) \setminus \{x\} \) the unique lower cover of \( \gamma(x) \), so that \( \gamma(x) \) is completely join irreducible. \( \square \)

4. **Join semidistributive SCLCC lattices**

Next we generalize some equivalences of join semidistributivity which are well-known for finite lattices.

The implication

\[(\text{SD}_\vee) \quad w = x \vee y = x \vee z \quad \text{implies} \quad w = x \lor (y \land z)\]

is known as the *join semidistributive law*. In view of Lemma 4 and the results in Jónsson and Kiefer [14], we consider two potentially infinite versions of the law:

\[(\text{SD}'_\vee) \quad w = x \lor y \quad \text{for all} \ y \in Y \quad \text{implies} \quad w = x \lor \bigwedge Y\]

\[(\text{SD}^*_\vee) \quad w = \bigvee Y = \bigvee Z \quad \text{implies} \quad w = \bigvee (y \land z)\]

It is an elementary exercise that join semidistributive, lower continuous, complete lattices satisfy \( \text{SD}'_\vee \), and we shall use that at will.
For subsets $A, B \subseteq L$ we say that $A$ refines $B$, denoted $A \ll B$, if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. Note that $A \ll B$ implies $\bigvee A \leq \bigvee B$.

We say that $w = \bigvee A$ is a canonical join decomposition if the join is irredundant, and $w = \bigvee B$ implies $A \ll B$.

**Theorem 6.** The following are equivalent for an SCLLC lattice $L$.

1. $L$ satisfies $SD_\gamma$.
2. $L$ satisfies $SD_\gamma'$.
3. Every element of $L$ has a canonical join decomposition.
4. If $w \succ c$ in $L$, then there exists a join irreducible $k$ which is the unique minimal element such that $k \leq w$ but $k \not\leq c$.

**Proof.** To see that (1) $\Rightarrow$ (4), we invoke Lemma 4 (2), which says that there is at least one such element $k$. If there were two or more, say $k_1$ and $k_2$, then $c \lor k_1 = w = c \lor k_2 > c \lor (k_1 \land k_2)$ by the minimality of each $k_i$, contradicting (SD_\gamma). Hence such an element $k$ is unique.

(4) $\Rightarrow$ (3). Assume that property (4) holds, and fix an element $w \in L$. Let $C = \{c \in L : w \succ c\}$. For each $c \in C$, we can find an element $k_c$ such that $k_c \leq w$ but $k_c \not\leq c$. We claim that $w = \bigvee_{c \in C} k_c$ canonically. Clearly $\bigvee_{c \in C} k_c = w$, since each $k_c$ is below $w$, while the join is below no lower cover of $w$.

If $c \neq d \in C$, then $c \lor d = w$. Hence by SD_\gamma' we have $c \lor \overline{c} = w$, where $\overline{c} = \bigwedge (C \setminus \{c\})$. By the unique minimality of $k_c$, this implies $k_c \leq \overline{c}$, i.e., $k_c \leq d$ whenever $c \neq d \in C$. It follows that the join representation $w = \bigvee_{c \in C} k_c$ is irredundant.

Suppose $w = \bigvee A$ and consider $c \prec w$. There exists some $a_0 \in A$ such that $a_0 \not\leq c$, though $a_0 \leq w$, whereupon $k_c \leq a_0$. Since this holds for all $c \in C$, we have $\{k_c : c \in C\} \ll A$, as desired.

(3) $\Rightarrow$ (2). Suppose that $w = \bigvee S = \bigvee T$. If there is a canonical join decomposition $w = \bigvee U$ in $L$, then $U$ refines both $S$ and $T$, so that for each $u \in U$ there exist $s \in S$ with $u \leq s$, and $t \in T$ with $u \leq t$. Hence each $u \leq s \land t$ for some pair, and it follows that $w = \bigvee (s \land t)$.

