(1) Find the mass of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x+y+z = 1$ if the density is given by $\delta(x, y, z) = 1 - z$.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 - z dz\, dy\, dx = \frac{1}{8}$$

(2) Find the centroid of the region between the xy-plane and $z = 1 - x^2 - y^2$.

The volume is

$$\int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r\, dz\, dr\, d\theta = \frac{\pi}{2}$$

while

$$V\overline{z} = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} z \cdot r\, dz\, dr\, d\theta = \frac{\pi}{6}$$

so $\overline{z} = \frac{\pi/6}{\pi/2} = \frac{1}{3}$. By symmetry $\overline{x} = \overline{y} = 0$.

(3) Find the mass of the earth if the density is given by $\delta = \frac{K}{\rho}$ and the radius is R.

$$M = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{K}{\rho} \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi K R^2$$

(4) Find the length of the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$.

With the parameterization $x(t) = t^2$, $y(t) = t$, $0 \le t \le 1$ you get

$$s = \int_0^1 \sqrt{1 + 4t^2}\, dt = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2 + \sqrt{5})$$

With the parameterization $x(t) = t$, $y(t) = \sqrt{t}$, $0 \le t \le 1$ you will need to do a substitution such as $4t = u^2$, or $4t = \tan^2\theta$, to evaluate the integral.

(5) Find the line integral $\int_L (2x + 2y)\, dx + (2x + 9y^2)\, dy$ along the line from $(0, 1)$ to $(2, 4)$ both directly and by using the fundamental theorem.

Directly, use the parameterization $x(t) = 2t$, $y(t) = 1+3t$, $0 \le t \le 1$. For the fundamental theorem, use the fact that $(2x + 2y)\mathbf{i} + (2x + 9y^2)\mathbf{j} = \nabla(x^2 + 2xy + 3y^3)$ and $\int_{A\to B} \nabla\varphi \cdot dr = \varphi(B) - \varphi(A)$. Either way, the answer is 209.

(6) Find the line integral $\int_L (2x\cos y + 5)\, dx + (x - x^2 \sin y)\, dy$ counterclockwise around the ellipse $\frac{x^2}{2} + \frac{y^2}{5} = 1$ both directly and by Green's Theorem.

With Green's Theorem, the integral is $\int\int 1\, dA = \pi\sqrt{10}$. It seems nigh impossible to do this integral directly, but perhaps that shows the value of Green's Theorem!

(7) Find φ such that $f = \nabla\varphi$: $f(x, y) = (x + \sin y)\mathbf{i} + (x\cos y + 5)\mathbf{j}$.

$\varphi = \frac{x^2}{2} + x\sin y + 5y$

(8) Find φ such that $f = \nabla\varphi$: $f(x, y, z) = (7yz^2 + 1)\mathbf{i} + (7xz^2 + y)\mathbf{j} + (14xyz + \cos z)\mathbf{k}$.

$\varphi = 7xyz^2 + x + \frac{y^2}{2} + \sin z$

(9) How much work is required to stretch a spring with spring constant k from equilibrium to L units beyond equilibruim?

Measuring the extension x on an axis with the equilibrium at $x = 0$, the spring force is $F = -kx = -\frac{dV}{dx}$. So $V = \frac{1}{2}kx^2$ and the work done in the problem, which is the change in potential energy for a conservative force like this, is $W = \frac{1}{2}kL^2$.