THE RIGHT PROOF OF THE LAGRANGE REMAINDER FORMULA

Most calculus texts follow a version of Lagrange’s original proof of the remainder formula for Taylor polynomials. This proof is ingenious, but not very intuitive, and not easily understood by undergraduates. On the other hand, the proof of the remainder formula for interpolatory polynomials uses only Rolle’s theorem, and is easy to understand (see any numerical analysis text). A slight variation of the latter proof gives an easy proof of the remainder formula for Taylor polynomials.

**Lemma 1.** Let \( h(t) \) be a function which is \( n \) times differentiable on \([0, x]\). If \( h(x) = 0 \) and \( h^{(k)}(0) = 0 \) for \( 0 \leq k < n \), then there exists \( \eta \in (0, x) \) such that \( h^{(n)}(\eta) = 0 \).

**Proof.** Since \( h(0) = 0 = h(x) \), by Rolle’s theorem there exists \( c \in (0, x) \) such that \( h'(c) = 0 \). Applying induction to \( h'(t) \) on the interval \([0, c]\) yields the conclusion of the lemma. \( \quad \square \)

For \( x < 0 \), replace \([0, x]\) and \((0, x)\) by \([x, 0]\) and \((x, 0)\), respectively.

Let \( p(x) \) be the \((n - 1)\)-st Taylor polynomial for \( f(x) \). Fixing \( x \), apply the lemma to

\[
 h(t) = f(t) - p(t) - \left[ \frac{f(x) - p(x)}{x^n} \right] t^n
\]

to obtain the Lagrange remainder formula

\[
 f(x) - p(x) = \frac{f^{(n)}(\eta)}{n!} x^n.
\]

(I found this proof while teaching numerical analysis. It is the proof given in Apostol’s *Calculus*, but does not seem to be well known. - JBN)

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*Date: January 22, 2007.*