Recall that a *ring* is an algebra \( R = \langle R, +, -, \times, 0 \rangle \) satisfying the axioms

1. \( x + y = y + x \)
2. \( (x + y) + z = x + (y + z) \)
3. \( x + 0 = x \)
4. \( x + (-x) = 0 \)
5. \( (xy)z = x(yz) \)
6. \( x(y + z) = xy + xz \) and \( (y + z)x = yx + zx \).

Some are commutative, some are not. Some have a multiplicative identity 1 or \( e \); some do not.

A. Give two examples of rings. (Catch: no example should be the same as someone else’s!)

B. Prove the following statements about rings.

1. If \( x + y = x + z \), then \( y = z \).
2. The additive identity of \( R \) is unique.
3. The additive inverse of an element \( x \) is unique.
4. If \( xy = xz \) and \( x \neq 0 \), what can we conclude?
5. What if \( xy = xz \) and \( x \) has a multiplicative inverse? (In this case, \( R \) must have an identity element 1.)
6. \( x0 = 0 \)
7. \( x(-y) = -xy \)
8. \( -x(y) = -xy \)

C. Let \( R \) be a ring and \( a \in R \). Prove that \( \{ x \in R : ax = 0 \} \) is a subring of \( R \).

D. Consider the integers with the operations

\[
a \oplus b = a + b - 1 \quad a \odot b = a + b - ab
\]

Show that this defines a commutative ring with unit.

E. Can every ring be embedded into a ring with a multiplicative identity? How?