(2) $\Rightarrow$ (1) clearly, as SD_\gamma is a special case of $SD_\gamma^*$.

5. **Characterizations of SCLCC convex geometries**

First, we introduce a few more terms.

A strongly coatomic complete lattice is said to be locally distributive (or lower locally distributive or meet distributive) if for any $x \in L$ the interval $[\mu(x), x]$ where $\mu(x) = \bigwedge \{y : y \prec x\}$ is a distributive lattice (and hence for SCLCC lattices a boolean algebra).

For any $A \subseteq X$, $x \in A$ is called an extreme point of $A$ if $x \notin \gamma(A \setminus \{x\})$. The set of extreme points of $A$ is denoted $\text{Ex}(A)$. In lattice
terms, for a strongly spatial lattice $L$, we identify the element $a$ with the set $\text{Ji}(a) = \{ p \in \text{Ji}(L) : p \leq a \}$. Then $x \in \text{Ji}(a)$ is an extreme point of $a$ if $a > \bigvee (\text{Ji}(a) \setminus \{x\})$. This means that (i) $x$ is join prime in the ideal $\text{id}(a)$, and (ii) there is no other join irreducible $y$ with $x < y \leq a$.

Recall that a lattice $L$ is called lower semimodular if $a \prec b$ implies $a \land c \preceq b \land c$ for all $a, b, c \in L$. Equivalently, a lattice is lower semimodular if $a \prec a \lor c$ implies $a \land c \prec c$.

We now extend some characterizations of finite convex geometries to SCLCC geometries. In this setting we want to think of a lattice in terms of its standard representation as a closure system on its set of join irreducibles. Note that properties (1)–(2) of the next theorem are about closure systems, while (3)–(6) are lattice properties. For the finite case, various parts of the theorem are due to R.P. Dilworth, S.P. Avann, P. Edelman and R. Jamison, and V. Duquenne.

**Theorem 7.** Let $L$ be an SCLCC lattice. Then the following are equivalent.

1. $L$ is the closure lattice $\text{Cld}(X, \gamma)$ of a closure system $(X, \gamma)$ with the AEP.
2. $L$ is the closure lattice $\text{Cld}(X, \gamma)$ of a closure system $(X, \gamma)$ with the property that if $A \prec B$ in $\text{Cld}(X, \gamma)$, then $|B \setminus A| = 1$.
3. $L$ is locally distributive and lower semimodular.
4. $L$ is join semidistributive and lower semimodular.
5. Every element $a \in L$ is the join of $\text{Ex}(a)$.
6. Every element of $L$ has a unique irredundant join decomposition.

In fact, most of these equivalences can be proved in a weaker setting, without lower continuity. Whether or not this can be extended to include property (3) remains an open question at this time.

**Lemma 8.** Properties (1), (2), (4), (5) and (6) of Theorem 7 are equivalent for complete lattices that are strongly coatomic and spatial.

**Proof.** The equivalence of (1) and (2) is Theorem 3.

To see that (2) $\Rightarrow$ (4), consider a closure system $(X, \gamma)$ satisfying (2). Clearly (2) implies that $\text{Cld}(X, \gamma)$ is lower semimodular; we want to show that it is join semidistributive. This can be done by proving that every closed set $B$ has a canonical join decomposition in $\text{Cld}(X, \gamma)$. (Note that (3) $\Rightarrow$ (2) $\Rightarrow$ (1) of Theorem 6 holds in all complete lattices.)

