INFLATION OF FINITE LATTICES ALONG ALL-OR-NOTHING SETS

TRISTAN HOLMES

J. B. NATION

Department of Mathematics, University of Hawaii, Honolulu, HI 96822, USA
Phone:(808)956-4655

Abstract. We introduce a new generalization of Alan Day's doubling construction. For ordered sets $\mathcal{L}$ and $\mathcal{K}$ and a subset $E \subseteq \leq_{\mathcal{L}}$ we define the ordered set $\mathcal{L} \star_E \mathcal{K}$ arising from inflation of $\mathcal{L}$ along E by $\mathcal{K}$. Under the restriction that $\mathcal{L}$ and $\mathcal{K}$ are finite lattices, we find those subsets $E \subseteq \leq_{\mathcal{L}}$ such that the ordered set $\mathcal{L} \star_E \mathcal{K}$ is a lattice. Finite lattices that can be constructed in this way are classified in terms of their congruence lattices.

A finite lattice is binary cut-through codable if and only if there exists a $0-1$ spanning chain $\{\theta_i \colon 0 \leq i \leq n\}$ in $Con(\mathcal{L})$ such that the cardinality of the largest block of θ_i/θ_{i-1} is 2 for every i with $1 \leq i \leq n$. These are exactly the lattices that can be constructed by inflation from the 1-element lattice using only the 2-element lattice. We investigate the structure of binary cut-through codable lattices and describe an infinite class of lattices that generate binary cut-through codable varieties.

MSC2010 Classification 06B05, 06B10

Keywords: finite lattice; doubling construction; inflation construction; cut-through codable lattice

E-mail addresses: `tristanh314@math.hawaii.edu, jb@math.hawaii.edu`.
Date: July 3, 2016.

Introduction

Allen Day introduced the doubling construction in [2]. In [3], he showed that lower bounded images of free lattices are precisely those that can be constructed from the 1-element lattice by sequentially doubling a certain class of convex sets.

Day's construction is applicable to any convex subset of a finite lattice. In the years after the classification of finite bounded lattices, some progress was made in expanding the scope of Day's construction. Day and Geyer classified those finite lattices that are constructible by doubling arbitrary convex sets in [4], [7], and [11]. In [12] Nation observed that it was possible to construct new lattices by doubling sets that were not necessarily convex, but he did not classify such sets.

Progress toward any such characterization was not made until Heiko Reppe [16] classified those subsets of a lattice that would yield a lattice when doubled according to Day's doubling construction, and called them *municipal* sets. The present study uses a modification of Day's doubling construction to classify those finite lattices that are sublattices of lattices constructible by doubling municipal sets.

In the first section we present Day's doubling construction, and the classes of finite lattices that can be obtained using it. Section 2 introduces our modification of Day's doubling construction that allows us to obtain additional finite lattices. This modified construction is called *inflation*. The results presented in Section 3 classify those finite lattices that can be obtained using the inflation construction. We restrict our attention to inflation by the 2-element lattice in Section 4 in order to classify lattices that generate varieties in which every finite lattice can be constructed by inflation using only the 2-element lattice.

1. Day's Doubling Construction

Let $\mathcal{P} = \langle P, \leq \rangle$ be an ordered set. A subset C of P is *convex* if whenever a and c are in C and $a \leq b \leq c$, then $b \in C$. An interval is a convex set. Other examples of convex sets include *lower pseudo-intervals*, which are unions of intervals that share the same least element. Dually, an *upper pseudo-interval* is a union of intervals with a common greatest element.

Let $\mathcal{L}$ and $\mathcal{K}$ be ordered sets, and let S be a subset of L. Let $L[K, S]$ denote the disjoint union $(L \setminus S) \cup (S \times K)$. Order $L[K, S]$ as follows.

(1) $x \leq y$ if x, $y \in L \setminus S$ and $x \leq_{\mathcal{L}} y$,
(2) $x \leq (y, j)$ if $x \in L \setminus S$, $y \in S \times K$, and $x \leq_{\mathcal{L}} y$,
(3) $(x, i) \leq y$ if $(x, i) \in S \times K$, $y \in L \setminus S$, and $x \leq_{\mathcal{L}} y$,
(4) $(x, i) \leq (y, j)$ if (x, i), $(y, j) \in S \times K$, and $(x, i) \leq_{\mathcal{S}\times\mathcal{K}} (y, j)$,
(5) $(x, i) \leq (y, j)$ if (x, i), $(y, j) \in S \times K$, and there exists $t \in L \setminus S$ such that $x < t < y$.

With this ordering, $\langle L[K, S], \leq \rangle$ is a structure denoted $\mathcal{L}[\mathcal{K}, S]$. Day's basic result can be stated thusly.

Theorem 1.1. [2] *If $\mathcal{L}$ and $\mathcal{K}$ are ordered sets and $S \subseteq L$, then $\mathcal{L}[\mathcal{K}, S]$ is an ordered set. Moreover, if $\mathcal{L}$ is a lattice, $\mathcal{K}$ a lattice with 0 and 1, and S is a convex subset of L, then $\mathcal{L}[\mathcal{K}, S]$ is a lattice.*

Originally, Day used $\mathcal{K} = \mathbf{2}$, but the extension to arbitrary $\mathcal{K}$ has long been known, e.g, [5]. It does simplify matters to assume that $\mathcal{K}$ is nontrivial ($|K| > 1$) and connected, which we will do throughout. Day also assumed that S was convex, and so did not need (5). However, if S is not convex, then we can have $(x,1) \leq t \leq (y,0)$, so (5) is required for transitivity. Moreover, $\mathcal{L}[\mathcal{K},S]$ may be a lattice for some non-convex subsets.

There is a natural order-preserving map $h : \mathcal{L}[\mathcal{K},S] \to \mathcal{L}$ defined by

(1) $h(x) = x$ if $x \in L \setminus S$,
(2) $h(x,i) = x$ if $x \in S$.

If $\mathcal{L}$, $\mathcal{K}$ and $\mathcal{L}[\mathcal{K},S]$ are lattices, then h is a lattice homomorphism.

Let $\mathcal{K}$ and $\mathcal{L}$ be lattices. A homomorphism $h\colon \mathcal{K} \to \mathcal{L}$ is called a *lower bounded homomorphism* if for every $a \in \mathcal{L}$, the set $h^{-1}(\uparrow a)$ is either empty or has a least element. The least element of a nonempty $h^{-1}(\uparrow a)$ will be denoted $\beta_h(a)$, or if the context is clear $\beta(a)$. If h is lower bounded, $\beta_h\colon \mathcal{L} \to \mathcal{K}$ is a partial mapping whose domain is an ideal of $\mathcal{L}$. Dually, h is called *upper bounded* if $h^{-1}(\downarrow a)$ is empty or has a greatest element, which is denoted $\alpha_h(a)$ or $\alpha(a)$, whenever it is nonempty. For an upper bounded homomorphism, the domain of α_h is a filter of $\mathcal{L}$. A homomorphism which is both upper and lower bounded is called *bounded*.

When h is surjective, h is lower bounded if and only if $h^{-1}(a)$ has a least element for every $a \in \mathcal{L}$. Likewise, when $\mathcal{L}$ is finite, $h\colon \mathcal{K} \to \mathcal{L}$ is lower bounded if and only if $h^{-1}(a)$ has a least element whenever it is nonempty. On the other hand, every homomorphism with domain $\mathcal{K}$, where $\mathcal{K}$ is finite, is bounded.

Note that β is monotonic and a left adjoint for h, that is, $a \leq h(x)$ if and only if $\beta(a) \leq x$. It follows that β is join preserving on its domain, i.e., if $h^{-1}(\uparrow a) \neq \emptyset$ and $h^{-1}(\uparrow b) \neq \emptyset$, then $\beta(a \vee b) = \beta(a) \vee \beta(b)$. Similarly, α is a right adjoint of h, i.e., $h(y) \leq a$ if and only if $y \leq \alpha(a)$, and α is a meet preserving map on its domain. In particular, if h is a homomorphism, then α and β are meet and join homomorphisms respectively. We say that a lattice $\mathcal{L}$ is a *lower bounded lattice* if every homomorphism from a finitely generated free lattice to $\mathcal{L}$ is a lower bounded homomorphism.

Following Day, we define the *join dependency relation* D on the set of join irreducible elements $JI(\mathcal{L})$ by $p\ D\ q$ if $p \neq q$ and there exists $x \in \mathcal{L}$ with $p \leq q \vee x$, and $p \nleq a \vee x$ for every $a < q$. When $\mathcal{L}$ is finite we can replace the final condition with $p \nleq q_* \vee x$, where q_* denotes the unique lower cover of q. Note that $p \in JI(\mathcal{L})$ is join prime if and only if there exists no $q \in JI(\mathcal{L})$ such that $p\ D\ q$.

We use the notation $Cg(a,b)$ for the smallest congruence θ of $\mathcal{L}$ such that $a\ \theta\ b$. For a finite lattice $\mathcal{L}$ and a join irreducible $p \in JI(\mathcal{L})$, we use the notation $\Phi_p := Cg(p,p_*)$. Similarly for $m \in MI(\mathcal{L})$ define $\Phi^m := Cg(m,m^*)$. Since the congruence lattice of a lattice is distributive [6], for finite lattices every congruence is the unique join of join prime congruences, and the join prime congruences are precisely those that can be expressed in the form Φ_p or dually Φ^m. For $p \in JI(\mathcal{L})$, or $m \in MI(\mathcal{L})$, there exists a unique congruence $\Psi_p = \bigvee\{\Phi_q \colon \Phi_p \nleq \Phi_q\}$, respectively $\Psi^n = \bigvee\{\Phi^m \colon \Phi^n \nleq \Phi^m\}$, which is maximal with respect to the condition that $(p,p_*) \notin \theta$, or respectively $(m,m^*) \notin \theta$.

We say that $p\ D^n\ q$ if there exists a finite sequence of elements $r_i \in \mathcal{L}$ such that

$$p = r_0\ D\ r_1\ D \cdots D\ r_n = q.$$

In [3] Day established the connection between this relation, the congruences of a finite lattice, and lower bounded lattices.

Lemma 1.2. *For a finite lattice $\mathcal{L}$ and $p,q \in JI(\mathcal{L})$, we have that $\Phi_p \leq \Phi_q$ if and only if $p = q$ or $p\ D^n\ q$ for some n.*

A *D-cycle* is a sequence witnessing $p\ D^n\ p$ for some $p \in JI(\mathcal{L})$ and $n \geq 2$. Finite lower bounded lattices can be classified in terms of omitting D-cycles.

