LATTICES OF QUASI-EQUATIONAL THEORIES AS CONGRUENCE LATTICES OF SEMILATTICES WITH OPERATORS, PART II

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Abstract. Part I proved that for every quasivariety $\mathcal{K}$ of relational structures there is a semilattice $\mathbf{S}$ with operators such that the lattice of quasi-equational theories of $\mathcal{K}$ (the dual of the lattice of sub-quasivarieties of $\mathcal{K}$) is isomorphic to $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$. It is known that if $\mathbf{S}$ is a join semilattice with 0 (and no operators), then there is a quasivariety $\mathcal{Q}$ such that the lattice of theories of $\mathcal{Q}$ is isomorphic to $\mathrm{Con}(\mathbf{S}, +, 0)$. We prove that if $\mathbf{S}$ is a semilattice having both 0 and 1 with a group $\mathcal{G}$ of operators acting on $\mathbf{S}$, then there is a quasivariety $\mathcal{W}$ such that the lattice of theories of $\mathcal{W}$ is isomorphic to $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{G})$.

1. Introduction

In Part I, we proved that for every quasivariety $\mathcal{K}$ of relational structures there is a semilattice $\mathbf{S}$ with operators such that the lattice of quasi-equational theories containing the theory of $\mathcal{K}$ is isomorphic to $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$. (An *operator* is a $+, 0$-endomorphism.) In this second part, we will be concerned with trying to represent a congruence lattice $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$ as a lattice of quasi-equational theories. This is not always possible, because the theory generated by the equation $x \approx y$ has special properties, with no analogue in the congruence lattice of an arbitrary semilattice with operators. For example, the congruence lattice of $\mathbf{\Omega} = (\omega, \vee, 0, p)$, where $p(0) = 0$ and $p(x) = x - 1$ for $x > 0$, is isomorphic to $\omega + 1$. This is not a lattice of quasi-equational theories; see Section 5 of Part I.

Nonetheless, there are the following positive results.

- Gorbunov and Tumanov proved that if $\mathbf{S}$ is a join semilattice with 0, then there is a quasivariety $\mathcal{Q}$ such that the lattice of theories of $\mathcal{Q}$ is isomorphic to $\mathrm{Con}(\mathbf{S}, +, 0)$ with no operators; see [6].
- In this paper, we prove that if $\mathbf{S}$ is a semilattice having both 0 and 1 with a group $\mathcal{G}$ of operators acting on $\mathbf{S}$, then there is a quasivariety $\mathcal{W}$ such that the lattice of theories of $\mathcal{W}$ is isomorphic

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to $\text{Con}(\mathbf{S}, +, 0, \mathcal{G})$. In fact, the construction works for a slightly more general class of operators than groups, but still a rather special type of monoid.

- In a third part of this study, the second author shows that the congruence lattice $\text{Con}(\mathbf{S}, +, 0, \mathcal{F})$ of a semilattice with operators can be represented as a lattice of implicational theories in a language that may not contain a primitive equality relation [8].

A fourth part of the study looks at the structure of lattices of atomic theories in languages without equality [7].

2. Representations of $\text{Con}(\mathbf{S}, +, 0)$

This section describes ways to represent a coatomistic lattice $\mathbf{L} = \text{Con}(\mathbf{S}, +, 0)$ as the lattice of theories $\text{QTh}(\mathcal{B})$ of a quasivariety $\mathcal{B}$. Later on, we will modify the representations to fit $\text{Con}(\mathbf{S}, +, 0, \mathcal{F})$ where $\mathcal{F}$ is a sufficiently nice set of operators.

As in Part I, it will be convenient to use a closely related type of relation, rather than congruences. For an algebra $\mathbf{A}$ with a join semilattice reduct, let Eon $\mathbf{A}$ be the lattice of all reflexive, transitive, compatible relations R such that

(1) $R \subseteq \leq$, i.e., $x\,R\,y$ implies $x \leq y$, and
(2) if $x \leq y \leq z$ and $x\,R\,z$, then $x\,R\,y$.

Lemma 1. *If* $\mathbf{A} = \langle A, \vee, 0, \mathcal{F}\rangle$ *is a semilattice with operators, then* Con $\mathbf{A} \cong$ Eon $\mathbf{A}$.

The general setup is described as follows. Some simplifications are possible in the finite case, and the second representation requires that the semilattice have a greatest element 1.

1. Label $S = \{0, a, b, c, \dots\}$. Include 1 if there is one.

2. Construct $\text{Eon}(\mathbf{S}, +, 0)$. For $a < b$ in $\mathbf{S}$, let $\langle a, b\rangle$ denote the principal eon-relation generated by (a, b). Compacts are joins of finitely many of these. Coatoms of $\text{Eon}(\mathbf{S}, +, 0)$ correspond to congruences with two blocks, an ideal and its complement.

Eon relations of the form $\bigvee_{b\in I}\langle 0, b\rangle$ for an ideal I of $\mathbf{S}$ will be termed *equational*. The equational relations include 0 and 1, and are closed under joins.

An example is given in Figure 1, where for space purposes $\langle a, b\rangle$ is abbreviated as ab. The solid points indicate the equational eon-relations.

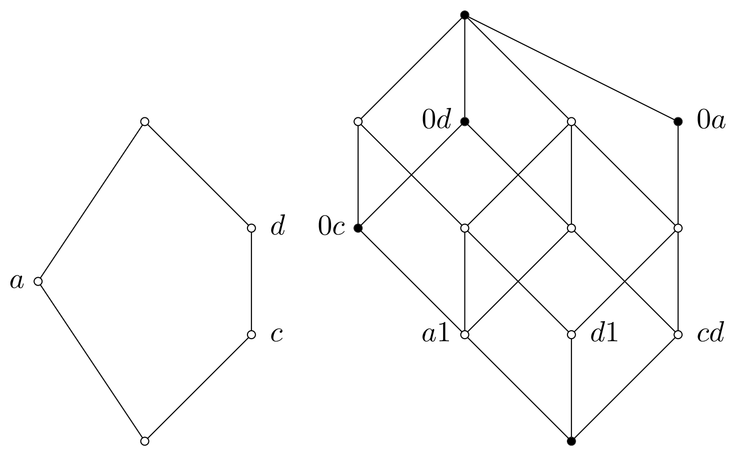
3. The plan is now to assign predicates A, B, $C, \dots$ to the elements of $\mathbf{S}$, and to define a quasivariety $\mathcal{B}$ in this language that will represent $\mathbf{L}$ as the lattice of theories of $\mathcal{B}$.

