MATH 411 WORKSHEET #8

(1) (a) Let \(d(x, y)\) be a distance function on a set \(V\). Define a new function \(d'(x, y) = \min(1, d(x, y))\). Show that \(d'(x, y)\) is a distance function.

(b) Assuming that \(V\) is a real vector space, show that there is not a norm \(\| \cdot \|\) such that \(d'(x, y) = \|x - y\|\).

(2) (a) Illustrate the parallelogram law with \(\| \cdot \|_2\).

(b) Show that \(\| \cdot \|_1\), \(\| \cdot \|_3\) and \(\| \cdot \|_\infty\) are not induced by an inner product.

(3) (a) Let \(\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx\) be the standard inner product on \(C[0, 1]\). Consider \(\{1, x, x^2, x^3\}\) as the basis for the subspace of (real) polynomials of degree at most 3. Find the Gram matrix for the inner product on this subspace.

(b) On this subspace, find the matrix for the differentiation operator \(D\) and its adjoint \(D^*\).

(4) Let \(V = \mathbb{R}^n\) with the standard inner product, and let \(T : V \to \mathbb{R}\) be a linear functional. Show that there is a vector \(x_0\) in \(V\) such that \(T(v) = x_0 \cdot v\) for all \(v \in V\).

(5) For each of these symmetric matrices, find an orthogonal matrix \(P\) such that \(P^TMP\) is diagonal.

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L = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}
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M = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}
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\[
N = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ -3 & -1 & 4 \end{bmatrix}
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