Theorem 1.3. *A finite lattice $\mathcal{L}$ is lower bounded if and only if it contains no D-cycle.*

Theorem 1.4. *A finite lattice $\mathcal{L}$ is lower bounded if and only if $\mathcal{L}$ can be constructed from the 1-element lattice by doubling lower pseudo-intervals.*

There is an alternative way to characterize finite lower bounded lattices which will be particularly important in the sequel. Following Jónsson in [8], for a finite lattice $\mathcal{L}$ define the sets

$$D_0 = \{p \in JI(\mathcal{L})\colon p \text{ is join prime}\},$$
$$D_{k+1} = \{p \in JI(\mathcal{L})\colon p\ D\ q \text{ implies } q \in D_k\}.$$

In those finite lattices where the concept is well-defined, the *D-rank* of $p \in JI(\mathcal{L})$ is the least integer k such that $p \in D_k$ and $p \notin D_{k-1}$, and is denoted by $rk_D(p)$. It is important to note however that there are finite lattices where the D-rank is undefined, e.g., $\mathcal{M}_3$. It is not difficult to prove that $D_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ if and only if $\mathcal{L}$ admits no D-cycles, whence a finite lattice is lower bounded if and only if every join irreducible element has a well-defined D-rank.

Define the *depth* of an element x in a finite ordered set as the length of a maximal chain in $\uparrow x$. Given Lemma 1.2 and Theorem 1.4, we have Day's result that the depth of Φ_p in the ordered set $JI(Con(\mathcal{L}))$ is equal to the D-rank of p for finite lower bounded lattice $\mathcal{L}$. In considering a lower bounded lattice constructed from the 1-element lattice by doubling lower pseudo-intervals, as in Theorem 1.4, if b is the unique minimum element of a lower pseudo-interval C, then $q\ D\ (b,1)$ holds for no $q \in JI(\mathcal{L}[\mathbf{2}, C])$. It follows $\Phi_{(b,1)}$ is a minimal element in the ordered set $JI(Con(\mathcal{L}[\mathbf{2}, C]))$.

Recall that every join prime congruence of a finite lattice $\mathcal{L}$ is of the form Φ_p for some $p \in JI(\mathcal{L})$. Thus, there exists a surjection from the set of join irreducible elements of $\mathcal{L}$ onto the join irreducible congruences of $\mathcal{L}$, all of which are join prime since $Con(\mathcal{L})$ is distributive. Combining Lemma 1.2 and Theorem 1.3 we deduce the well known result of P. Pudlák and J. Tůma that classifies those finite lattices such that the correspondence is one-to-one.

Theorem 1.5. [14] *The following are equivalent for a finite lattice $\mathcal{L}$:*

(1) *$\mathcal{L}$ is lower bounded.*
(2) *$JI(\mathcal{L}) = D_k(\mathcal{L})$ for some $k \in \omega$.*
(3) *$\Phi_p = \Phi_q$ implies $p = q$ for all $p, q \in JI(\mathcal{L})$.*

Several important classes of lattices are examples of lower bounded lattices. The class of splitting lattices introduced in [10], which are vital in studying the equational theory of lattices, are (both lower and upper) bounded. Projective lattices are also bounded [9]. The lattice of all subsemilattices of a finite semilattice is lower bounded, see [1] and [15].

Recall that if $\mathcal{L}$ is a lattice and $C \subseteq \mathcal{L}$ is a convex subset, then $\mathcal{L}[\mathbf{2}, C]$ is a lattice. So far we have seen that if we restrict C to intervals or pseudo-intervals then there is a characterization of those lattices that are constructible by doubling such classes of subsets. The question arises: is there such a characterization for lattices constructible from arbitrary convex sets? The answer to this question was first discovered by Winfried Geyer in [7], who used Formal Concept Analysis. Alan Day quickly followed this with a solution using more traditional lattice theoretical language in [4].

To understand this class of lattices, we first need to introduce some new concepts. Let $a \leq b$ and $u \leq v$ in a lattice $\mathcal{L}$. We say that the intervals b/a and v/u are *associates* if $b \vee u = v$ and $b \wedge u = a$, or if $v \vee a = b$ and $v \wedge a = u$. We say that b/a and v/u are *projective* and write $b/a \approx v/u$ if there exist sequences $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = u$ and $b = y_0, y_1, y_2, \ldots, y_{n-1}, y_n = v$ such that for every $1 \leq i \leq n$ we have that y_{i-1}/x_{i-1} and y_i/x_i are associates. Note that $b/a \approx v/u$ implies $Cg(b, a) = Cg(v, u)$.

Of particular interest is the case when $\mathcal{L}$ is finite and we consider intervals that consist of covering pairs of the form p/p_* for $p \in JI(\mathcal{L})$ and m^*/m for $m \in MI(\mathcal{L})$. For $p, q \in JI(\mathcal{L})$ we say p is projective to q, and write $p \approx q$, to mean $p/p_* \approx q/q_*$. Similarly for $m, n \in MI(\mathcal{L})$ we say m is projective to n, and write $m \approx n$, to mean $m^*/m \approx n^*/n$. Lastly, we say p is projective to m, written $p \approx m$ to mean $p/p_* \approx m^*/m$. When interpreted in this way, projectivity is an equivalence relation on $JI(\mathcal{L}) \cup MI(\mathcal{L})$. Note that for $p, q \in JI(\mathcal{L})$ we have that $p \approx q$ implies $\Phi_p = \Phi_q$, but not conversely in general.

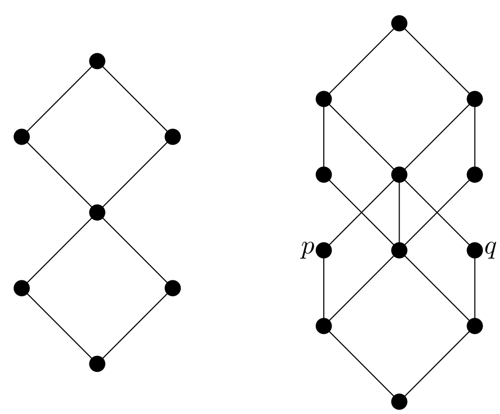

FIGURE 1. $\mathcal{L}[\mathbf{2}, C]$ is neither lower nor upper bounded.

There is a classification of finite lattices constructible by doubling arbitrary convex subsets that is analogous to the classification of finite lattices constructible by doubling pseudo-intervals. Consider the example in Figure 1. The first lattice is clearly bounded, indeed, it is distributive. However in the second lattice the elements p, q witness a D-cycle. Notice that both of the elements in this D-cycle are contained within a single projectivity class, and are "new" join and meet irreducible elements that were produced by the doubling.

Up until now we have been working with subclasses of convex sets that are connected by definition. It is important to note that if C_1, C_2 are sets that are connected and convex such that $C_1 \cup C_2$ is not connected, then $\mathcal{L}[\mathbf{2}, C_1][\mathbf{2}, C_2] \cong \mathcal{L}[\mathbf{2}, C_2][\mathbf{2}, C_1]$. From this point forward we restrict our attention to doubling sets that are connected and convex. For simplicity, we use the term *connected convex set* to refer to such sets.

If C is convex, then in $\mathcal{L}[\mathbf{2}, C]$ it is not possible for $(x, 1)$ to be below $(y, 0)$ for any $x, y \in C$. This fact, along with observations of the previous example, motivate the following definition: a lattice $\mathcal{L}$ is called *congruence normal* if for all $p \in JI(\mathcal{L})$ and $m \in MI(\mathcal{L})$, $\Phi_p = \Phi^m$ implies $p \not\leq m$.

Theorem 1.6. [7, 4] *A finite lattice $\mathcal{L}$ is constructible from the one element lattice by doubling convex sets if and only if $\mathcal{L}$ is congruence normal.*

The proof is a series of technical lemmas. The specific techniques used will not be directly applicable in the sequel, and so we refer the reader to Day's original paper [4]. There are however, concepts in the appendix to that article [11] that are directly relevant in the sequel, which we discuss presently.

Recall that a finite lattice is lower bounded, i.e., constructible from the one element lattice by doubling lower pseudo-intervals, if and only if $D_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$. There is an analogous result for congruence normal lattices. We define a relationship that is based on that used by Nation in [11], but which we have extended for the present work.

$$F_0 = \{p \in JI(\mathcal{L})\colon p \; D \; q \text{ implies } p \approx q\}$$
$$F_{k+1} = \{p \in JI(\mathcal{L})\colon p \; D \; q \text{ implies } q \in F_k \text{ or } p \approx q\}\,.$$

We define the *F-rank* of a join irreducible element p in a finite lattice $\mathcal{L}$ as we did the D-rank, and use the notation $rk_F(p)$. As with D-rank, this may not be defined for every join irrreducible in an arbitrary finite lattice.

Lemma 1.7. *The following are equivalent for a finite lattice $\mathcal{L}$.*

(1) $F_k(\mathcal{L}) = JI(\mathcal{L})$ *for some* $k \in \omega$.
(2) $\Phi_p = \Phi_q$ *implies* $p \approx q$ *for all* $p, q \in JI(\mathcal{L})$.

Proof. Assume that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$. Let $\Phi_p = \Phi_q$ for some $p, q \in JI(\mathcal{L})$. By Lemma 1.2 we have that $p = r_0 \; D \; r_1 \; D \; r_2 \; D \cdots D \; r_n = q$ where $n \in \omega$ and $r_i \in JI(\mathcal{L})$ for every i such that $0 \leq i \leq n$. By assumption this implies that $r_{i-1} \approx r_i$ or $rk_F(r_{i-1}) > rk_F(r_i)$. In the latter case $r_i \; D^m \; r_{i-1}$ fails for all $m \in \omega$, whence $\Phi_{r_i} \not\leq \Phi_{r_{i-1}}$, contrary to our choice of p and q. We conclude that $r_{i-1} \approx r_i$ for all i, whence $p \approx q$.