For each finite join $c = \bigvee a_j$ in $\mathbf{S}$, let C be identified with the conjunction $\& A_j$, so that $C(x) \iff \& A_j(x)$ will be part of the theory of $\mathcal{B}$. (When $\mathbf{S}$ is finite, we only need predicates for the join irreducible elements of $\mathbf{S}$.) It is sometimes convenient to have special predicate symbols T and E reserved corresponding to 0 and 1, respectively.

First representation. The simplest representation (Gorbunov and Tumanov [6]) has unary predicates $A(x)$, $B(x)$ etc. for the elements of $\mathbf{S}$. The quasivariety $\mathcal{B}$ satisfies the laws

$$x \approx y$$
$$A(x) \implies B(x) \text{ whenever } a \geq b$$
$$\&_i\, A_i(x) \implies B(x) \text{ whenever } \bigvee_i a_i \geq b.$$

The isomorphism with $\text{Eon}(\mathbf{S}, +, 0)$ has $\langle 0, b\rangle$ corresponding to the law $B(x)$, and $\langle a, b\rangle$ corresponding to $A(x) \implies B(x)$. More generally, $\langle \bigvee a_i, \bigvee b_j\rangle$ corresponds to the conjunction (over j) of laws $\&_i A_i(x) \implies B_j(x)$, and the join of a set of principal eon-relations corresponds to their conjunction.



$\mathbf{N}_5$ $\qquad$ $\text{Eon}(\mathbf{N}_5, +, 0)$

FIGURE 1. Example of $\text{Eon}(\mathbf{S}, +, 0)$

Second representation. If we assume that $\mathbf{S}$ has a top element 1, then there is another kind of representation of $\text{Con}(\mathbf{S}, +, 0)$. This incorporates some ideas from a different representation due to Gorbunov (see Theorem 5.2.10 of Gorbunov [5], though it goes back to [4]).

We again use unary predicates $A(x)$ corresponding to elements of $\mathbf{S}$ (or just the join irreducibles in the finite case), and a predicate $E(x)$ corresponding to 1. Let e be a constant symbol. The representation is given by the quasivariety $\mathcal{B}$ with the laws

$$A(e) \text{ holds for all } A$$
$$E(x) \implies x \approx e$$
$$A(x) \implies B(x) \text{ whenever } a \geq b$$
$$\&_i\, A_i(x) \implies B(x) \text{ whenever } \bigvee_i a_i \geq b.$$

Using the first law above, we obtain the following.

Lemma 2. *Every quasi-identity of $\mathcal{B}$ is equivalent to one with at most one variable.*

Now one can see that the order and join dependency relation on the semilattice of compact elements of $\mathrm{Eon}(\mathbf{S}, \vee, 0)$ are determined as follows.

- $\langle a, b\rangle \le \langle c, d\rangle$ iff $c \le a$ and $a \vee d \ge b$,
- $\langle a, b\rangle \le \bigvee_j \langle c_j, d_j\rangle$ iff there exists a sequence $e_1 < f_1 = e_2 < f_2 = e_3 < \cdots < f_k$ such that

$$e_1 \le a$$
$$\forall i \exists j \ \langle e_i, f_i\rangle \le \langle c_j, d_j\rangle$$
$$a \vee f_k \ge b.$$

The proof is straightforward checking. That is, we have $(a, b) \in \langle c, d\rangle$ iff $a = b$ or $a < b$, $c \le a$, $a \vee d \ge b$. Similarly, check that the description of the join is correct.

Again, the isomorphism of $\mathrm{Eon}(\mathbf{S}, +, 0)$ with the quasi-equational theories of $\mathcal{B}$ has $\langle 0, b\rangle$ corresponding to the law $B(x)$, and $\langle a, b\rangle$ corresponding to $A(x) \implies B(x)$. More generally, $\langle \bigvee a_i, \bigvee b_j\rangle$ corresponds to the conjunction over j of laws $\&_i A_i(x) \implies B_j(x)$.

The point is that the quasi-equational theory of $\mathcal{B}$, involving only one-variable laws, has exactly the same rules for deduction as $\mathrm{Eon}(\mathbf{S}, \vee, 0)$, again interpreting $a \le b$ as $A(x) \implies B(x)$. That is, for one-variable quasi-identities with unary predicates (where substitution is not a factor), the only rule of deduction is that $(P \implies Q)$ AND $(Q \implies R)$ entails $P \implies R$. Here P, Q, R may be conjunctions of atomic predicates $A(x)$, or empty, and of course we have to take the transitive closure of entailment.

Equally important, the map from Eon-relations to quasi-equational theories is surjective because quasi-identities in this language must have the form $\&_i A_i(x) \implies B(x)$. Otherwise the preceding remarks would still only yield an embedding.

3. The dual leaf as the congruence lattice of a semilattice with operators

The *leaf* is the dual of the lattice in Figure 2; it is $\mathbf{1} \dot{+} \mathrm{Co}(\mathbf{4})$. The leaf is join semidistributive but not lower bounded, and supports a (two point) equaclosure operator. It is an open question whether the leaf is a Q-lattice, but the dual leaf has a natural representation as $\mathrm{Con}(\mathbf{S}, \vee, 0, f, g)$ where $\mathbf{S}$ is given in Figure 3, $f(x_k) = x_{k+1}$ and $g(x_k) = x_{k-1}$ for $x \in \{a, b, c, d\}$ and $k \in \mathbb{Z}$. (Compare the representation of $\mathrm{Co}(\mathbf{4})$ as a lattice of ε-closed algebraic subsets in Example 5.5.10 of Gorbunov [5].)