For each $A \prec B$ in $\text{Cld}(X, \gamma)$, let $\{x_A\} = B \setminus A$. The claim is that $B = \bigvee_{A \prec B} \gamma(x_A)$ canonically. Let $R$ denote the right hand side, and note that $B \supseteq R$. If $B \supset R$ properly, then there would exist $C$ such that $R \leq C \prec B$. That would imply $x_C \in R$, a contradiction. Thus
Now suppose $B = \bigvee_{i \in I} D_i$ for some closed sets $D_i$ in $\text{Cld}(X, \gamma)$. For each $A \prec B$, we have $\bigcup D_i \not\subseteq A$, so that there exists an $i_0$ with $D_{i_0} \not\subseteq A$. Since $D_{i_0} \subseteq B$, this implies $x_A \in D_{i_0}$, whence $\gamma(x_A) \leq D_{i_0}$. We have shown that $\{\gamma(x_A) : A \prec B\} \ll \{D_i : i \in I\}$, as required for a canonical join decomposition.

Before proving that (4) implies (5), let us interpret the set of extreme points in a more traditional way.

**Sublemma 9.** In an $\text{SCLCC}$ lattice, $w = \bigvee \text{Ex}(w)$ iff for each element $c$ with $c \prec w$, there exists a unique $j_c$ in $\text{Ji} L$ such that $j_c \leq w$ and $j_c \not\prec c$.

**Proof.** Assume $w = \bigvee \text{Ex}(w)$, and let $c \prec w$. There exists $j \in \text{Ex}(w)$ such that $j \not\leq c$, and $j \not\leq \bigvee (\text{Ji}(w) \setminus \{j\})$ because $j$ is extreme. In that case, the latter join must in fact be $c$, since $c$ is a join of join irreducibles below $w$. Hence $j$ is unique.

Conversely, suppose that for each $c \prec w$ such a $j_c$ exists. Then $\bigvee \{j_c : c \prec w\} = w$, since the join is below $w$ but not below any of its lower covers. On the other hand, each such $j_c$ is extreme, as $c = \bigvee (\text{Ji}(w) \setminus \{j_c\})$, since $j_c$ is the only join irreducible below $w$ that not below $c$.

Now assume (4), that $L$ is join semidistributive and lower semimodular. Let $w \succ c$ in $L$. The assumption that $L$ is spatial means that there is a completely join irreducible element $j$ such that $j \leq w$ but $j \not\leq c$. By lower semimodularity, $j \succ j \wedge c$, i.e., $j_i \leq c$, so $j$ is minimal with this property. By SD$_\lor$, there can be only one such minimal element. Thus $j$ is the unique minimal element with $j \leq w$, $j \not\leq c$.

If $j < y \leq w$, then by lower semimodularity $y \succ y \wedge c$, whence $y = j \lor (y \wedge c)$ is join reducible. Thus $j$ is an extreme element for $w$.

As before, $w$ is the join of all these elements, yielding (5).

The equivalence of (5) and (6), and that (6) implies (2) for the standard representation of a spatial lattice $L$ as a closure operator on its join irreducibles, are both easy.

So far we have proved the lemma, giving the equivalence of (1), (2), (4), (5) and (6) for strongly coatomic, spatial complete lattices. To complete the proof of Theorem 7, it remains to show that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) for $\text{SCLCC}$ lattices, allowing the use of lower continuity.

To see that (2) implies (3), assume that $L \cong \text{Cld}(X, \gamma)$ where the closure system $(X, \gamma)$ has the property (2). Let $T \subseteq X$ be a $\gamma$-closed set, and let $M = \bigcap\{S \in \text{Cld}(X, \gamma) : S \prec T\}$. By hypothesis, for
each $S \prec T$ there is a unique element $x_S \in T \setminus S$, and $T \setminus M$ consists precisely of those elements.

Consider any set $Z$ with $M \subseteq Z \subseteq T$. For every $x \in T \setminus Z$, the set $T \setminus \{x\}$ is $\gamma$-closed, so that $Z = \bigcap \{T \setminus \{x\} : x \in T \setminus Z\}$ is also closed. Hence the entire boolean algebra of subsets $[M, T]$ is contained in $\text{Cld}(X, \gamma)$. Thus (2) implies local distributivity, and it is clear that (2) implies lower semimodularity.