Assume that for all $p, q \in JI(\mathcal{L})$, if $\Phi_p = \Phi_q$ then $p \approx q$. Let $p \in JI(\mathcal{L})$. We proceed by induction on the depth of Φ_p. If Φ_p is maximal in the ordered set of join prime congruences of $\mathcal{L}$, then $p \; D \; q$ implies that $\Phi_p = \Phi_q$, whence by assumption $p \approx q$. It follows that $p \in F_0$. Let $rk_F(p) = k \geq 1$. If $p \; D \; q$, then $\Phi_p \leq \Phi_q$. If $\Phi_p = \Phi_q$ then $p \approx q$. Otherwise $\Phi_p < \Phi_q$ whence, by the inductive assumption, $q \in F_{k-1}(\mathcal{L})$. We conclude that $p \in F_k(\mathcal{L})$, as desired. □

The proof that (2) implies (1) in Lemma 1.7 give us the following fact that will be frequently referred to, so we state it as a corollary.

Corollary 1.8. *Let $\mathcal{L}$ be a finite lattice. If $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$, then $rk_F(p)$ is equal to the depth of Φ_p in $JI(Con(\mathcal{L}))$ for every $p \in JI(\mathcal{L})$.*

Lemma 1.7 is a partial generalization of Theorem 1.5. However, we are crucially missing a characterization of finite lattices that satisfy these conditions that is analogous to lower boundedness. Clearly lower bounded lattices satisfy the conditions of Lemma 1.7, but so do finite modular lattices, and finite geometric lattices. Every class of finite lattices classified in the remaining sections will be subclasses of the class of finite lattices that satisfy these conditions as well. Clearly $\mathcal{M}_3$ cannot be obtained from the one element lattice by doubling convex sets, so we will need additional assumptions to characterize such lattices.

We say that a finite lattice $\mathcal{L}$ is **2**-*simple* if the only simple lattice that is a homomorphic image of $\mathcal{L}$ is the 2-element lattice. All simple homomorphic images of a lattice $\mathcal{L}$ are those of the form $\mathcal{L}/\theta$, were θ is a maximal proper congruence of $\mathcal{L}$. Recall that the meet irreducible congruences of $\mathcal{L}$ are those of the form Ψ_p for some $p \in JI(\mathcal{L})$, where Ψ_p is the maximal congruence that does not contain the pair (p, p_*). Thus every simple homomorphic image of $\mathcal{L}$ is of the form $\mathcal{L}/\Psi_p$ for some $p \in JI(\mathcal{L})$. Hence a finite lattice $\mathcal{L}$ is **2**-simple if and only if for every $p \in JI(\mathcal{L})$ either p is join prime, or there exists some join prime q such that $p\ D^n\ q$.

The conditions given in Lemma 1.7 do not guarantee that a finite lattice $\mathcal{L}$ is **2**-simple. If a finite lattice $\mathcal{L}$ satisfies the conditions of Lemma 1.7 it is not difficult to see that any simple homomorphic image $\mathcal{K}$ of $\mathcal{L}$ will satisfy $F_0(\mathcal{K}) = JI(\mathcal{K})$. However, there exist simple lattices other than **2**, such as $\mathcal{M}_3$, for which $F_0(\mathcal{K}) = JI(\mathcal{K})$. In [11], Nation introduced the following condition for a finite lattice $\mathcal{L}$ to exclude such possibilities:

T: for all $p, q \in JI(\mathcal{L})$, if $p \approx q$ then $q \leq q_* \vee p$.

Nation did not address the issue of **2**-simplicity. The following lemma corrects this omission.

Lemma 1.9. *Let $\mathcal{L}$ be a finite lattice such that $F_k(\mathcal{L}) = JI(\mathcal{L})$ and $\mathcal{L}$ satisfies* **T**. *Then $\mathcal{L}$ is* **2**-*simple.*

Proof. The simple homomorphic images of $\mathcal{L}$ are those isomorphic to $\mathcal{L}/\theta$ where θ is a maximal proper congruence of $\mathcal{L}$. Recall that $\Psi_p = \bigvee\{\Phi_q \colon \Phi_q \not\geq \Phi_p\}$. Thus Ψ_p is maximal in $Con(\mathcal{L})$ if and only if Φ_p is maximal in $JI(Con(\mathcal{L}))$. If p is join prime then the quotient map $f\colon \mathcal{L} \to \mathbf{2}$ with kernel Ψ_p maps $\uparrow p$ to 1, and all other elements to 0. Thus it suffices to prove that for $p \in JI(\mathcal{L})$, if $rk_F(p) > 0$ then Φ_p is not maximal in $JI(Con(\mathcal{L}))$.

Assume $rk_F(p) > 0$. As p is not join prime, there exists $q \in JI(\mathcal{L})$ such that $p\ D\ q$, whence $\Phi_p \leq \Phi_q$. By hypothesis this implies $rk_F(q) < rk(p)$ or $p \approx q$. If $rk_F(q) < rk_F(p)$ then $rk_F(p) \not\leq rk_F(q)$, whence $p\ D^n\ q$ fails to hold for all n. This implies that $\Phi_q \not\leq \Phi_p$, which combined with the fact that $\Phi_p \leq \Phi_q$ leads us to conclude $\Phi_p < \Phi_q$. On the other hand if $p \approx q$, we have by **T** that $p \leq p_* \vee q$. As such there exists $r \in JI(\mathcal{L})$ such that $r \leq p_*$ and $p\ D\ r$. Suppose $p \approx r$. By **T** it follows that

$$p \leq p_* \vee r = p_* < p,$$

which is a contradiction. Thus $rk_F(r) < rk_F(p)$ and we can apply the previous argument to conclude $\Phi_p < \Phi_r$. □

Finally we can give Nation's classification of congruence normal lattices.

Theorem 1.10. [11] *A finite lattice $\mathcal{L}$ is congruence normal if and only if $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ and $\mathcal{L}$ satisfies* **T**.

Nation observed in [12] that Day's doubling construction need not be limited to convex subsets of a lattice. In 2002 McNulty, Nation and Freese used a generalization of doubling as a critical tool in their work in [5]. The departure here was that instead of replacing old lattice elements with a 2-element interval, other lattices such as $\mathcal{M}_3$ were used. Heiko Reppe classified those subsets that can be used in Day's doubling construction in [16]. Reppe's result can likewise be extended to apply to inflation by other finite lattices. In the next section, we will combine these ideas to introduce a new way of looking at Day's doubling construction and its generalizations.

2. Constructing Finite Lattices by Inflation

One cannot double arbitrary subsets in a finite lattice and produce a lattice. This is seen in the example given in Figure 2. Clearly the elements a and b have no least upper bound. Note however that while the doubling does not produce a lattice, it does produce a well defined partially ordered set.

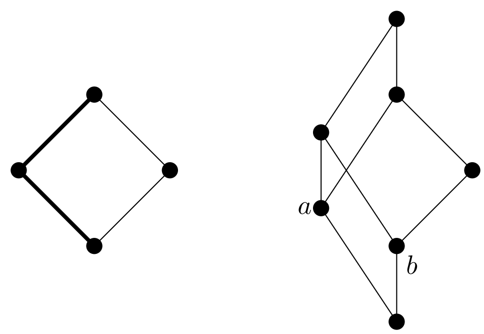

Figure 2. A doubling that fails to produce a lattice.

The question then is what kinds of non-convex sets can be doubled and still produce a lattice? A canonical example is due to Heiko Reppe and is reproduced in Figure 3.

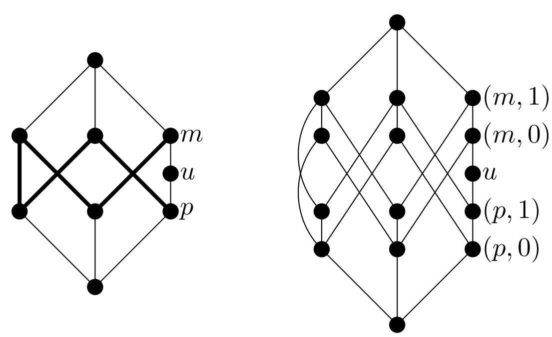

Figure 3. Reppe's lattice.

It is not hard to check that the doubling produces a lattice despite the fact that u witnesses the set that is doubled fails to be convex. Because the element u is not in the set that is doubled, we get $(p, 1) \leq u$ and $u \leq (m, 0)$, and thus $(p, 1) \leq (m, 0)$ in the transitive closure of the defining relation. This sort of "twisting" does not occur when convex sets are doubled.

We would like the class of lattices obtained by doubling to be closed under sublattices. Suppose we omit the element u but, as illustrated in Figure 4, maintain

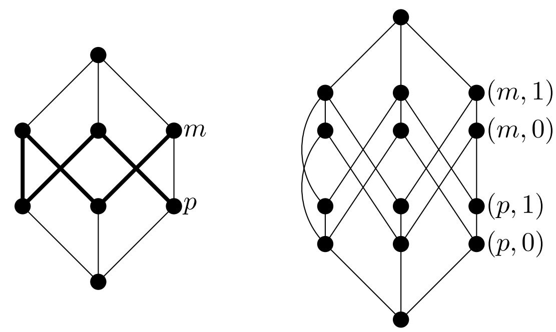

FIGURE 4. A, so far, unexplained doubling.

the ordering of the previous example with $(p, 1) \leq (m, 0)$. Even though the elements p and m are in the set of elements that is doubled, we ignore the covering edge $p \prec m$ during the construction.

Our first task is to define the new partial order created for a generalized type of doubling. We will also introduce the concept of replacing elements of the original lattice with ordered sets other than **2**.

Let $\mathcal{L}$ and $\mathcal{K}$ be ordered sets. Let E be a subset of the partial order $\leq_{\mathcal{L}}$ that is transitive, i.e., if $(x, y) \in E$ and $(y, z) \in E$ then $(x, z) \in E$. We will normally construct such sets by specifying a set of covering pairs and then taking their transitive closure. One can define a new partially ordered set $\mathcal{L} \star_E \mathcal{K}$ by "inflating" E as follows. Let E' denote the set of elements of $\mathcal{L}$ that are included in the relations of E. The universe of $\mathcal{L} \star_E \mathcal{K}$ is $(L \setminus E') \cup (E' \times K)$. Let $x, y \in L$ and $a, b \in K$. We define a binary relation $\sqsubseteq$ on $(L \setminus E') \cup (E' \times K)$ by

(1) $x \sqsubseteq y$ if $x \leq_{\mathcal{L}} y$,
(2) $(x, a) \sqsubseteq y$ if $x \leq_{\mathcal{L}} y$,
(3) $x \sqsubseteq (y, b)$ if $x \leq_{\mathcal{L}} y$,
(4) $(x, a) \sqsubseteq (y, b)$ if $x \leq_{\mathcal{L}} y$ and $a \leq_{\mathcal{K}} b$,
(5) $(x, a) \sqsubseteq (y, b)$ if $x \leq_{\mathcal{L}} y$, and there exists $(u, v) \notin E$ such that $x \leq u < v \leq y$.