In Section 6, we will modify this example to represent the dual near-leaf of Figure 4 as a lattice of quasi-equational theories.

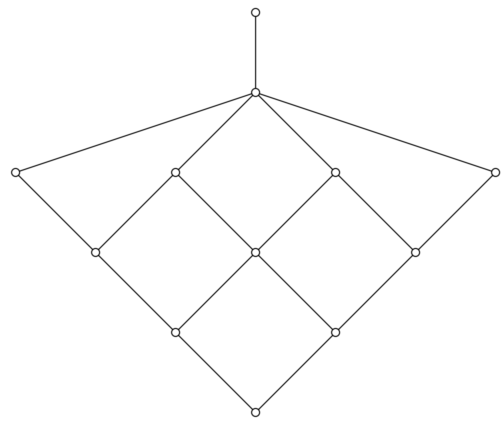


FIGURE 2. Dual leaf

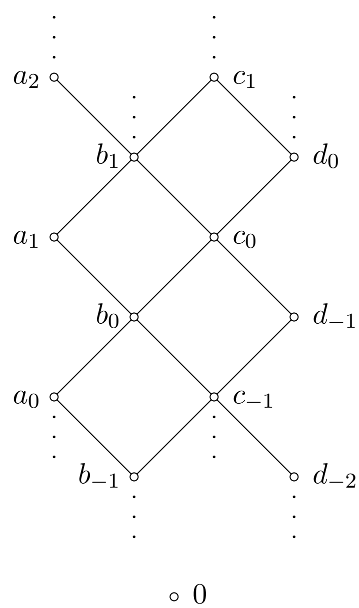


FIGURE 3. **S** to represent the dual leaf

4. Sufficient conditions for reduction to one variable

An important objective is to find conditions that allow us to represent certain congruence lattices of semilattices with operators as lattices of quasi-equational theories. The motivating case is when the semilattice has a largest element and the operators form a group, but the results are slightly more general than that. We begin with properties that permit us to consider only quasi-identities in one variable.

Let us say that a monoid **M** is *reductive* if

(R) for every pair f, $g \in \mathbf{M}$ there is an element $h \in \mathbf{M}$ such that either $f = hg$ or $hf = g$.

This is a rather strong property, but in particular, every group is reductive.

Let $\star$ denote the operation in $\mathbf{M}^{\mathrm{opp}}$, so that $f \star g = gf$.

Theorem 3. *Let $\mathcal{B}$ be a quasivariety in a language $\mathcal{L}$ with the following restrictions and laws.*

1. *$\mathcal{L}$ has only unary predicate symbols (except for $\approx$).*
2. *$\mathcal{L}$ has only unary function symbols, corresponding to the elements of a fixed reductive monoid $\mathbf{M}$.*
3. *$\mathcal{L}$ has one constant symbol w.*
4. *$\mathcal{B}$ satisfies the law $A(x) \implies A(w)$ for every predicate of $\mathcal{L}$.*
5. *$\mathcal{B}$ satisfies the law $f(w) \approx w$ for every function of $\mathcal{L}$.*
6. *$\mathcal{B}$ satisfies identities saying that the functions act as the monoid $\mathbf{M}^{\mathrm{opp}}$, that is, $i(x) \approx x$ and $f(g(x)) \approx h(x)$ whenever $h = f \star g = gf$.*
7. *$\mathcal{B}$ satisfies the laws*

$$f(x) \approx f(y) \implies x \approx y$$

for every function of $\mathcal{L}$.

Then every quasi-identity holding in a theory extending the theory of $\mathcal{B}$ is equivalent (modulo the laws of $\mathcal{B}$) to a set of quasi-identities in only one variable. Hence the lattice of theories of $\mathcal{B}$ is isomorphic to $\mathrm{Con}(\mathbf{S})$*, where* $\mathbf{S} = \langle \mathbf{T}, \vee, 0, \widehat{\mathcal{E}} \rangle$ *with $\mathbf{T}$ the semilattice of compact $\mathcal{B}$-congruences of $\mathbf{F}_{\mathcal{B}}(1)$ and $\widehat{\mathcal{E}}$ the monoid of endomorphisms of $\mathbf{T}$ induced by* $\mathrm{Sbn}(\mathbf{F})$.

Here $\mathrm{Sbn}(\mathbf{F})$ denotes the monoid of all *substitution* endomorphisms of a $\mathcal{B}$-free algebra, generated by extending maps $\sigma : X \to \mathbf{F}$, extending the relations minimally. (In this case, $X = \{x\}$.) See Theorem 7 of Part I.

Proof. First note that, by properties (5) and (7), for each function symbol f, the quasivariety $\mathcal{B}$ satisfies the law $f(x) \approx w \iff x \approx w$. Secondly, by properties (R) and (7), each law of the form $f(x) \approx g(y)$ is equivalent to one of the forms $x \approx h(y)$ or $h(x) \approx y$. For if $f = hg$, then

$$f(x) \approx g(y) \quad \text{iff} \quad (hg)(x) \approx g(y) \quad \text{iff} \quad g(h(x)) \approx g(y) \quad \text{iff} \quad h(x) \approx y$$

and a similar calculation applies if $hf = g$.