Next assume (3), that $L$ is locally distributive and lower semimodular, and we want to show that $L$ is join semidistributive by means of property (4) of Theorem 6, which yields (4) of this theorem.

So let $w \succ c$, and let $a$ be minimal w.r.t. $a \leq w$, $a \not\preceq c$. Suppose by way of contradiction that there exists another such element $b$. Choose an element $b'$ with $b \leq b' \prec a \lor b$. Using strong coatomicity, form a (possibly transfinite) sequence $a_\alpha$ for $\alpha \leq \kappa$ such that

(i) $a_0 = a \lor b$ and $a_\kappa = a$,
(ii) $a_{\alpha+1} \prec a_\alpha$,
(iii) $a_\lambda = \bigwedge_{\alpha < \lambda} a_\alpha$ for limit ordinals $\lambda$.

We show by induction that for all $\alpha \leq \kappa$,

(a) $a_\alpha \lor (b' \land c) = a_0$,
(b) $b' \lor (a_\alpha \land c) = a_0$,
(c) $c \lor (a_\alpha \land b') = w$.

When $\alpha = 0$, parts (a) and (c) are immediate. Since $a_0 \leq w$ but $a_0 \not\leq c$, by lower semimodularity we have $a_0 \succ a_0 \land c$, whence (b) also follows.

Now assume that the claims hold for $a_\alpha$ and that $a_\alpha > a$. Using lower semimodularity, parts (b) and (c) imply that $a_\alpha \land c$ and $a_\alpha \land b'$ are lower covers of $a_\alpha$, neither of which is above $a$. Choose $a_{\alpha+1}$ so that $a \leq a_{\alpha+1} \prec a_\alpha$. Then $a_{\alpha+1}$, $a_\alpha \land b'$ and $a_\alpha \land c$ are three distinct lower covers of $a_\alpha$, which by local distributivity generate an eight element boolean algebra. Claims (a)–(c) follow from this.

At limit ordinals, the three claims follow by lower continuity. But for $\alpha = \kappa$, part (c) contradicts the minimality of $a$. We conclude that no such element $b$ exists, and hence $L$ is join semidistributive.

This completes the proof of Theorem 7.

A lattice $L$ is called atomistic if every nonzero $a \in L$ is a join of atoms. Atomistic convex geometries were characterized in Proposition 3.1 of Adaricheva, Gorbunov and Tumanov [3]. For the SCLCC case, the proof is particularly easy.

**Corollary 10.** Any SCLCC atomistic join semidistributive lattice is the closure lattice of some convex geometry.
Proof. It is enough to show that a SCLCC atomistic join semidistributive lattice is lower semimodular. Indeed, it follows from join semidistributivity that if \( a \prec b \), then there exists a unique atom \( t \) such that \( t \leq b \), \( t \nleq a \). If \( c \leq b \), then either \( t \leq c \) and thus \( c \) and \( c \wedge a \) differ by the single atom \( t \), so that \( c \wedge a \prec c \), or else \( t \nleq c \) in which case every atom below \( c \) is below \( a \), whence \( c \leq a \). Thus \( a \prec b \) implies \( c \wedge a \preceq c \wedge b \), which is lower semimodularity. \( \square \)

6. Examples of SCLCC convex geometries

There are natural examples of the kind of geometries described in Corollary 10, obtained by taking standard convex geometries and adding finiteness conditions to ensure strong coatomicity and lower continuity.

Theorem 11. (1) If \( P \) is an ordered set such that both the ideal \( \text{id}(x) \) and the filter \( \text{fil}(x) \) are finite for every \( x \in P \), then the lattice of convex subsets \( \text{Co}(P) \) is an atomistic SCLCC convex geometry.

(2) If \( S \) is a meet semilattice such that the filter \( \text{fil}(x) \) is finite for every \( x \in S \), then the lattice of subsemilattices \( \text{Sub} \cap(S) \) is an atomistic SCLCC convex geometry.

