Math 215 First Exam Solutions Friday, September 28, 2007

Name:____________________________ Student ID:____________________

Note. In some cases answers can be simplified in more than one way.

(1) Find the limits if they exist. If a limit does not exist, explain why.

(a) $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \frac{5}{4}$

(b) $\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} = \frac{1}{2\sqrt{2}}$

(c) $\lim_{x\to 0} x\sin\frac{1}{x} = 0$

(d) $\lim_{x\to 0} \frac{\sin x}{x^2+2x} = \frac{1}{2}$

(2) Differentiate. You need not simplify complicated expressions.

(a) $f(x) = 5x^4 + 4.2 + \frac{1}{x^2}$

$f'(x) = 20x^3 - \frac{2}{x^3}$

(b) $g(x) = x^{45}\tan x$

$g'(x) = 45x^{44}\tan x + x^{45}\sec^2 x$

(c) $h(t) = 7^t + t^7$

$h'(t) = 7^t \ln 7 + 7t^6$

(d) $y = \frac{x^2}{x^4+x+1}$

$y' = \frac{-2x^5+x^2+2x}{(x^4+x+1)^2}$

(e) $m(t) = \sqrt{t+\cos t}$

$m'(t) = \frac{1}{2}(t+\cos t)^{-\frac{1}{2}}(1-\sin t)$

(f) $n(x) = \ln(x^2+\pi x+4)$

$\frac{dn}{dx} = \frac{2x+\pi}{x^2+\pi x+4}$

(g) $q(t) = \arctan(t+1)$

$\frac{dq}{dt} = \frac{1}{t^2+2t+2}$

(h) $r(\theta) = \sin^3 \theta$
$\frac{dr}{d\theta} = 3 \sin^2 \theta \cdot \cos \theta$

(3) Find the tangent line to the curve $y = 2^x$ at $x = 0$.

$$y = \ln 2 \cdot x + 1$$

(4) Find $\frac{dy}{dx}$ for the curve $x^2 y + y^2 = 4$. At what points is the tangent line horizontal?

$$y' = \frac{-2xy}{x^2 + 2y}$$

Thus $y' = 0$ if $x = 0$ or $y = 0$. If $x = 0$ then $y = \pm 2$, while $y = 0$ is impossible.

(5) Consider the function

$$h(x) = \begin{cases} (1+x)^{\frac{1}{x}} & \text{if } x \neq 0 \\ e & \text{if } x = 0 \end{cases}$$

What limit would you take to find $h'(0)$?

$$h'(0) = \lim_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x - 0}$$

(6) Complete the definition: $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

State an important theorem about differentiable functions.
Every differentiable function is continuous.

(7) Prove that $\lim_{x \to 2} 5x = 10$, OR for extra credit, prove that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.
Let $\epsilon > 0$. If $|x - 2| < \frac{\epsilon}{5}$, then

$$|5x - 10| = 5|x - 2| < 5\frac{\epsilon}{5} = \epsilon.$$

Thus the limit is correct.

Let $\epsilon > 0$. If $0 < |x| < \epsilon$, then

$$|x \sin \frac{1}{x}| = |x| \, |\sin \frac{1}{x}| < |x| = \epsilon$$

since $|\sin \frac{1}{x}| \leq 1$. This verfies the limit.