Thus every atomic formula is equivalent to one of the following forms:

$$A(x) \qquad A(f(x)) \qquad A(w) \qquad x \approx y \qquad x \approx f(x) \qquad x \approx f(y) \qquad x \approx w.$$

We want to show that any law $\lambda \implies \rho$ involving more than one variable is equivalent to a set of laws involving fewer variables. We may assume that ρ is an atomic formula involving either x or x, y and possibly w. Using z to denote an arbitrary variable not appearing in ρ (there may be more than one of these), then λ is a conjunction involving some (including none or all) of the following forms:

$$\begin{array}{llll}
A(x) & A(y) & A(z) & A(w) \\
A(f(x)) & A(f(y)) & A(f(z)) & \\
x \approx y & y \approx z & z \approx z' & \\
x \approx z & y \approx f(x) & z \approx f(x) & \\
x \approx f(x) & y \approx f(y) & z \approx f(y) & \\
x \approx f(y) & y \approx f(z) & z \approx f(z) & \\
x \approx f(z) & y \approx w & z \approx f(z') & \\
x \approx w & & z \approx w. &
\end{array}$$

The conclusion ρ, on the other hand, by symmetry can be assumed to have one of the forms:

$$A(x) \qquad A(f(x)) \qquad A(w) \qquad x \approx y \qquad x \approx f(x) \qquad x \approx f(y) \qquad x \approx w.$$

If perchance the hypothesis λ includes any one of the following forms:

$$\begin{array}{lll}
x \approx y & y \approx z & z \approx z' \\
x \approx z & y \approx f(x) & z \approx f(x) \\
x \approx f(y) & y \approx f(z) & z \approx f(y) \\
x \approx f(z) & y \approx w & z \approx f(z') \\
x \approx w & & z \approx w.
\end{array}$$

then we can replace one of the variables by the corresponding expression and obtain an equivalent law with fewer variables. So we may assume that none of these forms appears in λ, and thus λ involves only these forms:

$$\begin{array}{llll}
A(x) & A(y) & A(z) & A(w) \\
A(f(x)) & A(f(y)) & A(f(z)) & \\
x \approx f(x) & y \approx f(y) & z \approx f(z) &
\end{array}$$

If, after the previous substitutions, the conclusion ρ involves only x, i.e., it has one of the forms:

$$A(x) \qquad A(f(x)) \qquad A(w) \qquad x \approx f(x) \qquad x \approx w$$

then we may replace all the remaining variables by w, using the fact that λ involves only relational forms or $t \approx f(t)$ with t a variable, thereby obtaining a one-variable law that is equivalent to the original. (Conditions (4) and (5) are used here.) Thus we may assume that ρ is either $x \approx y$ or $x \approx f(y)$.

If, after the previous reductions, any variable z not equal to x or y still occurs on the LHS, we may now replace it by w and obtain an equivalent law containing only x and y. Hence we may assume that λ is a conjunction of these forms:

$$\begin{array}{rl} A(x) & A(y) \\ A(f(x)) & A(f(y)) \\ x \approx f(x) & y \approx f(y). \end{array}$$

Now, replacing y by w, we obtain a law of the form $H(x) \implies x \approx w$, where $H(x)$ is the conjunction of the terms on the LHS involving only x and w. Similarly, replacing x by w yields a law $G(y) \implies y \approx w$ where $G(y)$ is the conjunction of the terms on the LHS involving only y and w. Thus we have derived two laws, in one variable each. Conversely,

$$H(x) \ \& \ G(y) \implies x \approx w \ \& \ y \approx w$$

which in turn yields the original law, with the same hypothesis and conclusion either $x \approx y$ or $x \approx f(y)$.

Note that any theory extending the theory of $\mathcal{B}$ includes (1)–(7), and thus is determined by its laws in one variable. The last statement of the theorem is then a consequence of Theorem 7 of Part I. □

The elements of $\mathbf{F}_{\mathcal{B}}(1)$ are w and $f(x)$ for $f \in \mathbf{M}$. The substitution endomorphisms are the constant map $\varepsilon_w : t \mapsto w$ for all t, and the maps ε_f for $f \in \mathbf{M}$ with $\varepsilon_f(w) = w$ and $\varepsilon_f(g(x)) = g(f(x)) = (fg)x$. Note that the correspondence $f \mapsto \varepsilon_f$ embeds $\mathbf{M}$ into $\mathrm{End}(\mathbf{F}_{\mathcal{B}}(1))$, as $\varepsilon_f \varepsilon_g = \varepsilon_{fg}$. The calculation: $\varepsilon_f \varepsilon_g(x) = \varepsilon_f(g(x)) = g(f(x)) = (fg)(x) = \varepsilon_{fg}(x)$.

5. Lattices of 1-variable quasi-equational theories

We turn to the problem of trying to represent the congruence lattice of a semilattice with operators as a lattice of quasi-equational theories. This requires that we deal with the pseudo-one at the semilattice level.

Let $\mathcal{K}$ be a quasivariety with $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(X)$. Consider the least $\mathcal{K}$-congruence Υ on $\mathbf{F}$ with $|\mathbf{F}/\Upsilon| = 1$. Now Υ may or may not be compact; below we will deal with the case when it is. The crucial observation is that if θ is a $\mathcal{K}$-congruence and ε a substitution endomorphism, then $\hat{\varepsilon}(\theta) \vee \Upsilon = \theta \vee \Upsilon$.

The following result represents certain congruence lattices of semilattices with operators as the lattice of 1-variable quasi-equational theories of a quasivariety.

Theorem 4. *Let* $\mathbf{S}$ *be a join semilattice with* 0*, and let* $\mathbf{M}$ *be a monoid of operators acting on* $\mathbf{S}$*. Assume there is an element* $u \in \mathbf{S}$ *such that*

1. $f(t) + u = t + u$ *whenever* $t \in S$ *and* $f \in \mathbf{M}$*,*
2. *there exists* $k \in \mathbf{M}$ *such that, for all* $s, t \in S$ *we have* $s + u \geq t$ *iff* $s \geq k(t)$*.*

Then there is a quasivariety $\mathcal{C}$ *such that* $\mathrm{Con}(\mathbf{S}, +, 0, \mathbf{M})$ *is isomorphic to* $\mathrm{Con}(\mathbf{T}, \vee, 0, \widehat{\mathcal{E}})$*, where* $\mathbf{T}$ *is the semilattice of compact* $\mathcal{C}$*-congruences of* $\mathbf{F}_{\mathcal{C}}(1)$ *and* $\widehat{\mathcal{E}}$ *is the monoid of endomorphisms of* $\mathbf{T}$ *induced by* $\mathrm{Sbn}(\mathbf{F})$*.*

A map k with the given property will automatically be a join homomorphism preserving 0. We are requiring that it be in $\mathbf{M}$. In view of (1), $k(t) \leq t \leq t+u$ and $k(t+u) = k(t)$ and $k(t)+u = t+u$.

