(1) Complete the definitions.

(a) An algebra $\mathbf{V} = \langle V, +, -, 0, \mu_c : c \in F \rangle$ is a *vector space* if it satisfies the closure axioms and the following eight equational axioms:

(b) The *null space* or *kernel* of a matrix $\mathbf{M}$ is

(c) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are *independent* if

(d) B is a *basis* for a vector space $\mathbf{V}$ if

(e) The *dimension* of a vector space $\mathbf{V}$ is

(f) The real-valued function $\| \cdot \|$ is a *vector norm* if

(g) A *linear transformation* from U to V is a map $T : U \to V$ satisfying

(2) Which of the following are subspaces of $\mathbb{R}^n$? Take $\mathbf{M}$ to be an arbitrary matrix and $\mathbf{b}$ an arbitrary (nonzero) vector.

(a) $Q = \{\mathbf{x} : x_1^2 - x_2^2 = 1\}$.
(b) $R = \{\mathbf{x} : x_1^2 - x_2^2 = 0\}$.
(c) $S = \{\mathbf{x} : \mathbf{Mx} = \mathbf{b}\}$.
(d) $T = \{\mathbf{x} : \mathbf{Mx} = \mathbf{0}\}$.

(3) Which of the following are subspaces of $C(\mathbb{R})$?

(a) $Q = \mathrm{Span}(\emptyset)$.
(b) $R = \mathrm{Span}(1, t, t^2, t^3)$.
(c) $S = \mathrm{Span}(1, t, t^2, t^3, t^4, \ldots)$.
(d) $T = \mathrm{Span}(\cos t, \sin t)$.

(4) List five equivalents to the statement $\mathbf{M}$ *is nonsingular.*

(5) Find a basis for $\mathbf{W} = \{\mathbf{x} \in \mathbb{R}^4 : x_1 + 2x_2 - x_3 - x_4 = 0\}$.

(6) Find an orthonormal basis for $\mathbf{W} = \{\mathbf{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$.

(7) Find bases for the kernel and range of this matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 1 & -4 \end{bmatrix}$$

(8) For $\mathbf{A}$ from the preceding problem and $\mathbf{b} = [1 \ -5 \ 1 \ 3]^T$, find $\|\mathbf{b}\|_1$, $\|\mathbf{A}\|_1$, $\|\mathbf{b}\|_\infty$, $\|\mathbf{A}\|_\infty$.

(9) Prove that if $\mathbf{y}$ and $\mathbf{z}$ are two solutions to the equation $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{y} - \mathbf{z}$ is in the kernel of $\mathbf{A}$. Then show that if $\mathbf{z}$ is a solution and $\mathbf{n} \in \ker \mathbf{A}$, then $\mathbf{z} + \mathbf{n}$ is a solution.

(10) Prove that if B is a basis for a vector space V, then every vector in V has a unique expression as a linear combination of elements of B.