While our primary interest is the case when $\mathcal{L}$ and $\mathcal{K}$ are finite lattices, this is not required for the construction to produce an ordered set.

Lemma 2.1. *The binary relation $\sqsubseteq$ is a partial order on $(L \setminus E') \cup (E' \times K)$.*

The proof of Lemma 2.1 is straightforward. It is convenient to note that if U and V are connected subsets of $\leq_{\mathcal{L}}$ such that $U \cup V$ is not connected, then

$$\mathcal{L} \star_{U \cup V} \mathcal{K} \cong (\mathcal{L} \star_U \mathcal{K}) \star_V \mathcal{K} \cong (\mathcal{L} \star_V \mathcal{K}) \star_U \mathcal{K}.$$

When inflating, we therefore restrict our attention to connected sets.

There exists a canonical map $h : \mathcal{L} \star_E \mathcal{K} \to \mathcal{L}$ that behaves as projection onto the first coordinate for $E' \times K$, and acts as the identity on $L \setminus E'$. Note that h is order preserving.

We take this opportunity to consider an example that illustrates the subtle, but extremely important, ways that this definition of inflation differs from Day's doubling construction in the previous section. Let $\mathcal{L}$ be the ordered set illustrated in Figure 5.

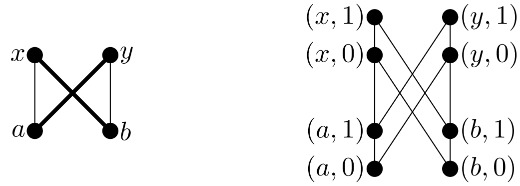

FIGURE 5. The ordered set $\mathcal{L}$ (left), and the ordered set $\mathcal{L} \star_E \mathbf{2}$.

Let $E := \{(a, y), (b, x)\}$. Inflating E by the 2-element lattice, which is illustrated by the bold edges on the left hand side of Figure 5, results in the ordered set illustrated on the right hand side of Figure 5. Unlike previously, there exist $u, v \in E'$ such that $(u, 1) \leq (v, 0)$. We need not restrict ourselves to inflating by $\mathbf{2}$: $\mathcal{L} \star_E \mathcal{M}_3$ is illustrated in Figure 6.

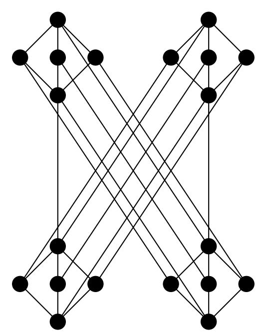

FIGURE 6. Inflation by the ordered set $\mathcal{M}_3$.

The next step is to determine those sets E such that $\mathcal{L} \star_E \mathcal{K}$ is a lattice. Given a lattice $\mathcal{L}$ and $E \subseteq \leq_{\mathcal{L}}$, for $x \leq y$ in $\mathcal{L}$ we write $[y/x] \subseteq E$ to mean that $(u, v) \in E$ whenever $x \leq u < v \leq y$.

Definition 2.2. *A subset $E \subseteq \leq_{\mathcal{L}}$ is called* all-or-nothing *if E' is connected and whenever $x < t < y$ are such that $[t/x] \subseteq E$ and $[y/t] \subseteq E$, then $[y/x] \subseteq E$.*

Note that if C is a convex subset of $\mathcal{L}$, then $E_C := \{(a, b) \in \leq_{\mathcal{L}} : a, b \in C\}$ is all-or-nothing. In this case we have that $\mathcal{L}[\mathbf{2}, C] \cong \mathcal{L} \star_{E_C} \mathbf{2}$.

A straightforward induction gives a useful equivalent formulation.

Lemma 2.3. *A subset $E \subseteq \leq_{\mathcal{L}}$ is* all-or-nothing *if and only if E' is connected and for all $n \geq 1$, whenever $x = x_0 < x_1 < \cdots < x_n = y$ are such that $[x_{i+1}/x_i] \subseteq E$ for $0 \leq i < n$, then $[y/x] \subseteq E$.*

Corollary 2.4. *Let $\mathcal{L}$ be a finite lattice. A subset $E \subseteq \leq_{\mathcal{L}}$ is* all-or-nothing *if and only if E' is connected and whenever $x = x_0 \prec x_1 \prec \cdots \prec x_n = y$ is a covering chain with $(x_i, x_{i+1}) \in E$ for $0 \leq i < n$, then $[y/x] \subseteq E$.*

Now we can specify when inflation yields a lattice.

Theorem 2.5. *Given a lattice $\mathcal{L}$, a subset $E \subseteq \leq_{\mathcal{L}}$, and a nontrivial lattice $\mathcal{K}$ with 0 and 1, the ordered set $\mathcal{L} \star_E \mathcal{K}$ is a lattice if and only if E is all-or-nothing.*

Proof. First, assume that E is not an all-or-nothing subset and let $x < t < y$ witness this failure, so that $[t/x] \subseteq E$ and $[y/t] \subseteq E$ but $[y/x] \nsubseteq E$. In $\mathcal{L} \star_E \mathcal{K}$ consider the pair $(x, 1)$ and $(t, 0)$. Now $(t, 1)$ is a minimal common upper bound, because $[t/x] \subseteq E$. However, $(y, 0)$ is also above both elements, because $[y/x] \nsubseteq E$, while $(t, 1) \nleq (y, 0)$ since $[t/y] \subseteq E$. Hence $(x, 1)$ and $(t, 0)$ have no join in $\mathcal{L} \star_E \mathcal{K}$.

Conversely, assume that E is an all-or-nothing set. We need to show that the meet and join operations are well defined for every pair of elements in $\mathcal{L} \star_E \mathcal{K}$. We prove that joins are well defined and meets follow by duality. Let $x, y \in \mathcal{L}$ and $a, b \in \mathcal{K}$. Note that in any case below if $x \vee y \notin E'$ then the join will be $x \vee y$ in $\mathcal{L} \star_E \mathcal{K}$, so assume below that $x \vee y \in E'$.

Observe that if $(z, c) \geq (x, a)$ and $(z, c) \geq (y, b)$, then $z \geq x \vee y$. Moreover, if $c \ngeq a$, say, then either $[(x \vee y)/x] \nsubseteq E$ or $[z/(x \vee y)] \nsubseteq E$. Using these facts repeatedly, we calculate as follows.

(1) $x \vee y = (x \vee y, 0)$.
(2) $x \vee (y, b) = (x \vee y, 0)$ if $[(x \vee y)/y] \nsubseteq E$.
(3) $x \vee (y, b) = (x \vee y, b)$ if $[(x \vee y)/y] \subseteq E$.
(4) $(x, a) \vee (y, b) = (x \vee y, 0)$ if $[(x \vee y)/x] \nsubseteq E$ and $[(x \vee y)/y] \nsubseteq E$.
(5) $(x, a) \vee (y, b) = (x \vee y, a)$ if $[(x \vee y)/x] \subseteq E$ and $[(x \vee y)/y] \nsubseteq E$.
(6) $(x, a) \vee (y, b) = (x \vee y, b)$ if $[(x \vee y)/x] \nsubseteq E$ and $[(x \vee y)/y] \subseteq E$.
(7) $(x, a) \vee (y, b) = (x \vee y, a \vee b)$ if $[(x \vee y)/x] \subseteq E$ and $[(x \vee y)/y] \subseteq E$.

This completes the proof. □

Before proceeding we take some time to compare this new inflation construction to the method used by Reppe in [16], using sets of points rather than sets of edges.

Definition 2.6. *Let $\mathcal{L}$ be a lattice. A connected subset $M \subseteq \mathcal{L}$ is called* municipal *if and only if the following condition is satisfied for all x in $\mathcal{L}$: if there exists $a \in M$ such that $a \wedge x \in M$ and $a \vee x \in M$, then $x \in M$.*

Theorem 2.7. [16] *Let $\mathcal{L}$ and $\mathcal{K}$ be finite lattices, and let $S \subseteq L$. Then $\mathcal{L}[\mathcal{K}, S]$ is a lattice if and only if S is municipal.*

Given a connected subset S in a finite lattice $\mathcal{L}$, let E_S be the transitive closure of the set of edges $\{(x, y) \in \leq_{\mathcal{L}} : x, y \in S \text{ and } x \prec y\}$. Using Corollary 2.4, one can show that if S is municipal then E_S is all-or-nothing. On the other hand, an all-or-nothing set E need not be E_S for $S = E'$. Thus, for example, the all-or-nothing set that is inflated in Figure 4 does not correspond to E_M for a municipal set M.

Sufficient for our purposes is the following observation.

Lemma 2.8. *Let $\mathcal{L}$ and $\mathcal{K}$ be finite lattices, and let E be an all-or-nothing set in $\mathcal{L}$. If*

(‡) *$x, y \in E'$ with $x \leq y$ and $[y/x] \nsubseteq E$ implies that there exists $t \in y/x$ such that $t \notin E'$*

then E' is municipal and $\mathcal{L}[\mathcal{K}, E'] = \mathcal{L} \star_E \mathcal{K}$.

Indeed, the condition (‡) ensures that the ordering is the same for both constructions.

Theorem 2.9. *Let $\mathcal{L}$ and $\mathcal{K}$ be finite lattices, and let E be an all-or-nothing set in $\mathcal{L}$. Then there exists a lattice $\widehat{\mathcal{L}}$ and a municipal set $M \subseteq \widehat{L}$ such that $\mathcal{L} \star_E \mathcal{K}$ is a sublattice of $\widehat{\mathcal{L}}[\mathcal{K}, M]$.*

Proof. Let $U = \{(u,v) \in \leq_{\mathcal{L}} \colon u,v \in E',\ (u,v) \notin E\}$. We construct $\widehat{\mathcal{L}}$ as follows. The universe of $\widehat{\mathcal{L}}$ is the set of elements $L \cup \{t_{(u,v)} : (u,v) \in U\}$. Each $t_{(u,v)}$ is both join and meet irreducible in $\widehat{\mathcal{L}}$, with unique lower cover u and unique upper cover v. The rest of $\leq_{\widehat{\mathcal{L}}}$ is inherited from $\leq_{\mathcal{L}}$. Then let $M = E'$. As a subset of $\widehat{\mathcal{L}}$, E' satisfies the condition (‡) of Lemma 2.8, and thus $\mathcal{L} \star_E \mathcal{K} \leq \widehat{\mathcal{L}} \star_E \mathcal{K} = \widehat{\mathcal{L}}[\mathcal{K}, M]$. (By Corollary 2.4, it would suffice to add new elements for pairs $(u,v) \in U$ with $u \prec v$.) □

Since the set of edges E_S is all-or-nothing whenever S is municipal, any finite lattice that can be constructed by inflating municipal sets can be produced by inflating all-or-nothing sets. The other way around, Theorem 2 says that we could forgo dealing with edges in a lattice by adding new doubly irreducible elements, inflating a municipal set of points, and then taking a sublattice by removing the elements that were just added. We adopt the approach of using edges and all-or-nothing sets as it is easier to classify lattices constructible in this way than it is to directly classify sublattices of finite lattices that can be obtained by inflating municipal sets of points.