Note 1: The two conditions of the theorem are a generalization of the situation when $\mathbf{S}$ has a greatest element 1, for in that case we can take $u = 1$ and k to be the zero map. However, they can also reflect properties of a quasivariety.

Condition (1) is satisfied when the congruence Υ described above is compact. If, for example, we could take the generating set X for the free algebra to be finite, and $\mathbf{F}_{\mathcal{K}}(1)$ is finite, that would be the case. Generally, the lattice of quasi-equational theories is determined by the compact $\mathcal{K}$-congruences of $\mathbf{F}_{\mathcal{K}}(\omega)$, unless there is an integer n such that every quasi-variety containing $\mathcal{K}$ is n-based with respect to $\mathcal{K}$. This can happen in various ways, though.

Condition (2) holds when Υ is compact and the language has a single constant w which is idempotent in $\mathcal{K}$, i.e., $\mathcal{K}$ satisfies $f(w,\ldots,w) \approx w$ for every operation symbol f. For in that case we can take k to be the operator induced by the substitution $x \mapsto w$ for every $x \in X$. This latter property will hold in our construction.

These are strong conditions, but not unreasonably so.

Note 2: However, the laws of $\mathcal{B}$ in the previous theorem and $\mathcal{C}$ in this one are not always compatible, which is an issue to be addressed in the next section.

Proof. Our language will include unary predicates A for each nonzero element a of $\mathbf{S}$, operations f for each $f \in \mathbf{M}$, and a constant w. The predicate U corresponds to the element u.

The construction begins by assigning sets of predicates $\mathcal{P}(s)$ and $\mathcal{Q}(s)$ to each nonzero element s of $\mathbf{S}$. First, for each $a \in \mathbf{S}$ and $f \in \mathbf{M}$, assign the predicate $A(f(x))$ to $f(a)$. In this way each element of $\mathbf{S}$ may be assigned multiple predicates, but they will all be of the form $B(g(x))$ for different predicates B and operations $g \in \mathbf{M}$.

Anticipating law (9a) below, we close this downward: for $s \in S$, let $\mathcal{P}(s) = \{A(f(x)) : f(a) = s\}$. These are the predicates involving x.

The second set of predicates involves the constant w: for $s \in S$, let $\mathcal{Q}(s) = \{P(w) : p \leq u+s\}$. Note that $\mathcal{Q}(s) = \mathcal{Q}(u+s)$, and that the latter is an $\mathbf{M}$-closed ideal by the observation above.

Define $\mathcal{C}$ to be the quasivariety determined by these laws.

(4) $A(x) \implies A(w)$ for every predicate A.
(5) $f(w) \approx w$ for every $f \in \mathbf{M}$.
(6) $i(x) \approx x$ and $f(g(x)) \approx h(x)$ whenever $h = f \star g = gf$.
(8) $U(f(x)) \implies x \approx w$ for every $f \in \mathbf{M}$.
(9a) $\beta \implies \alpha$ whenever $a \leq b$, $\alpha \in \mathcal{P}(a)$, $\beta \in \mathcal{P}(b)$.
(9b) $\beta \implies \alpha$ whenever $a \leq b$, $\alpha \in \mathcal{Q}(a)$, $\beta \in \mathcal{Q}(b)$.
(10a) $\&\beta_j \implies \alpha$ whenever $a \leq \sum b_j$, $\alpha \in \mathcal{P}(a)$, $\beta_j \in \mathcal{P}(b_j)$ for each j.
(10b) $\&\beta_j \implies \alpha$ whenever $a \leq \sum b_j$, $\alpha \in \mathcal{Q}(a)$, $\beta_j \in \mathcal{Q}(b_j)$ for each j.

(11) $f(x) \approx g(x) \implies x \approx w$ for each pair $f \neq g \in \mathbf{M}$.
(12a) $P(w)$ for all $p \leq u$.
(12b) $C(w) \implies P(w)$ for all $p \leq c + u$.
(13) $T(w) \implies K_t(x)$ for all $t \geq u$.

Note: The language is specified to satisfy (1)–(3) of Theorem 3. Law (7) is missing here, while the last six (8)–(13) are new; (9) is redundant as a special case of (10). When the theorems are combined below, we of course want all the laws (1)–(13).

The universe of $\mathbf{F} = \mathbf{F}_{\mathcal{C}}(1)$ is $\{f(x) : f \in \mathbf{M}\} \cup \{w\}$. The operations correspond to elements of $\mathbf{M}$, and there is a unary predicate for each nonzero element of $\mathbf{S}$. Note that $A(t)$ holds in the free structure only for $t = w$ and $a \leq u$. The substitution endomorphisms are again the constant map ε_w and the maps ε_f with $\varepsilon_f(g(x)) = g(f(x)) = (fg)x$ for $f \in \mathbf{M}$.

Claim 1: The following are $\mathcal{C}$-congruences on $\mathbf{F}$:

- the equality relation and the relations $\mathcal{Q}(u) = \{P(w) : p \leq u\}$,
- for any nonzero ideal I of $\mathbf{S}$ with $u \notin I$, the equality relation and the relations $\bigcup_{s \in I}(\mathcal{P}(s) \cup \mathcal{Q}(s))$,
- for any nonzero ideal I of $\mathbf{S}$ with $u \in I$, the universal equivalence and the relations $\bigcup_{s \in I}(\mathcal{P}(s) \cup \mathcal{Q}(s))$.