Note that in each example, the closure operator is algebraic, and an algebraic lattice is dually algebraic if and only if it is lower continuous.

Proof. (1) Recall that a subset \( A \) of an ordered set \( P \) is convex if \( a_1, a_2 \in A \) and \( a_1 \leq x \leq a_2 \) implies \( x \in A \). We know that the convex hull operator on an ordered set satisfies the AEP; it remains to show that if \( P \) has the property that every principal ideal and filter is finite, then \( \text{Co}(P) \) is strongly coatomic and lower continuous.

Suppose that \( P \) has that property, and that \( A \prec B \) in \( \text{Co}(P) \). Let \( b_0 \in B \setminus A \). Then either \( \text{id}(b) \cap A = \emptyset \) or \( \text{fil}(b) \cap A = \emptyset \), w.l.o.g. the former. Choose \( b_1 \) minimal in \( B \cap \text{id}(b_0) \). Then \( A \subseteq B \setminus \{b_1\} \prec B \) in \( \text{Co}(P) \). Thus \( \text{Co}(P) \) is strongly coatomic.

For lower continuity, it is convenient to use the equivalent formulation in terms of chains; see e.g. Theorem 3.8 of [17]. Let \( X \) be convex, and let \( \mathcal{C} = \{C_i : i \in I\} \) be a chain of convex subsets of \( P \). We want to show that \( \bigwedge_i (X \vee C_i) \subseteq X \vee \bigwedge_i C_i \). Let \( w \) be in the left-hand side, and w.l.o.g. \( w \notin X \). Either there exists a co-initial subset \( \mathcal{D} = \{C_j : j \in J\} \) for some \( J \subseteq I \), and elements \( x_j \in X \) and \( y_j \in C_j \), such that \( x_j \leq w \leq y_j \) for all \( j \in J \), or dually; assume the first. Because \( \text{fil}(w) \) is finite, there exists a co-initial subset \( \mathcal{E} = \{C_k : k \in K\} \) with \( K \subseteq J \) such that \( y_k = y_{k'} \) for all \( k, k' \in K \). Then \( y_k \in \bigcap_{k \in K} C_k = \bigcap_{i \in I} C_i \), and fixing any \( k_0 \in K \) we have
Thus \( w \) is in the right-hand side, and \( \text{Co}(P) \) is lower continuous.

(2) Now consider \( \text{Sub}_\land(S) \) for a meet semilattice \( S \). Again, we know that the subsemilattice operator satisfies the AEP, and it remains to show that if \( S \) has the property that every principal filter is finite, then \( \text{Sub}_\land(S) \) is strongly coatomic and lower continuous.

If \( A < B \) in \( \text{Sub}_\land(S) \), then we can choose \( b_0 \) maximal in \( B \setminus A \) to obtain \( A \leq B \setminus \{b_0\} \prec B \).

To show lower continuity, let \( X \) be a subsemilattice and let \( C = \{C_i : i \in I\} \) be a chain of subsemilattices of \( S \). Again, we want to show that \( \bigwedge_i (X \lor C_i) \subseteq X \lor \bigwedge_i C_i \), so let \( w \) be in the left-hand side. That means that for each \( i \in I \) there exist \( x_i \in X \) and \( y_i \in C_i \) such that \( w = x_i \land y_i \). Let \( x = \bigwedge_i x_i \), which really is a finite meet. If \( x = w \), we are done. Otherwise, we continue: because \( \text{fil}(w) \) is finite, there exists a co-initial subset \( \mathcal{E} = \{C_k : k \in K\} \) for some \( K \subseteq I \) such that \( y_k = y_{k'} \) for all \( k, k' \in K \). Then \( w = x \land y_k \), and \( y_k \in \bigcap_{k \in K} C_k = \bigcap_{i \in I} C_i \), so \( w \in X \lor \bigwedge_i C_i \), as desired. \( \square \)

We can even combine these examples: if \( S \) is a meet semilattice in which every principal filter \( \text{fil}(x) \) is a finite tree, then the lattice of convex subsemilattices of \( S \) is an SCLCC convex geometry. See Adaricheva [1] and Cheong and Jones [7].