3. Inflation and Congruences

Having defined a new construction involving inflation by finite lattices, our goal is to classify those finite lattices that can be constructed in this manner. In the case of Day's doubling construction, such lattices can be classified by the properties of their congruences. This in turn allows these lattices to be described in terms of a ranking system on their join irreducible elements that measures the depth of Φ_p in the ordered set of join irreducible congruences of such a lattice.

The new inflation construction allows us to inflate by any finite lattice $\mathcal{K}$. Clearly $\mathcal{K} \cong \mathbf{1} \star_{\{(1,1)\}} \mathcal{K}$, where $\mathbf{1}$ is the 1-element lattice and 1 denotes its single element, so in a trivial way we can say any finite lattice can be constructed by inflation. As we have seen, some finite lattices can be constructed by inflation using smaller finite lattices. We say that a finite lattice $\mathcal{L}$ is *inflatable* if

(1) $\mathcal{L}$ is simple, or
(2) There is a sequence of lattices $\mathcal{L}_i$, $0 \leq i \leq n$, such that $\mathcal{L}_0 = \mathbf{1}$, $\mathcal{L}_n \cong \mathcal{L}$, and $\mathcal{L}_i = \mathcal{L}_{i-1}$ where $E_{i-1} \subseteq \leq_{\mathcal{L}_{i-1}}$ is all-or-nothing and $|\mathcal{K}_{i-1}| < |\mathcal{L}|$ for every i.

Not every finite lattice can be constructed in such a manner. Indeed, consider the lattice shown in Figure 7. It will be proven later that this lattice is not inflatable. This can also be seen by trial and error.

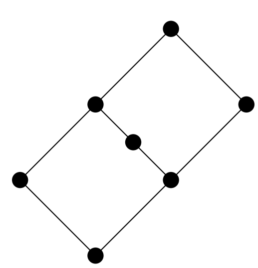

Figure 7. A lattice that is not inflatable.

As with those lattices constructible by doubling convex sets, we will classify inflatable lattices by the properties of their congruences. We begin by examining what kind of congruences can arise from inflation.

Observe that if $\mathcal{L}$ and $\mathcal{K}$ are finite lattices, $E \subseteq \leq_{\mathcal{L}}$ is an all-or-nothing set and $h\colon \mathcal{L} \star_E \mathcal{K} \to \mathcal{L}$ is the canonical homomorphism, then

$$\alpha_h(x)/\beta_h(x) \cong \mathcal{K}$$

for every $x \in E'$. A *weaving polynomial* on a lattice $\mathcal{L}$ is a member of the set W of unary functions defined recursively by the rules

(1) $id_{\mathcal{L}} \in W$,
(2) if $w \in W$ and $a \in \mathcal{L}$, then the functions u, v such that $u(x) = w(x) \wedge a$ and $v(x) = w(x) \vee a$ are in W,
(3) only these functions are in W.

Thus every weaving polynomial looks something like

$$w(x) = (\cdots (x \vee s_1) \wedge s_2) \vee \cdots \vee s_{n-1}) \wedge s_n),$$

or dually, or with some successive joins and meets.

We say that intervals y/x and t/s in a lattice $\mathcal{L}$ are *projectively isomorphic* if $y/x \approx t/s$ and there exists a weaving polynomial w such that when restricted y/x we have that w is an isomorphism of the sublattices y/x and t/s.

Lemma 3.1. *Let $\mathcal{L}$ and $\mathcal{K}$ be finite lattices and let $E \subseteq \leq_{\mathcal{L}}$ be an all-or-nothing set. If $x, y \in E'$, then $\alpha_h(x)/\beta_h(x)$ is projectively isomorphic to $\alpha_h(y)/\beta_h(y)$ in $\mathcal{L} \star_E \mathcal{K}$.*

Proof. The canonical homomorphism $h\colon \mathcal{L} \star_E \mathcal{K} \to \mathcal{L}$ is bounded as $\mathcal{L}$ and $\mathcal{K}$ are finite. Let $\alpha := \alpha_h$ and $\beta := \beta_h$. As E' is connected there exists a sequence of elements of E' such that

$$\beta(x) = u_0 \leq v_1 \geq u_1 \leq \cdots \leq v_n \geq u_n = \beta(y),$$

$(u_{i-1}, v_i) \in E$, and $(u_i, v_i) \in E$ for every i. Note that for every i we have $\beta(v_i) = (t_i, 0)$ and $\alpha(u_i) = (s_i, 1)$ for some $s_i, t_i \in E'$. From the definition of the join operation in $\mathcal{L} \star_E \mathcal{K}$ it follows that for every $1 \leq i \leq n$ and $a \in \mathcal{K}$

$$(u_{i-1}, a) \vee \beta(v_i) = (v_i, a),$$
$$(v_i, a) \wedge \alpha(u_i) = (u_i, a).$$

We conclude that the weaving polynomial

$$w(x) = (\cdots (x \vee \beta(v_1)) \wedge (\alpha(u_1)) \vee \cdots \vee \beta(v_n)) \wedge \alpha(u_n)).$$

witnesses that $\alpha(x)/\beta(x)$ is projectively isomorphic to $\alpha(y)/\beta(y)$. $\square$

For a lattice $\mathcal{L}$, let $Con(\mathcal{L})$ be the lattice of congruence relations of $\mathcal{L}$ ordered by inclusion. For a congruence $\theta \in Con(\mathcal{L})$ and $x \in L$, let $\alpha(x)$ and $\beta(x)$ denote the greatest and least members, respectively, of the θ-class $[x]$.

Definition 3.2. *A congruence $\theta \in Con(\mathcal{L})$ is an* inflation congruence *if*

(1) *for any two nontrivial θ-classes $[x]$, $[y]$ there is a weaving polynomial w on $\mathcal{L}$ such that the restriction $w : [x] \to [y]$ is a projective isomorphism,*
(2) *any two such weaving polynomials w, w' satisfy $w(t) = w'(t)$ for all $t \in [x]$,*

(3) *if* $[x] \leq [y]$ *in* $\mathcal{L}/\theta$, *then either* $\alpha(x) \leq \beta(y)$ *or the maps*

$$u : [x] \to [y] \text{ via } u(s) = s \vee \beta(y)$$
$$v : [y] \to [x] \text{ via } v(t) = t \wedge \alpha(x)$$

are inverse isomorphisms.

In view of Lemma 3.1 and the calculation in its proof, if E is a connected all-or-nothing set in a finite lattice, then the kernel $\ker h$ of the standard map $h : \mathcal{L} \star_E \mathcal{K}$ is an inflation congruence. The converse is also true.

Lemma 3.3. *If* θ *is an inflation congruence on a finite lattice* $\mathcal{L}$, *then there are a finite lattice* $\mathcal{K}$ *and a connected all-or-nothing set* E *such that* $\mathcal{L} \cong \mathcal{L}/\theta \star_E \mathcal{K}$.

Proof. Let $[a]$ be any nontrivial block of θ, and set $\mathcal{K} = [a]$. Let E be the set of all pairs $([b], [c])$ in $\mathcal{L}/\theta$ such that $[b]$ is nontrivial, $[b] \leq [c]$ and $\alpha(b) \nleq \beta(c)$. For these pairs, by (3) the polynomials $u(s) = s \vee \beta(c)$ and $v(t) = t \wedge \alpha(b)$ are inverse projective isomorphisms between $[b]$ and $[c]$. It follows from this that $\mathcal{E}$ is an all-or-nothing set. For if $[b] \leq [a] \leq [d] \leq [c]$, then $\alpha(b) \leq \alpha(a)$ and $\beta(d) \leq \beta(c)$. Hence $\alpha(a) \nleq \beta(d)$ because $\alpha(b) \nleq \beta(c)$, and so $([a], [d]) \in E$.

We want to construct an isomorphism $g : \mathcal{L} \cong \mathcal{L}/\theta \star_E \mathcal{K}$. Consider an element $d \in L$. If $|[d]| = 1$ then $d \notin E'$ and set $g(d) = [d]$. On the other hand, if $|[d]| > 1$ then $d \in E'$ and there is a weaving polynomial giving a projective isomorphism $w : [d] \to [a]$. In this case, set $g(d) = ([d], w(d))$. By (2), the map g is well-defined.

It is straightforward that g is an order-preserving bijection, but we must also check that its inverse is order-preserving, i.e., that the order relation on $\mathcal{L}$ and $\mathcal{L}/\theta \star_E \mathcal{K}$ are the same.

Assume $g(d) \leq g(e)$. The crucial case is when $[d]$, $[e]$ are both nontrivial. Then, for the appropriate weaving polynomials, $([d], w(d)) \leq_{\mathcal{L}/\theta \star_E \mathcal{K}} ([e], w'(e))$. In particular, $[d] \leq [e]$. If $\alpha(d) \leq \beta(e)$, then in $\mathcal{L}$ we have $d \leq \alpha(d) \leq \beta(d) \leq e$, as desired. Otherwise, $([d], [e]) \in E$ and $w(d) \leq w'(e)$ in $\mathcal{K} = [a]$. As above, let $v(t) = t \wedge \alpha(d)$ for $t \in [e]$. Then $d = v(w')^{-1}w(d)$ by (2), whence

$$d = v(w')^{-1}w(d) \leq v(w')^{-1}w'(e) = v(e) \leq e,$$

so that again $d \leq e$, as required. □

Note that (3) of Definition 3.2 is a local version of (2). Examples show that in general both parts are needed. However, if $\mathcal{K}$ is simple and has no proper automorphisms, e.g., $\mathcal{K} = \mathbf{2}$, then (1) and (3) suffice.