Clearly equality and the universal relations respect the operations (join and $\mathbf{M}$), but one must also check the laws of $\mathcal{C}$. Here are some of them.

(4) Suppose $A(f(x)) \in \mathcal{P}(s)$ and $s \in I$. (Note: The property reads $A(x)$ but you must allow substitution.) Then $f(a) \leq s$ so $f(a) \in I$, and $\mathcal{Q}(f(a)) = \mathcal{Q}(f(a) + u) = \mathcal{Q}(a + u) = \mathcal{Q}(a)$, whence $A(w) \in \mathcal{Q}(s)$, as desired.

(12b) Assume $C(w) \in \mathcal{Q}(s)$ and $s \in I$, whence $c \leq s + u$. If $p \leq c + u$ then $p \leq s + u$, so $P(w) \in \mathcal{Q}(s)$.

(13) Suppose $T(w) \in Q(s)$ for some $s \in I$. Then $t \leq s + u$, whence $s \geq k_t$ and $k_t \in I$. Thus $K_t(x)$ is in the congruence.

Let us refer to the congruence described above as θ_I, or if $I = \downarrow s$, then θ_s.

Claim 2: Every $\mathcal{C}$-congruence on $\mathbf{F}$ is θ_I for some I. Indeed, given a $\mathcal{C}$-congruence θ, by law (11) its equivalence θ_0 is either the identity or universal relation. Let

$$I = \{s \in \mathbf{S} : \mathcal{P}(s) \subseteq \theta\}$$
$$J = \{s \in \mathbf{S} : \mathcal{Q}(s) \subseteq \theta\}.$$

Then the following hold.

- I is an ideal by (9a) and (10a).
- J is an ideal by (9b) and (10b).
- $J \subseteq I + u$ by law (13).
- $J \supseteq I + u$ by laws (4) and (12b).

Thus $J = I + u$, and it follows that $\theta = \theta_I$. Note that θ_I is compact exactly when I is principal.

Claim 3: For $s \in \mathbf{S}$, $\widehat{\varepsilon}_w(\theta_s) = \theta_{k(s)}$.

Claim 4: For a substitution endomorphism ε_h, $\widehat{\varepsilon}_h(\theta_s) = \theta_{h(s)}$. Now if $f(a) \leq s$, then $\widehat{\varepsilon}_h(A(f(x)) = A(\widehat{\varepsilon}_h f(x)) = A((hf)(x))$, and $hf(a) \leq h(s)$, so these relations are all in θ_s. Moreover, with $f = i$ and $a = s$ they include $S(h(x))$, which is a generator for the congruence. Thus equality holds.

Thus we see that the action of $\widehat{\mathcal{E}}$ on $\mathbf{T}$, the semilattice of compact $\mathcal{C}$-congruences of $\mathbf{F}_{\mathcal{C}}(1)$, mimics the action of $\mathbf{M}$ on $\mathbf{S}$.

This completes the proof of the theorem. □

For groups of operators, we can get by with only one predicate per orbit, and we could use inverses instead of $\mathbf{G}^{\text{opp}}$.

6. Semilattices with groups of operators

To combine the previous two theorems, we want to consider a quasivariety $\mathcal{D}$ satisfying all the laws (1)–(13). The difficulty that we encounter is that laws (7) and (11) could combine to collapse the free algebra on one generator. To avoid this, we assume that $\mathbf{M}$ is right cancellative: *if $gf = hf$ then $g = h$.* If that holds, then by (11) we have

$$f(g(x)) = f(h(x)) \quad \text{iff} \quad (gf)(x) = (hf)(x) \quad \text{iff} \quad g(x) = h(x)$$

in the free algebra, so that (7) and (11) are consistent.

Corollary 5. *Let $\mathbf{S}$ be a join semilattice with 0, and let $\mathbf{M}$ be a reductive, right cancellative monoid of operators acting on $\mathbf{S}$. Assume there is an element $u \in \mathbf{S}$ such that*

(1) *$f(t) + u = t + u$ whenever $t \in S$ and $f \in \mathbf{M}$,*
(2) *there exists $k \in \mathbf{M}$ such that, for all s, $t \in S$ we have $s + u \geq t$ iff $s \geq k(t)$.*

Then there is a quasivariety $\mathcal{D}$ such that the lattice of quasi-equational theories of $\mathcal{D}$ is isomorphic to $\mathrm{Con}(\mathbf{S}, +, 0, \mathbf{M})$.

The statement of the Corollary is somewhat technical, but it does include at the extremes

- congruence lattices of semilattices, with $u = 0$ and $\mathbf{M}$ having only the identity map,
- congruence lattices of semilattices with both 0 and 1, with a reductive, right cancellative monoid of operators, where $u = 1$ and k is the zero map.

It also includes the linear sum of the first type over the second, or a direct product of the first type and the second.

The following special case yields some new examples.

Corollary 6. *Let $\mathbf{S}$ be a join semilattice with both 0 and 1, and let $\mathbf{M}$ be a reductive, right cancellative monoid of operators acting on $\mathbf{S}$. Then there is a quasivariety $\mathcal{D}$ such that the lattice of quasi-equational theories of $\mathcal{D}$ is isomorphic to* $\mathrm{Con}(\mathbf{S}, +, 0, \mathbf{M})$.

This corollary applies to the following situations, so that in each case the congruence lattice of the semilattice with operators is representable as a lattice of quasi-equational theories.

