Math 215 Final Exam December 2007

Name: Solutions Student ID:

(1) Find the limits if they exist. If a limit does not exist, explain why.

(a) $\lim_{x \to -1} \frac{x^2+5x+4}{x^2+3x+2}$ $= \lim_{x \to -1} \frac{\cancel{(x+1)}(x+4)}{\cancel{(x+1)}(x+2)} = \frac{3}{1} = 3$

(b) $\lim_{x \to 0} x \sin \frac{1}{x}$ $= 0$

(c) $\lim_{x \to 0} \frac{\sin x}{x}$ $= 1$

(2) Give approximations of the following.

(a) $\sin 0.2$ $\approx .2$

(b) $\cos 0.2$ $= 1 - \frac{(.2)^2}{2} = 1 - \frac{.04}{2} = 1 - .02 = .98$

(c) $\csc 0.2$ $= \frac{1}{\sin .2} \approx 5$

(3) Differentiate.

(a) $f(x) = 2x^5 - \pi + \frac{2}{x^5}$

$f'(x) = 10x^4 - 2 \cdot 5x^{-6} = 10x^4 - \frac{10}{x^6}$

(b) $y = x^7 \sin x$

$y' = 7x^6 \sin x + x^7 \cos x$

(c) $y = \ln(x^4 + x + 1)$

$y' = \frac{4x^3+1}{x^4+x+1}$

(d) $h(x) = \frac{\tan x}{x}$

$h'(x) = \frac{x \sec^2 x - \tan x}{x^2}$

(e) $m(t) = \sqrt{t^2+4}$ $= (t^2+4)^{1/2}$

$\frac{dm}{dt} = \frac{1}{2}(t^2+4)^{-1/2} \cdot 2t = \frac{t}{\sqrt{t^2+4}}$

(f) $h(t) = t^7 + 7^t$

$$\frac{dh}{dt} = 7t^6 + 7^t \ln 7$$

(4) Find the equation of the tangent line to the curve $y = x^2 + x + 1$ at $x = 2$.

$x_0 = 2$

$y_0 = 7$

$y' = 2x+1$ so $m = 5$

tangent line:

$y - 7 = 5(x-2)$

$y = 5x - 3$.

(5) It costs \$2 apiece to make tidgets. The number n of tidgets that you can sell is related to the price x by $n = 500 - x$. Your fixed costs are \$200. What price should you charge to maximize your profit?

$$P = (500-x)(x-2) = -x^2 + 502x - 1000$$

$$\frac{dP}{dx} = -2x + 502$$

$$= 0 \text{ if } \boxed{x = \$251}$$

(6) The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. A conical pile of dinosaur excrement is settling, so the radius and height are changing but the volume is not. Give the formula that relates the rates $\frac{dr}{dt}$ and $\frac{dh}{dt}$ during this process.

$$\frac{dV}{dt} = 0 = \frac{\pi}{3} \cdot 2r \cdot \frac{dr}{dt} \cdot h + \frac{\pi}{3} r^2 \frac{dh}{dt}.$$

(7) For a young growing dinosaur, the length of the skull S and the length of the backbone B are related by $S = 1.2B^{0.93}$. Give the formula relating the rates of growth of these two parts. (There are two correct ways to write this; choose either.)

$$\ln S = \ln 1.2 + 0.93 \ln B$$

$$\frac{dS/dt}{S} = .93 \frac{dB/dt}{B}$$

(8) Integrate.

(a) $\int_1^2 x + \frac{1}{x}\, dx =$ $\frac{x^2}{2} + \ln x \Big|_1^2 = \frac{4}{2} + \ln 2 - \left(\frac{1}{2} + 0\right) = \frac{3}{2} + \ln 2.$

(b) $\int_1^5 x\sqrt{x-1}\, dx =$ $\int_0^4 (u+1)u^{1/2}\, du = \int_0^4 u^{3/2} + u^{1/2}\, du$

$u = x - 1$

$u + 1 = x$

$du = dx$

$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} \Big|_0^4 = \frac{2}{5} \cdot 32 + \frac{2}{3} \cdot 8$

$= \frac{64}{5} + \frac{16}{3}$

(c) $\int x\cos x^2\, dx =$ $\int \frac{1}{2} \cos u\, du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin x^2 + C.$

$u = x^2$

$du = 2x\, dx$

(d) $\int xe^{5x}\, dx =$ $\frac{1}{5}e^{5x} - \int \frac{1}{5}e^{5x}\, dx = x\frac{1}{5}e^{5x} - \frac{1}{25}e^{5x} + C.$

$u = x$ $\quad du = dx$

$v = \frac{1}{5}e^{5x}$ $\quad dv = e^{5x}\, dx$

(e) $\int \frac{x+1}{x^2+4}\, dx =$ $\int \frac{x}{x^2+4} + \frac{1}{x^2+4}\, dx = \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + C.$

(f) $\int \frac{x+1}{x^2-4}\, dx =$ $\int \frac{3/4}{x-2} + \frac{1/4}{x+2}\, dx = \frac{3}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C.$

$\frac{x+1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$

$x + 1 = A(x+2) + B(x-2)$

$3 = 4A$ so $A = 3/4$

$-1 = -4B$ so $B = 1/4$

(9) Solve.