In a similar vein, if \((P, \leq)\) is an ordered set in which every chain is finite and every interval is finite, then the lattice of suborders of \( \leq \) on \( P \) is an SCLCC convex geometry. See Semenova [19].

Another construction yields SCLCC convex geometries that need not be atomistic. Our inspiration is the fact that a geometric lattice is isomorphic to the ideal lattice of its finite dimensional elements. (There is no chance for a similar characterization here, since for any non-limit ordinal \( \alpha \), the dual \( \alpha^d \) is an SCLCC convex geometry.) Our construction uses Jónsson and Rival’s characterization of join semidistributive varieties [15].

Define certain lattice terms recursively: for \( k \geq 0 \),

\[
\begin{align*}
y_0 &= y \\
y_{k+1} &= y \land (x \lor z_k) \\
z_0 &= z \\
z_{k+1} &= z \land (x \lor y_k).
\end{align*}
\]

Then consider the lattice inclusions

\[
\text{SD}_\lor(k) \quad y_k \leq x \lor (y \land z) .
\]

These are equivalent to the corresponding identities \( x \lor y_k \approx x \lor (y \land z) \). For example, \( \text{SD}_\lor(1) \) is equivalent to the distributive law.
Lemma 12. Let $\mathcal{V}$ be a lattice variety. Then every lattice in $\mathcal{V}$ is join semidistributive if and only if $\mathcal{V}$ satisfies $SD_\vee(n)$ for some $n < \omega$.

Let $SD_\vee(n)$ be the variety of all lattices satisfying $SD_\vee(n)$.

Theorem 13. Let $L_0$ be a lattice with the following properties.

- $\text{fil}(x)$ is finite for each $x \in L_0$.
- $L_0 \in SD_\vee(n)$ for some $n < \omega$.
- $L_0$ is lower semimodular.

Then the filter lattice $L = \text{Fil}(L_0)$ is an SCLCC convex geometry.

Proof. As usual, we order the filter lattice by reverse set inclusion: $F \leq G$ if and only if $F \supseteq G$. The filter lattice of any lattice is lower continuous and satisfies the equations of the original, in particular $SD_\vee(n)$ in this case. It remains to show that $L$ is strongly coatomic and lower semimodular.

Suppose $F < G$ in $L$, i.e., $F \supset G$. Let $k$ be an element maximal in $F \setminus G$, and note that $k$ is meet irreducible. We claim that the filter generated by $G \cup \{k\}$, say $H = \text{fil}(G, k)$, satisfies $F \leq H \prec G$. Let $\ell$ be any element in $H \setminus G$. Then $\ell \geq g \wedge k$ for some $g \in G$, and we may take $g \leq k^*$, where $k^*$ denotes the unique upper cover of $k$ in $L_0$. In that case, by lower semimodularity, $g \succ g \wedge k$, whence also $g \wedge k = g \wedge \ell$. It follows that $H = \text{fil}(G, \ell)$, and since $\ell$ is arbitrary, $H \prec G$. Thus $L$ is strongly coatomic.