- $\mathbf{S}$ is a semilattice with 0 and 1 and a group of operators.
- $\mathbf{S} = -\mathbb{N} \cup \{-\infty\}$, where $-\mathbb{N}$ denotes the non-positive integers, as a join semilattice with the operators $p_k(x) = k + x$ for $k \in -\mathbb{N}$.
- $\mathbb{N}$ can be replaced by non-positive rationals, or reals, in the preceding example.
- More generally, we can take $-\mathbf{P}$ to be the negative cone of a totally ordered group, with $\{-\infty\}$ adjoined, and say the left translations $p_k(x) = kx$ for $k \in -\mathbf{P}$ as operators.
- The operations can be restricted to a submonoid, so long as the reductive property is maintained. In particular, we can restrict to a cyclic or quasicylic monoid.
- These representations can be combined as follows. Let $\mathbf{S}$ be a fixed semilattice with 0, 1, and a group $\mathbf{G}$ of operators. Let $-\mathbf{P}$ be a negative cone as above, and let $\mathbf{K}$ denote its operators. Now replace each point of $-\mathbf{P}$ by $\mathbf{S}$, i.e., take $\mathbf{Q} = -\mathbf{P} \times \mathbf{S}$ with the lexicographic order, and adjoin $-\infty$. Then $\mathbf{K} \times \mathbf{G}$ operates on $\mathbf{Q}$ naturally: $(p_k, f)(x, y) = (kx, f(y))$. Moreover, it is reductive and cancellative, and so the corollary applies.
- If the chain is discrete, we could identify the 0 and 1 of consecutive semilattices in the chain.

There are a lot of details to be checked there, but they are routine.

For example, with the negative cone $-\mathbf{P}$ of a totally ordered group and its full complement of operators p_k, we obtain a representation of $\mathcal{I}^*(-\mathbf{P}) \times \mathbf{2}$ (where $\mathcal{I}^*$ denotes nonempty ideals) with a least element adjoined.

The Universal Algebra Calculator of Ralph Freese and Emil Kiss [3] has proved to be useful in calculating the congruence lattice of a finite semilattice with a group of operators. Even when $\mathbf{S}$ is fairly small, its congruence lattice can be rather large.

Now let us represent the dual near-leaf of Figure 4. For $\mathbf{S}$ we take the semilattice in Figure 3 and add a top element 1. There is a natural action of the integers $\mathbb{Z}$ on $\mathbf{S}$ as noted in Section 3. With the new 1 added, we obtain the dual near-leaf as $\mathrm{Con}(\mathbf{S}, +, 0, \mathbb{Z})$. Following the prescription given above, this is the lattice of theories of the quasivariety satisfying the laws below. As before, we denote the action of $\mathbb{Z}$ by $f(x_k) = x_{k+1}$ and $g(x_k) = x_{k-1}$ for

$x \in \{a, b, c, d\}$ and $k \in \omega$.

$$fg(x) \approx x \text{ and } gf(x) \approx x$$
$$A(e) \qquad B(e) \qquad C(e) \qquad D(e)$$
$$f(e) \approx e \qquad\qquad g(e) \approx e$$
$$D(x) \implies C(x) \implies B(x) \implies A(x)$$
$$C(x) \implies D(g(x))$$
$$B(x) \implies C(g(x))$$
$$A(x) \implies B(g(x))$$
$$A(x) \ \&\ C(g(x)) \implies B(x)$$
$$B(x) \ \&\ D(g(x)) \implies C(x)$$
$$x \approx f^k(x) \implies x \approx e \quad \text{for all } k > 0$$
$$x \approx g^k(x) \implies x \approx e \quad \text{for all } k > 0.$$

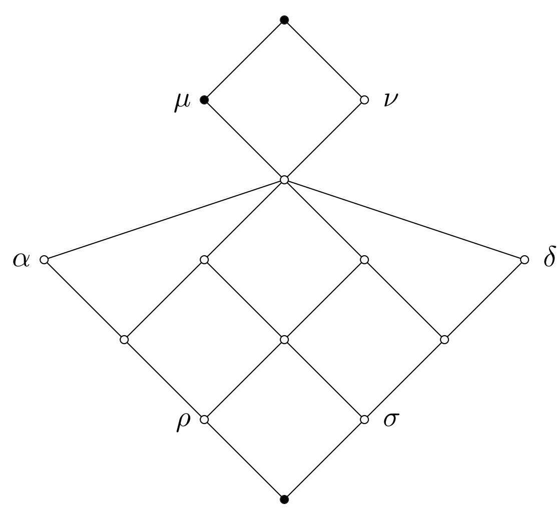


FIGURE 4. Dual near-leaf

Again, it is instructive to make a chart of the dual near-leaf with the corresponding congruence generators and theories (where $\sim$ indicates that laws are equivalent modulo the defining relations).

α	$\langle a_k, b_k\rangle \vee \langle b_k, c_k\rangle \vee \langle c_k, d_k\rangle$	$A(x) \Rightarrow B(x) \Rightarrow C(x) \Rightarrow D(x)$
δ	$\langle d_k, c_{k+1}\rangle \vee \langle c_{k+1}, b_{k+2}\rangle \vee \langle b_{k+2}, a_{k+3}\rangle$	$D(x) \Rightarrow C(fx) \Rightarrow B(f^2x) \Rightarrow A(f^3x)$
ρ	$\langle d_k, c_{k+1}\rangle$	$D(x) \implies C(f(x))$
σ	$\langle a_k, b_{k-1}\rangle$	$A(x) \implies B(g(x))$
μ	$\langle 0, a_k\rangle = \langle 0, b_k\rangle = \langle 0, c_k\rangle = \langle 0, d_k\rangle$	$A(x) \sim B(x) \sim C(x) \sim D(x)$
ν	$\langle a_k, 1\rangle = \langle b_k, 1\rangle = \langle c_k, 1\rangle = \langle d_k, 1\rangle$	$A(x) \Rightarrow x \approx e \sim \cdots \sim D(x) \Rightarrow x \approx e$

You can figure the omitted ones. Note that only 0, 1 and μ are equational.

The dual near-leaf is not an upper bounded lattice, and thus answers in the negative Question 3 from Adams, Adaricheva, Dziobiak and Kravchenko [1]. Dually, not every finite Q-lattice is lower bounded.

