(1) Complete the definitions.
  (a) An algebra $\mathbf{V} = \langle V, +, -, 0, \mu_c : c \in F \rangle$ is a *vector space* if it satisfies the closure axioms and the following eight equational axioms:
    (i) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
    (ii) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
    (iii) $\mathbf{x} + 0 = \mathbf{x}$
    (iv) $\mathbf{x} + (-\mathbf{x}) = 0$
    (v) $1 \cdot \mathbf{x} = \mathbf{x}$
    (vi) $c(d\mathbf{x}) = (cd)\mathbf{x}$
    (vii) $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$
    (viii) $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
  (b) The *null space* or *kernel* of a matrix $\mathbf{M}$ is $\{\mathbf{x} : \mathbf{M}\mathbf{x} = 0\}$.
  (c) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are *independent* if $\sum_{i=1}^{n} c_i \mathbf{v}_i = 0$ implies $c_i = 0$ for all i.
  (d) B is a *basis* for a vector space $\mathbf{V}$ if B is independent and $\mathrm{Span}(B) = \mathbf{V}$.
  (e) The *dimension* of a vector space $\mathbf{V}$ is the number of elements in a basis.
  (f) The real-valued function $\|\cdot\|$ is a *vector norm* if
    (i) $\|\mathbf{x}\| > 0$ if $\mathbf{x} \neq \mathbf{0}$, and $\|\mathbf{0}\| = 0$.
    (ii) $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$.
    (iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
  (g) A *linear transformation* from U to V is a map $T : U \to V$ satisfying $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(c\mathbf{x}) = cT(\mathbf{x})$.

(2) Which of the following are subspaces of $\mathbb{R}^n$? Take $\mathbf{M}$ to be an arbitrary matrix and $\mathbf{b}$ an arbitrary (nonzero) vector.
  (a) $Q = \{\mathbf{x} : x_1^2 - x_2^2 = 1\}$. Not a subspace.
  (b) $R = \{\mathbf{x} : x_1^2 - x_2^2 = 0\}$. Not a subspace.
  (c) $S = \{\mathbf{x} : \mathbf{M}\mathbf{x} = \mathbf{b}\}$. Not a subspace.
  (d) $T = \{\mathbf{x} : \mathbf{M}\mathbf{x} = \mathbf{0}\}$. Subspace.

(3) Which of the following are subspaces of $C(\mathbb{R})$?
  (a) $Q = \mathrm{Span}(\emptyset)$. Subspace.
  (b) $R = \mathrm{Span}(1, t, t^2, t^3)$. Subspace.
  (c) $S = \mathrm{Span}(1, t, t^2, t^3, t^4, \ldots)$. Subspace.
  (d) $T = \mathrm{Span}(\cos t, \sin t)$. Subspace.

  The span of any subset $X \subseteq V$ is the set of linear combinations of vectors from X, which is the smallest subspace of V containing X.

(4) List five equivalents to the statement $\mathbf{M}$ *is nonsingular.* (Here $\mathbf{M}$ should be a square $n \times n$ matrix.)
  (a) $\ker \mathbf{M} = \{0\}$, i.e., $\mathbf{M}\mathbf{x} = 0$ implies $\mathbf{x} = 0$.
  (b) $\mathbf{M}$ is left invertible, i.e., there exits $\mathbf{L}$ such that $\mathbf{L}\mathbf{M} = \mathbf{I}$.
  (c) The range of $\mathbf{M}$ is $\mathbb{R}^n$.
  (d) $\mathbf{M}$ is right invertible, i.e., there exits $\mathbf{R}$ such that $\mathbf{M}\mathbf{R} = \mathbf{I}$.
  (e) The columns of $\mathbf{M}$ are independent.
  (f) The columns of $\mathbf{M}$ span $\mathbb{R}^n$.
  (g) The rows of $\mathbf{M}$ are independent.
  (h) The rows of $\mathbf{M}$ span $\mathbb{R}^n$.
  (i) $\det \mathbf{M} \neq 0$.
  (j) Every equation $\mathbf{M}\mathbf{x} = \mathbf{b}$ has a solution.
  (k) Every equation $\mathbf{M}\mathbf{x} = \mathbf{b}$ has a unique solution.

(5) Find a basis for $\mathbf{W} = \{\mathbf{x} \in \mathbb{R}^4 : x_1 + 2x_2 - x_3 - x_4 = 0\}$.

Any three independent vectors in $\mathbf{W}$ form a basis, e.g.,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

(6) Find an orthonormal basis for $\mathbf{W} = \{\mathbf{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$.

Any two orthogonal unit vectors in $\mathbf{W}$ will work, e.g.,

$$\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

(7) Find bases for the kernel and range of this matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 1 & -4 \end{bmatrix}$$

A basis for the kernel is

$$\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 4 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

and a basis for the range is

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(8) For $\mathbf{A}$ from the preceding problem and $\mathbf{b} = [1 \ -5 \ 1 \ 3]^T$, find $\|\mathbf{b}\|_1$, $\|\mathbf{A}\|_1$, $\|\mathbf{b}\|_\infty$, $\|\mathbf{A}\|_\infty$.

$\|\mathbf{b}\|_1 = 10$, $\|\mathbf{A}\|_1 = 8$, $\|\mathbf{b}\|_\infty = 5$, $\|\mathbf{A}\|_\infty = 6$.

(9) Prove that if $\mathbf{y}$ and $\mathbf{z}$ are two solutions to the equation $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{y} - \mathbf{z}$ is in the kernel of $\mathbf{A}$. Then show that if $\mathbf{z}$ is a solution and $\mathbf{n} \in \ker \mathbf{A}$, then $\mathbf{z} + \mathbf{n}$ is a solution.

If $\mathbf{Ay} = \mathbf{b}$ and $\mathbf{Az} = \mathbf{b}$, then $\mathbf{A}(\mathbf{y} - \mathbf{z}) = \mathbf{Ay} - \mathbf{Az} = \ -\mathbf{b} = 0b$. Thus $\mathbf{y} - \mathbf{z} \in \ker \mathbf{A}$.

If $\mathbf{Az} = \mathbf{b}$ and $\mathbf{An} = 0$, then $\mathbf{A}(\mathbf{z} + \mathbf{n}) = \mathbf{Az} + \mathbf{An} = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

(10) Prove that if B is a basis for a vector space V, then every vector in V has a unique expression as a linear combination of elements of B.

A basis for V is an independent set that spans V. Let B be a basis and consider any vector $\mathbf{v} \in V$. Because B spans V, the vector $\mathbf{v}$ is a linear combination of elements of B, say $\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k$ with each $\mathbf{u}_i \in B$.

Suppose that also $\mathbf{v} = d_1\mathbf{u}_1 + \cdots + d_k\mathbf{u}_k$. Then $\sum c_i\mathbf{u}_i = \sum d_i\mathbf{u}_i$, whence $\sum (c_i - d_i)\mathbf{u}_i = 0$. Since B is independent, that implies $c_i - d_i = 0$ for $1 \le i \le k$, so that each $c_i = d_i$. Thus the coefficients in the expression of $\mathbf{v}$ as a linear combination of elements of B are unique.