The proof that $L$ is lower semimodular is an adaptation of that for the corresponding dual claim in Theorem 11.1 of [17]. Assume that $L$ is lower semimodular, and suppose that $F \prec F \vee G = F \cap G$ in $\text{Fil}(L_0)$. Choose an element $a$ maximal in $F \setminus G$, and note that $a$ is meet irreducible, thus by the finiteness of $\text{fil}(a)$ completely meet irreducible. Then $F = (F \vee G) \wedge \text{fil}(a)$, and hence $F \wedge G = \text{fil}(a) \wedge G$. Let $x$ be any element in $(F \wedge G) \setminus G$. Since $x \in F \wedge G$, there exists $g \in G$ such that $x \geq a \wedge g$. Because $L$ is lower semimodular, $a \wedge g \prec a^* \wedge g$. On the other hand, every element of $L$ is a meet of finitely many meet irreducibles, so $x \notin G$ implies there exists a meet irreducible element $b \geq x$ with $b \notin G$. Now $b \geq a \wedge g$ and $b \neq g$, so $b \wedge g = a \wedge g$, whence $a \geq b \wedge g$. Thus $\text{fil}(b) \wedge G = \text{fil}(a) \wedge G = F \wedge G$; if follows a fortiori that $\text{fil}(x) \wedge G = F \wedge G$. As this holds for every $x \in (F \wedge G) \setminus G$, we have $F \wedge G \prec G$, as desired. ☐

So in particular, we could take $L_0$ to be the elements of finite depth in a direct product of finite convex geometries that satisfy $SD_\vee(n)$ for some fixed $n$.

The examples so far have all been algebraic closure operators. For a non-algebraic example of an SCLCC convex geometry, we form a closure system $(\omega, \gamma)$ on the natural numbers $\omega$. Define a subset $S \subseteq \omega$
to be $\gamma$-closed if either $0 \in S$ or $S$ is finite. Clearly the closed sets are closed under arbitrary intersections, so $\text{Cld}(\omega, \gamma)$ is a complete lattice. Moreover, the lower covers of a nonempty closed set $S$ are all sets $S \setminus \{x\}$ with $x \in S$ if $S$ is finite, and all sets $S \setminus \{x\}$ with $0 \neq x \in S$ if $S$ is infinite. It follows easily that $\text{Cld}(\omega, \gamma)$ is strongly coatomic, and it has the property that $T \prec S$ implies $|S \setminus T| = 1$ of Theorem 7. Note that $\text{Cld}(\omega, \gamma)$ is a sublattice of the subset lattice $\text{Pow}(\omega)$, closed under arbitrary intersections and finite unions (but not infinite unions). This clearly makes $\text{Cld}(\omega, \gamma)$ lower continuous. To see that it is not algebraic, we show that it is not upper continuous. For $k \geq 1$, let $F_k = \{1, \ldots, k\}$. This is a chain with $\bigvee F_k = \omega$, and hence

$$\{0\} = \{0\} \land \bigvee F_k \supset \bigvee (\{0\} \land F_k) = \emptyset.$$  

Thus $\text{Cld}(\omega, \gamma)$ is a non-algebraic SCLCC convex geometry.

7. Discussion

In some sense, algebraic closure operators are the natural settings for any type of geometry. On the other hand, Crawley and Dilworth’s setting of dually algebraic and strongly coatomic gives the nice equivalence of local distributivity and unique representations. Since dually algebraic lattices are lower continuous, our hypothesis of SCLCC is slightly weaker, but means that we must assume lower semimodularity along with local distributivity. The lattice $(\omega + 1)^d \times 2$, with its atom doubled, shows that local distributivity by itself is not sufficient.

The question remains whether all the equivalences of Theorem 7 hold under the weaker hypothesis of a strongly coatomic, spatial complete lattice. The proof that (2) implies (3) does not use lower continuity, but our proof of (3) implies (4) depends on it. Thus the question can be phrased: Does every strongly coatomic, spatial, locally distributive and lower semimodular closure system satisfy the anti-exchange property?

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References


DEPARTMENT OF MATHEMATICAL SCIENCES, Yeshiva University, New York, NY 10016 USA

E-mail address: adariche@yu.edu

AND DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE AND TECHNOLOGY, NAZARBAEV UNIVERSITY, ASTANA, KAZAKHSTAN

E-mail address: kira.adaricheva@nu.edu.kz

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII, HONOLULU, HI 96822, USA

E-mail address: jb@math.hawaii.edu