Definition 3.4. *A strictly ascending sequence* $0_{Con(\mathcal{L})} < \theta_1 < \theta_2 < \cdots < \theta_n = 1_{Con(\mathcal{L})}$ *in* $Con(\mathcal{L})$ *is called an* inflation sequence *if* $n \geq 2$ *and for* $1 \leq i \leq n$, θ_i/θ_{i-1} *is an inflation congruence in* $\mathcal{L}/\theta_{i-1}$.

Recursive application of Lemma 3.3 yields our main result.

Theorem 3.5. *A finite lattice* $\mathcal{L}$ *is inflatable if and only if there exists an inflation sequence in* $Con(\mathcal{L})$.

The classification theorem for inflatable lattices looks, at first glance, quite different from the classification theorem for congruence normal lattices. Clearly, such lattices are inflatable. Given a congruence normal lattice $\mathcal{L}$, we extend the partial order on the set of join prime congruences of $\mathcal{L}$. To do so, we index $JI(\mathcal{L})$ as p_i for $1 \leq i \leq n$ in such a way that $i \leq j$ implies $rk_F(p_i) \geq rk_F(p_j)$. Note that $\Phi_{p_i} \leq \Phi_{p_j}$

implies $i \leq j$. We may define an ascending chain of congruences from $0_{Con(\mathcal{L})}$ to $1_{Con(\mathcal{L})}$ by defining $\theta_0 = 0_{Con(\mathcal{L})}$ and

$$\theta_i := \bigvee \{\Phi_{p_k} \colon 1 \leq k \leq i\}$$

for $1 \leq i \leq n$.

We know from the proof of Theorem 1.10 that if $x \in \mathcal{L}/\theta_{i-1}$ and $\alpha_i(x) \neq \beta_i(x)$ then $\alpha_i(x)/\beta_i(x)$ is projectively isomorphic to the interval $h_{i-1}(p_i)/h_{i-1}(p_i)_*$. We conclude that $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n$ is an inflation sequence.

Given any inflation sequence $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n$ in $Con(\mathcal{L})$ for congruence normal lattice $\mathcal{L}$, consider the interval θ_i/θ_{i-1} in $Con(\mathcal{L})$. If $\theta_{i-1} \leq \varphi \leq \theta_i$ then $(x, y) \in \varphi/\theta_{i-1}$ implies (x, y) is in the kernel of h_1. As such, the covering pairs that generate φ are all in $\alpha_i(x)/\beta_i(x)$ for some $x \in \mathcal{L}/\theta_i$. When such a pair is collapsed, it will collapse exactly those pairs that it is projectively isomorphic to as a result of the inflation sequence. It follows that the interval θ_i/θ_{i-1} in $Con(\mathcal{L})$ is isomorphic to $Con(\mathcal{K})$, where $\mathcal{K} \cong \alpha_i(x)/\beta_i(x)$ for any $x \in \mathcal{L}/\theta_{i-1}$ such that $\alpha_i(x) \neq \beta_i(x)$.

The class of congruence normal lattices form a *prevariety*: this class is closed under homomorphic images, sublattices, and finite direct products. It is natural to ask if the class of inflatable lattices is also a prevariety. The class of inflatable lattices is closed under finite direct products as a result of the fact that if $\mathcal{L}_1$ and $\mathcal{L}_2$ have inflation sequences $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_m$ and $\psi_0 < \psi_1 < \psi_2 < \cdots < \psi_n$, respectively, then we can construct an inflation sequence in $\mathcal{L}_1 \times \mathcal{L}_2$ consisting of congruences of the form $\theta_i \times \psi_j$.

To see that the class of inflatable lattices is closed under sublattices requires more care. Assume inductively that $\mathcal{L}$ is an inflatable lattice such that every sublattice $\mathcal{S}_{\mathcal{L}} \leq \mathcal{L}$ is inflatable. Let $E \subseteq \leq_{\mathcal{L}}$ be an all or nothing set and $\mathcal{K}$ a finite lattice. Given a sublattice $\mathcal{S} \leq \mathcal{L} \star_E \mathcal{K}$ we can construct $\mathcal{S}$ by beginning with some $\mathcal{S}_{\mathcal{L}} \leq \mathcal{L}$ and inflating an all-or-nothing subset of E by a sublattice $\mathcal{S}_{\mathcal{K}} \leq \mathcal{K}$. Note however that for an arbitrary sublattice $\mathcal{S}$ we may need to perform multiple inflations, as there is no guarantee that $\mathcal{S} \cap (E' \times \mathcal{K})$ is connected.

The class of inflatable lattices is, in general, not preserved under homomorphisms. Clearly if $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n$ in $Con(\mathcal{L})$ is an inflation sequence then $\mathcal{L}/\theta_i$ is inflatable for every $0 \leq i \leq n$. However, if $\theta \in Con(\mathcal{L})$ is not part of an inflation sequence, then $\mathcal{L}/\theta$ may not be inflatable, see Figure 9 below.

While inflatable lattices do not form a prevariety, when we restrict our choice of lattices $\mathcal{K}$ in the construction, there are still notable structural properties that are preserved by inflation. We say a finite lattice $\mathcal{L}$ is *projectively simple* if $\mathcal{L}$ is simple and $p \approx q$ for all $p, q \in JI(\mathcal{L})$. When we restrict ourselves to inflating by projectively simple lattices, we obtain a subclass of those lattices such that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$.

Theorem 3.6. *Let $\mathcal{L}$ be a finite lattice such that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$, $E \subseteq \leq_{\mathcal{L}}$ be an all-or-nothing set, and $\mathcal{K}$ be a finite projectively simple lattice. Then $F_{k+1}(\mathcal{L} \star_E \mathcal{K}) = JI(\mathcal{L} \star_E \mathcal{K})$.*

Proof. Note that $JI(\mathcal{L} \star_E \mathcal{K}) = J_{-1} \cup J_0 \cup J_1$ where

$$\begin{aligned} J_{-1} &:= \{x \in JI(\mathcal{L}) \colon x \notin E'\}, \\ J_0 &:= \{(p, 0) \colon p \in JI(\mathcal{L}) \cap E'\}, \\ J_1 &:= \{(x, a) \colon a \in JI(\mathcal{K}), w \prec x \Rightarrow (w, x) \notin E\}. \end{aligned}$$

The lattice $\mathcal{K}$ is simple, so it follows that if $p \in J_1$ then Φ_p is the kernel of the canonical homomorphism $h\colon \mathcal{L} \star_E \mathcal{K} \to \mathcal{L}$. Let $p, q \in J_1$ with $p \neq q$. We have that $p \in \alpha_h(x)/\beta_h(x)$ and $q \in \alpha_h(y)/\beta_h(y)$ for some $x, y \in \mathcal{L}$. Since $\mathcal{K}$ is projectively simple and $\alpha_h(x)/\beta_h(x)$ is projectively isomorphic to $\alpha_h(y)/\beta_h(y)$, we conclude that $p \approx q$. As $F_k(\mathcal{L}) = JI(\mathcal{L})$, it follows by Lemma 1.7 that $F_n(\mathcal{L} \star_E \mathcal{K}) = JI(\mathcal{L} \star_E \mathcal{K})$ for some $n \in \omega$. Moreover, by Corollary 1.8 we have that $rk_F(p)$ is equal to the depth of Φ_p for every $p \in JI(\mathcal{L} \star_E \mathcal{K})$. In particular, if $p \in J_1$, then $rk_F(p) \leq k+1$. This completes the proof. □

We see from the lattice shown in Figure 8 that not every lattice such that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ can be constructed from the 1-element lattice by inflating all-or-nothing sets by projectively simple lattices. The two minimal nontrivial congruences of $\mathcal{L}$ are Φ_p and Φ_q, where p and q are as indicated in Figure 8. Neither Φ_p nor Φ_q satisfies the conditions necessary to be part of an inflation sequence, whence $\mathcal{L}$ is not inflatable.

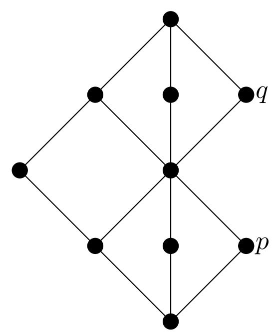

FIGURE 8. A lattice $\mathcal{L}$ disproving the converse of Theorem 3.6.

4. Binary Cut-Through Codable Lattices

For a lattice congruence θ we say the *norm of* θ, denoted $\|\theta\|$, is the cardinality of the largest block of θ.

Definition 4.1. *A finite lattice $\mathcal{L}$ is called n-ary cut-through codable if and only if there exists a chain*

$$0_{Con(\mathcal{L})} = \theta_0 < \theta_1 < \cdots < \theta_k = 1_{Con(\mathcal{L})}$$

such that $\|\theta_j/\theta_{j-1}\| \leq n$ for all $j \leq k$.

Lattices that are n-ary cut-through codable were studied by Sun and Li in [17]. Note that n-ary cut-through codability is preserved under the formation of sublattices and finite direct products. However, Sun and Li showed that for every n there exists a lattice that is n-ary cut-through codable and has a homomorphic image that fails to be n-ary cut-through codable.

There is a nice description of binary cut-through codable lattices in terms of inflation.

Theorem 4.2. *A finite lattice $\mathcal{L}$ is binary cut-through codable if and only if $\mathcal{L}$ is inflatable and $\mathcal{K}_i \cong \mathbf{2}$ for every $1 \leq i \leq n$.*

Proof. Assume that $\mathcal{L}$ is inflatable and $\mathcal{K}_i \cong \mathbf{2}$ for every $1 \leq i \leq n$. By Theorem 3.5 we know $\mathcal{L}$ has an inflation sequence $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n$. By assumption for every $h_i \colon \mathcal{L}/\theta_{i-1} \to \mathcal{L}/\theta_i$ we have that $\left|h_i^{-1}(x)\right| \leq 2$ for all $x \in \mathcal{L}/\theta_{i-1}$. It follows that the inflation sequence witnesses that $\mathcal{L}$ is binary cut-through codable.

Conversely, assume that $\mathcal{L}$ is binary cut-through codable. Let $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n$ in $Con(\mathcal{L})$ be a chain that witnesses this. We claim that this chain is an inflation sequence. If we prove that conditions that θ_1 is an inflation congruence, the desired conclusion holds by induction on n. Note that as $\mathbf{2}$ has a trivial automorphism group, we need only show θ_1 satisfies (1) and (3) of Definition 3.2.