(a) $y' - 5y = 0$, $y(0) = 4$

$$y' = 5y \Rightarrow y = Ce^{5t}$$
$$y(0) = 4 = C$$
$$y = 4e^{5t}$$

(b) $y' = 2t(y-5)$

$$\frac{y'}{y-5} = 2t$$
$$\ln|y-5| = t^2 + c$$
$$|y-5| = e^{t^2+c}$$
$$\to y - 5 = De^{t^2}$$
$$y = 5 + De^{t^2}$$

(10) The population of numbats in Western Australia is given (in millions) by $\frac{dn}{dt} = 0.1n - .02n^2$. In the year 2000 (count this as year zero) the population was 3 million. Find the population in year t. What is the carrying capacity?

$$n = \frac{.1/.02}{1 + Ce^{-.1t}} = \frac{5}{1 + Ce^{-.1t}}$$

So the carrying capacity is 5 (million numbats).

The initial condition is

$$3 = \frac{5}{1 + Ce^{0}} = \frac{5}{1+C}$$

$$3 + 3C = 5$$
$$3C = 2$$
$$C = \frac{2}{3}$$

So $$n = \frac{5}{1 + \frac{2}{3}e^{-.1t}}$$

(11) Give the statements of two of the following theorems.

(a) The fundamental theorem of algebra

Every real polynomial factors uniquely into factors of the form.

- a constant
- linear terms $x-a$
- quadratics x^2+bx+c with $b^2-4c<0$.

(b) The fundamental theorem of calculus

If $f(x)$ is continuous, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

(c) The mean value theorem

If $f(x)$ is differentiable on $[a,b]$,

then $f(b) = f(a) + f'(c)(b-a)$

for some c between a and b.

(12) What are the connections between the following types of functions?

(a) Differentiable functions

(b) Continuous functions

(c) Integrable functions

Every differentiable function is continuous, and every continuous function is integrable.

(1) Find the general solution to the following differential equations.

(a) $y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0$
$(r-3)(r-2) = 0$ $\to$ $y = Ae^{3t} + Be^{2t}$.

(b) $y'' + y' - 20y = 0$

$r^2 + r - 20 = 0$
$(r+5)(r-4) = 0$ $\to$ $y = Ae^{-5t} + Be^{4t}$

(c) $y'' - 2y' = 0$

$r^2 - 2r = 0$
$r(r-2) = 0$ $\to$ $y = A + Be^{2t}$

(d) $y'' - 2y' - 4y = 0$

$r^2 - 2r - 4 = 0$
$r = 1 \pm \sqrt{5}$ $\to$ $y = Ae^{(1+\sqrt{5})t} + Be^{(1-\sqrt{5})t}$.

(e) $y'' + 25y = 0$

$r^2 + 25 = 0$
$r = \pm 5i$

$y = A\cos(5t + \varphi)$
or
$y = A\cos(5t) + B\sin(5t)$.

(f) $y'' + 2y = 0$

$r^2 + 2 = 0$
$r = \pm\sqrt{2}i$

$y = A\cos(\sqrt{2}t + \varphi)$
or
$y = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$.

(g) $y'' + 5y = 0$

$r^2 + 5 = 0$
$r = \pm\sqrt{5}i$

$y = A\cos(\sqrt{5}t + \varphi)$ or $y = A\cos(\sqrt{5}t) + B\sin(\sqrt{5}t)$.

(h) $y'' + 2y' + 3y = 0$

$r^2 + 2r + 3 = 0$
$r = -1 \pm \sqrt{2}i$

$y = Ae^{-t}\cos(\sqrt{2}t + \varphi)$
or $y = Ae^{-t}\cos(\sqrt{2}t) + Be^{-t}\sin(\sqrt{2}t)$.

(i) $y'' + 6y' + 10y = 0$

$r^2 + 6r + 10 = 0$
$r = -3 \pm i$ $\to$ $y = Ae^{-3t}(\cos t + \varphi)$
OR
$y = Ae^{-3t}\cos t + Be^{-3t}\sin t$.

(2) Solve $y'' + y = 0$, $y(0) = 1$, $y'(0) = 3$.

$r^2 + 1 = 0$
$r^2 = -1$
$r = \pm i$

So the general solution is

$y = A\cos t + B\sin t$

Note $y' = -A\sin t + B\cos t$. $\to$

Substituting $t = 0$

$y(0) = 1 = A\cos 0 + B\sin 0$
$1 = A$

$y'(0) = 3 = -A\sin 0 + B\cos 0$
$3 = B$

So

$y = \cos t + 3\sin t$.