7. Discussion

The constant e played a crucial role in the reduction to one variable, Theorem 3. This was borrowed from Gorbunov, see [5], and some such device is necessary. For example, the law $A(x) \implies B(y)$ says that if there exists an x with $A(x)$, then $B(y)$ holds for all y. It cannot be reduced to one variable without introducing a constant.

On the other hand, the properties that we have given to e have the consequence of making the unit congruence of $\mathbf{F}_{\mathcal{C}}(1)$ compact, so that we must limit consideration to semilattices with both 0 and 1. In turn, if $\mathbf{S}$ has both 0 and 1, then the largest congruence of $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$ is compact.

Lemma 7. *Let $\mathbf{S} = (S, +, 0, \mathcal{F})$ be a semilattice with a monoid $\mathcal{F}$ of operators. The largest congruence of $\mathbf{S}$ is compact if and only if $\mathcal{F}u$ is cofinal for some $u \in S$.*

Question: Under what circumstances is it true that if the largest congruence of $\mathbf{S}$ is compact, then $\mathrm{Con}(\mathbf{S}) \cong \mathrm{Con}(\mathbf{T})$ for some $\mathbf{T}$ with both 0 and 1?

Now consider a semilattice with operators $\mathbf{S}$ that has 0 and 1. Let θ be any congruence on $\mathbf{S}$, and let F be the order-filter $1/\theta$. Then F satisfies

$$(*) \quad (\forall f \in \mathcal{F})(\forall a, b \in F)(\forall s \in S)\ fa + s \in F \implies fb + s \in F.$$

For any such order filter F, there is an interval $[\varphi(F), \psi(F)]$ in $\mathrm{Con}(\mathbf{S})$ of congruences θ such that $F = 1/\theta$. The congruences $\varphi(F)$ and $\psi(F)$ can be described thusly.

- $(x, y) \in \varphi(F)$ if $x = y$ or there are a sequence $x = x_0, x_1, \ldots, x_n = y$, operators $f_i \in \mathcal{F}$, elements a_i, $b_i \in F$ and $s_i \in S$ such that

$$x_i = f_i a_i + s_i$$
$$x_{i+1} = f_i b_i + s_i.$$

- $(x, y) \in \psi(F)$ if for all $f \in \mathcal{F}$ and all $s \in S$,

$$fx + s \in F \quad \text{iff} \quad fy + s \in F.$$

As before, these functions can be extended to $\varphi(x)$ and $\psi(x)$, and φ is an interior operator on $\mathrm{Con}(\mathbf{S})$. These operations should be useful, but at this point it is not clear how.

8. Summary and questions to pursue

Our results can be reasonably summarized, and compared to previous results, by considering four classes of lattices.

- $\mathcal{Q}$ is all Q-lattices $L_q(\mathcal{K})$ for a quasivariety $\mathcal{K}$.

- $\mathcal{C}^d$ is all duals of Con $(\mathbf{S}, +, 0, \mathcal{F})$ with $\mathcal{F}$ a set of operators.
- $\mathcal{S}$ is all $\text{Sp}(\mathbf{A}, \varepsilon)$ with A algebraic and ε a Brouwerian, filterable, continuous quasi-order.
- $\mathcal{J}$ is all join semidistributive, atomic, dually algebraic lattices supporting an equaclosure operator satisfying the duals of conditions (I1)–(I9) from Part I.

Each of the latter three classes contains $\mathcal{Q}$, and provides a different type of description.

This paper has focused on the representation of lattices of quasi-equational theories as congruence lattices of semilattices with operators. This viewpoint has several advantages.

- It represents lattices of quasi-equational theories in terms of familiar objects.
- It renders the representation of $\text{Con}(\mathbf{S}, +, 0)$ quite transparent.
- It allowed us to represent $\text{Con}(\mathbf{S}, +, 0, \mathcal{G})$ when $\mathcal{G}$ is a group and $\mathbf{S}$ has both 0 and 1.
- It led to the identification of new properties of the natural equa-interior operator.

A gap in our current understanding is that we have not found an effective way to deal with condition (I8) for equa-interior operators, the existence of a pseudo-one, which can fail in the congruence lattice of a semilattice with operators.

The older representations of Q-lattices as lattices of algebraic sets had problems with the duals of conditions (I8) and (I9). In that sense, the new representation may be preferable. But it is not clear at all that $\mathcal{C}^d$ and $\mathcal{S}$ are comparable: we don't know if the relation determining the congruence lattice of a semilattice with operators as a complete sublattice of Sp(Con $\mathbf{S}$) is continuous. (See Appendix I of Part I.) Since continuity is the only issue, it is true that if $\mathbf{S}$ is finite, then the dual of $\text{Con}(\mathbf{S}, +, 0, \mathcal{F})$ is in $\mathcal{S}$.

There is no reason to think that the properties describing $\mathcal{J}$ actually characterize Q-lattices, but we believe that they summarize what is known at this point. That leaves us with some interesting questions.

(1) Given a lattice $\mathbf{L} = \text{Con}(\mathbf{S}, +, 0, \mathcal{F})$, when can we represent $\mathbf{L}$ as the lattice of theories of a quasivariety?
(2) In particular, can we represent the dual leaf (Figure 2) as the lattice of theories of a quasivariety?
(3) Given a finite meet semidistributive lattice $\mathbf{L}$ with an equa-interior operator, when can we represent $\mathbf{L}$ as either the congruence lattice of a semilattice with operators or the lattice of theories of a quasivariety?
(4) Find an algorithm to determine whether a finite meet semidistributive lattice supports an equa-interior operator satisfying (I1)–(I9). (We have done this for the original conditions; see [2].)

(5) We know that the variety given by the law $x \approx y$ plays a special role in the lattice of quasivarieties. Find a good description of this behavior in the context of semilattices with operators.

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