We first reduce to the case that $0_{Con(\mathcal{L})} \prec \theta_1$. Assume there exists a congruence θ such that $0_{Con(\mathcal{L})} < \theta < \theta_1$. By the assumption that $\mathcal{L}$ is binary cut-through codable we have that $\left\|0_{Con(\mathcal{L})}/\theta\right\| = 2$ and $\|\theta/\theta_1\| = 2$. Relabeling as necessary, we may extend our chain to include θ. Thus we can assume, without loss of generality, that $0_{Con(\mathcal{L})} \prec \theta_1$.

We have that $\theta_1 = \Phi_p$ for some $p \in JI(\mathcal{L})$, whence by assumption $\|\Phi_p\| = 2$. Let $[x]$ denote the block of Φ_p containing x for any $x \in \mathcal{L}$. Given $[a]$ such that $|[a]| = 2$, if there is $b \in [a]$ that is incomparable to a this would imply that $|[a]| > 2$. So we assume $b \in [a]$ and $a \prec b$. We claim that p/p_* is projectively isomorphic to b/a.

If $(a, b) \in \Phi_p$ with $a < b$ then there exists a sequence of weaving polynomials $w_1, w_2, \ldots, w_k$ such that

$$a = w_1(p_*) < w_1(p) = w_2(p_*) < w_2(p) = \cdots = w_k(p_*) < w_k(p) = b.$$

For any weaving polynomial w we have that $w(p_*) \prec w(p)$ or $w(p_*) = w(p)$, once again by the fact that $\|\Phi_p\| = 2$. It follows that p/p_* is projectively isomorphic to b/a by the weaving polynomial w_1, whence $\mathcal{L}$ satisfies both (1) and (3). □

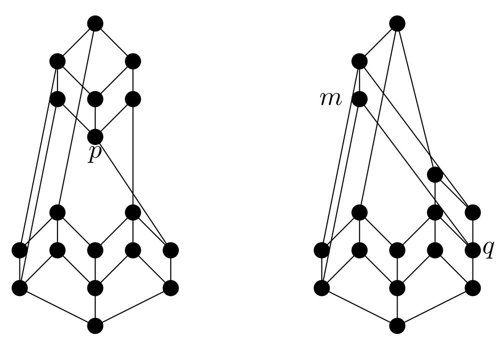

FIGURE 9. The lattice $\mathcal{L}$ (left), and its quotient $\mathcal{L}/\Phi_p$ (right).

In the case of binary cut-through codable lattices, because every inflation is by $\mathbf{2}$, we know that any inflation sequence may be extended to a covering chain. If $\mathcal{L}$ is binary cut-through codable and $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n$ in $Con(\mathcal{L})$ is a chain that witnesses this, we know from the proof of Theorem 4.2 that this chain can be extended to an inflation sequence. As such, we may assume $\theta_{i-1} \prec \theta_i$ for all $1 \leq i \leq n$. Thus $\theta_i = \theta_{i-1} \vee \Phi_p$ for some $p \in JI(\mathcal{L})$. This resembles the kind of inflation sequence that we saw with congruence normal lattices. Moreover, by Theorems 4.2 and 3.6, we see that if $\mathcal{L}$ is a binary cut-through codable lattice, then $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ and $\mathcal{L}$ is $\mathbf{2}$-simple.

Given a finite lattice $\mathcal{L}$ such that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ and $\mathcal{L}$ is **2**-simple, how can we insure that there exists an inflation sequence in $Con(\mathcal{L})$ that witnesses that $\mathcal{L}$ is binary cut-through codable? If $\mathcal{L}$ is congruence normal, any maximal chain in $Con(\mathcal{L})$ will witness that $\mathcal{L}$ is binary cut-through codable. Should there exist $p \in JI(\mathcal{L})$ and $m \in MI(\mathcal{L})$ such that $p \leq m$ and $p \approx m$, more care is needed. In Figure 9 the lattice $\mathcal{L}$ is binary cut-through codable, but $\mathcal{L}/\Phi_p$ is not because $q \approx m$.

Definition 4.3. *A lattice variety* $\mathbf{V}$ *is called n-ary cut-through codable if and only if every finite* $\mathcal{L} \in \mathbf{V}$ *is n-ary cut-through codable.*

As we have seen, n-ary codability is preserved by finite direct products and sublattices. It follows, using Jónsson's Lemma, that $Var(\mathcal{K})$ is n-ary cut-through codable if and only if every finite lattice $\mathcal{L}$ that is a homomorphic image of a sublattice of $\mathcal{K}$ is n-ary cut-through codable. The following was first shown by Sun and Li.

Lemma 4.4. [17] *The class of n-ary cut-through codable varieties forms a lattice ideal in* Λ*, the lattice of all lattice varieties ordered by inclusion.*

Clearly any finite lattice $\mathcal{L}$ such that $Var(\mathcal{L})$ is binary cut-through codable is itself binary cut-through codable. However, not every lattice $\mathcal{L}$ that is binary cut-through codable generates a variety $Var(\mathcal{L})$ that is binary cut-through codable, e.g., the lattice $\mathcal{L}$ illustrated in Figure 9.

Even a finite lattice $\mathcal{L}$ such that every quotient lattice of $\mathcal{L}$ is binary cut through codable may not generate a variety that is binary cut-through codable. Specifically, $\mathcal{L}$ may contain a sublattice $\mathcal{K}$ whose homomorphic images include lattices that are not binary cut-through codable. Such a lattice is depicted in Figure 10. In this case $\mathcal{L}$ and all its quotients are binary cut-through codable, but the sublattice consisting of the interval y/x in $\mathcal{L}$ is isomorphic to the lattice shown in Figure 9, which has homomorphic images that are not binary cut-through codable.

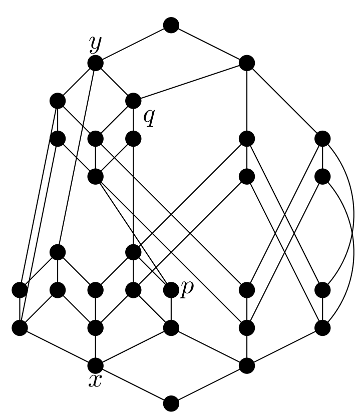

FIGURE 10. A binary cut-through lattice $\mathcal{L}$ such that every homomorphic image and sublattice is binary cut-through codable, but not every homomorphic image of a sublattice is. Thus $Var(\mathcal{L})$ is not binary cut-through codable.

There exists a rich class of finite lattices generating binary cut-through codable varieties that are not congruence normal. We say that a lattice $\mathcal{L}$ has the *binary square property* if for $\sigma, \tau \in Con(\mathcal{L})$ whenever $\sigma \wedge \tau \prec \sigma, \tau \prec \sigma \vee \tau$ and

$$\|\sigma/\sigma \wedge \tau\| \leq 2$$
$$\|\tau/\sigma \wedge \tau\| \leq 2$$
$$\|\sigma \vee \tau/\sigma\| \leq 2$$

then $\|\sigma \vee \tau/\tau\| \leq 2$.

This is a rather strong property, but we will construct an infinite class of lattices that satisfy it. It is straightforward to show that $\|\sigma/\sigma \wedge \tau\| \leq \|\sigma \vee \tau/\tau\|$, so if the latter is at most 2 then so is the former.

Lemma 4.5. *If $\mathcal{L}$ is a binary cut-through codable lattice and $\mathcal{S}$ is a sublattice of $\mathcal{L}$ that has the binary square property, then $\|\beta/\alpha\| = 2$ for every covering pair $\alpha \prec \beta$ in $Con(\mathcal{S})$.*

Proof. Since $\mathcal{L}$ is binary cut-through codable, there is a covering chain $0 \prec \gamma_1 \prec \cdots \prec \gamma_m = 1$ in $Con(\mathcal{L})$ with $\|\gamma_i/\gamma_{i-1}\| = 2$. The restriction of these congruences to $\mathcal{S}$ may not be a covering chain in $Con(\mathcal{S})$, but it can be extended to one, yielding a covering chain $0_{Con(\mathcal{L})} = \rho_0 \prec \rho_1 \prec \cdots \prec \rho_n = 1_{Con(\mathcal{L})}$ in $Con(\mathcal{S})$ with $\|\rho_i/\rho_{i-1}\| = 2$.

Let $\alpha \prec \beta$ in $Con(\mathcal{S})$. We show inductively that $\|\beta \wedge \rho_k/\alpha \wedge \rho_k\| \leq 2$. For $k = 0$ or $k = 1$ this is trivial, and for $k = n$ it yields the desired conclusion $\|\beta/\alpha\| \leq 2$.

In the free distributive lattice generated by the two chains $\alpha < \beta$ and $\rho_{k-1} < \rho_k$, we have

$$\rho_{k-1} \leq \rho_{k-1} \vee (\alpha \wedge \rho_k) \leq \rho_{k-1} \vee (\beta \wedge \rho_k) \leq \rho_k.$$

Since $\rho_{k-1} \prec \rho_k$, two of these three inclusions are equality and the other is strict.

First suppose $\rho_{k-1} < \rho_{k-1} \vee (\alpha \wedge \rho_k) = \rho_k$. Note that ρ_k/ρ_{k-1} projects down to $\beta \wedge \rho_k/\beta \wedge \rho_{k-1}$, which in turn projects down to $\alpha \wedge \rho_k/\alpha \wedge \rho_{k-1}$. Thus in $Con(\mathcal{S})$ we have the square $\alpha \wedge \rho_{k-1} \prec \alpha \wedge \rho_k,\ \beta \wedge \rho_{k-1} \prec \beta \wedge \rho_k$, with three of the four sides having norm at most 2 by the inductive assumption. By the binary square property for $\mathcal{S}$, the remaining side $\beta \wedge \rho_k/\alpha \wedge \rho_k$ also has norm at most 2, which was to be proved for the induction.

Next, assume the middle inequality is strict, so that $\rho_{k-1} = \rho_{k-1} \vee (\alpha \wedge \rho_k) < \rho_{k-1} \vee (\beta \wedge \rho_k) = \rho_k$. In this case, we claim that ρ_k/ρ_{k-1} projects down to $\beta \wedge \rho_k/\alpha \wedge \rho_k$, which will give the latter interval norm at most 2. The join relation for projectivity is the second equality of our assumption. So we consider $\rho_{k-1} \wedge (\beta \wedge \rho_k) = \beta \wedge \rho_{k-1}$. Now $\alpha \wedge \rho_k \preceq \beta \wedge \rho_k$. Using the first equality of our assumption, $\alpha \wedge \rho_k \leq \beta \wedge \rho_{k-1} \leq \beta \wedge \rho_k$. Our prior work insures that $\beta \wedge \rho_k \not\leq \rho_{k-1}$, so the second inequality is strict. We conclude that $\alpha \wedge \rho_k = \beta \wedge \rho_{k-1}$, yielding the desired projectivity and norm.

