NOTES ON THE $\ell_2$ MATRIX NORM

Let $M$ be a real $n \times n$ matrix, and let $\langle x, y \rangle = x^Ty = \sum x_ky_k$ denote the standard inner product. (All this business can be modified straightforwardly to the complex case, using the conjugate transpose.)

Recall these basic facts.

1. $\|x\|^2 = \langle x, x \rangle = x^Tx$.
2. $\|M\| = \max \{\|Mu\| : \|u\| = 1\}$.
3. So $\|M\|^2 = \max \{\langle Mu, Mu \rangle : \langle u, u \rangle = 1\}$.
4. $\langle x, My \rangle = x^TMy = (M^Tx)^Ty = \langle M^Tx, y \rangle$.

A matrix $U$ is orthogonal if $U^TU = I$, i.e., $U^{-1} = U^T$. Note that $U^T$ is also orthogonal.

Lemma 1. If $U$ is orthogonal, then $\langle Ux, Uy \rangle = \langle x, y \rangle$.

Lemma 2. If $S$ is symmetric, then its eigenvalues are real, and there is an orthogonal matrix $U$ such that $U^T SU$ is diagonal.

Lemma 3. $M^TM$ is symmetric. The eigenvalues of $M^TM$ satisfy $\lambda \geq 0$.

Now the crucial calculation, with $U^TM^TMU^T$:

$$\|M\|^2 = \max \{\langle Mu, Mu \rangle : \langle u, u \rangle = 1\}$$
$$= \max \{\langle M^Tu, u \rangle : \langle u, u \rangle = 1\}$$
$$= \max \{\langle UD^Tu, u \rangle : \langle u, u \rangle = 1\}$$
$$= \max \{\langle DU^Tu, U^Tu \rangle : \langle u, u \rangle = 1\}$$
$$= \max \{\langle Dv, v \rangle : \langle v, v \rangle = 1\}$$

since $U^Tu$ is again an arbitrary unit vector. But for diagonal matrices, this maximum is obtained from the largest diagonal entry.

Theorem 4. $\|M\|_2 = \sqrt{\lambda_{\text{max}}}$, where $\lambda_{\text{max}}$ is the largest eigenvalue of $M^TM$. 

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