Finally, we consider the possibility that the third inequality is strict. But in that case, we have $\beta \wedge \rho_k \leq \rho_{k-1}$, whence $\beta \wedge \rho_k = \beta \wedge \rho_{k-1}$ and $\alpha \wedge \rho_k = \alpha \wedge \rho_{k-1}$, with $\|\beta \wedge \rho_{k-1}/\alpha \wedge \rho_{k-1}\| \leq 2$ by induction. $\square$

Theorem 4.6. *Let $n \geq 3$ and $\mathcal{B}(n)$ be the boolean algebra on n elements, i.e., the lattice of subsets of an n element set ordered by inclusion. There exists $E \subseteq \leq_{\mathcal{B}(n)}$ such that $\mathcal{B}(n) \star_E \mathbf{2}$ generates a variety that is binary cut-through codable, and $\mathcal{B}(n) \star_E \mathbf{2}$ is not congruence normal.*

Proof. Given $\mathcal{B}(n)$ with $n \geq 3$, fix k such that $1 \leq k \leq n-2$. Let u be of height k and v be of height $k+1$ such that $u \prec v$. Define

$$E := \{(x,y) \colon x \text{ is of height } k,\ y \text{ is of height } k+1,\ x \prec y\} \setminus (u,v).$$

This set E consists of all covering relations in $\mathcal{B}(n)$ between elements of heights k and $k+1$ except $u \prec v$. By construction E is a connected all-or-nothing set. Consider $\mathcal{L} := \mathcal{B}(n) \star_E \mathbf{2}$. This new lattice is not congruence normal, as $(u,1) \approx (v,0)$ and $(u,1) \leq (v,0)$. The join irreducible elements of $\mathcal{L}$ are exactly the atoms of $\mathcal{L}$ and elements of the form $(x,1)$ for all $x \in \mathcal{L}$ of height k. Moreover, $(x_1,1) \approx (x_2,1)$ for all $x_1, x_2 \in \mathcal{B}(n)$ of height k. Let $\theta = \Phi_{(x,1)}$ where $x \in \mathcal{B}(n)$ is any element of height k.

We claim that θ is the unique least nontrivial congruence of $\mathcal{L}$. To prove this, note that every atom in $\mathcal{L}$ is join prime. We conclude that $\Phi_p \nleq \theta$ for any atom $p \in \mathcal{L}$. However, for any $x_1 \in \mathcal{B}(n)$ of height k, there exists $x_2 \neq x_1$ of height k in $\mathcal{B}(n)$ with the property that $(x_1,0) \vee (x_2,1) \geq (x_1,1)$ in $\mathcal{L}$. As $(x_1,0)$ has a unique minimal nontrivial join cover that consists entirely of the atoms below it, we have that if p is an atom such that $p \leq (x_1,0)$, then $(x_1,1)\ D\ p$. As x_1 was an arbitrary element of height k in $\mathcal{B}(n)$ this shows that $\theta \leq \Phi_p$ for any atom $p \in \mathcal{L}$. We conclude that $\mathcal{L}$ is subdirectly irreducible and its unique least nontrivial congruence is θ.

Any finite lattice in $Var(\mathcal{L})$ is isomorphic to a finite subdirect product of subdirectly irreducible lattices in $Var(\mathcal{L})$. As a consequence of Jónsson's Lemma any subdirectly irreducible lattice in $Var(\mathcal{L})$ is isomorphic to a homomorphic image of a sublattice of $\mathcal{L}$. The property of being binary cut-through codable is closed under taking sublattices of finite direct products. Thus, it suffices to show that any homomorphic image of a sublattice of $\mathcal{L}$ is binary cut-through codable.

Let $\mathcal{S}$ be a sublattice of $\mathcal{L}$. We will show that $\mathcal{S}$ has the binary square property, which will complete the proof by Lemma 4.5. Let $\sigma, \tau \in Con(\mathcal{S})$ be such that $\sigma \wedge \tau \prec \sigma, \tau \prec \sigma \vee \tau$ and

$$\|\sigma/\sigma\wedge\tau\| \leq 2$$
$$\|\tau/\sigma\wedge\tau\| \leq 2$$
$$\|\sigma\vee\tau/\sigma\| \leq 2.$$

This implies that $\|\sigma\vee\tau/\sigma\wedge\tau\| \leq 4$. If $\|\sigma\vee\tau/\sigma\wedge\tau\| \leq 3$ then the interval is a chain and the proof is straightforward. So assume $\|\sigma\vee\tau/\sigma\wedge\tau\| = 4$.

Let $[a] \in \mathcal{S}/(\sigma\wedge\tau)$ be such that the $\sigma\vee\tau/\sigma\wedge\tau$ block containing $[a]$ has cardinality 4. Say $[b],[c],[d] \in \mathcal{S}/(\sigma\wedge\tau)$ such that $([a],[d]) \in \sigma\vee\tau/\sigma\wedge\tau$, $[a] < [b] < [d]$, and $[a] < [c] < [d]$. There are two possible cases to consider. In both of these cases we may assume that the ordering on the elements $a,b,c,d \in \mathcal{S}$ is the same as the ordering on their respective $\sigma\wedge\tau$ classes and that meets and joins are preserved.

In the first case $[a] \prec [b] \prec [d]$ and $[a] \prec [c] \prec [d]$. Say $([a],[b]) \in \sigma/\sigma\wedge\tau$, whence $([c],[d]) \in \sigma/\sigma\wedge\tau$. We then have $([a],[c]),([b],[d]) \in \tau/\sigma\wedge\tau$ and the desired conclusion follows.

In the second case $[a] \prec [b] \prec [c] \prec [d]$. If $([a],[b]),([c],[d]) \in \tau/\sigma\wedge\tau$ then the desired conclusion follows immediately. Otherwise, we have that $([a],[b]),([c],[d]) \in \sigma/\sigma\wedge\tau$ and $([b],[c]) \in \tau/\sigma\wedge\tau$. The only interval in the original lattice $\mathcal{L}$ such that there exist elements $a < b < c < d$ with $b/a \approx d/c$ is the four element chain $(u,0) \prec (u,1) \prec (v,0) \prec (v,1)$. We conclude that these elements are all present in

$\mathcal{S}$. Moreover we may assume without loss of generality that $a = (u, 0)$, $b = (u, 1)$, $c = (v, 0)$, $d = (v, 1)$.

Given that $([a], [b]), ([c], [d]) \in \sigma/\sigma \wedge \tau$ it follows that $(v, 0) \approx (u, 1)$ in $\mathcal{S}$, whence there exists $(x, 1) \in \mathcal{S}$ such that $(x, 1)$ and $(v, 0)$ are associates. In $\mathcal{L}$ the only associates of $(v, 0)$ are such that $(v, 1)/(v, 0)$ projects down to $(x, 1)/(x, 0)$. It follows that in $\mathcal{S}$

$$\begin{aligned}(v, 0) \wedge (x, 0) &= (x, 0),\\ (u, 1) \wedge (x, 0) &= x \wedge u,\\ (x, 0) \vee (u, 0) &= (v, 0),\\ (x \wedge u) \vee (u, 0) &= (u, 0).\end{aligned}$$

These identities show that in $\mathcal{S}$, the interval $(v, 1)/(u, 1)$ projects down to to the interval $(x, 0)/(x \wedge v)$, which projects up to the interval $(v, 0)/(u, 0)$. This implies $\tau \geq \sigma$, contrary to assumption. Therefore, our second case does not occur for $\mathcal{S}$ and the proof is complete. □

As a lattice variety is determined by its subdirectly irreducible members, we obtain the following corollary.

Corollary 4.7. *There exist infinitely many binary cut-through codable lattice varieties containing finite lattices that are not congruence normal.*

References

[1] Kira V. Adaricheva, *A characterization of finite lattices of subsemilattices*, Algebra i Logika **30**, 385–404, 1991 (Russian)

[2] Alan Day, *A simple solution to the word problem for lattices*, Canad. Math. Bull. **13**, 253-254, 1970

[3] Alan Day, *Characterizations of finite lattices that are bounded-homomorphic images of sublattices of free lattices*, Canad. J. Math. **31**, 69–78, 1979

[4] Alan Day, *Congruence Normality: The characterization of the doubling class of convex sets*, Algebra Universalis **31** 397–406, 1994

[5] Ralph Freese, George McNulty, and J.B. Nation, *Inherently nonfinitely based lattices*, Annals of Pure and Applied Logic **115**, 175–193, 2002

[6] N. Funayama and T. Nakayama, *On the distributivity of a lattice of lattice-congruences*, Proc. Imp. Acad. Tokyo **18**, 553–554, 1942

[7] W. Geyer, *The generalized doubling construction and formal concept analysis*, Algebra Universalis **32**, 341–367, 1994

[8] Bjarni Jónsson and J.B. Nation, *A report on sublattices of a free lattice*, Contributions to universal algebra, Coll. Math. Soc. János Bolyai **17**, 233–257, 1977

[9] A. Kostinsky, *Projective lattices and bounded homomorphisms*, Pacific Journal of Mathematics **40**, 111-119, 1972

[10] Ralph McKenzie, *Equational bases and non-modular lattice varieties*, Trans. Amer. Math. Soc. **174**, 1–43, 1972

[11] J.B. Nation, *Addendum to 'Congruence Normality' of Alan Day*, Algebra Universalis **31**, 307–310, 1994

[12] J.B. Nation, *Alan Day's Doubling Construction*, Algebra Universalis **34**, 24–34, 1995

[13] J.B. Nation, *Notes on Lattice Theory*, unpublished, http://math.hawaii .edu/∼jb/books.html

[14] Pavel Pudlàk and Jı́ři Tůma, *Yeast graphs and fermentation of algebraic lattices*, Coll. Math. Soc. János Bolyai **14**, 301-341, 1974

[15] V. Repnitskiĭ, *On finite lattices which are embeddable in subsemigroup lattices*, Semigroup Forum **46**, 388-397, 1993

[16] Heiko Reppe, *Three Generalizations of Lattice Distributivity: An FCA Approach*, Shaker Verlag, Aachen, 2011

[17] Qifu Tyler Sun and Shuo-Yen Robert Li, *Lattice classification by cut-through coding*, Order **31**, 271-278, 2014