Appendix 2: The Axiom of Choice

In this appendix we want to prove Theorem 1.5.

Theorem 1.5. *The following set theoretic axioms are equivalent.*

(1) (Axiom of Choice) *If X is a nonempty set, then there is a map $\phi : \mathfrak{P}(X) \to X$ such that $\phi(A) \in A$ for every nonempty $A \subseteq X$.*
(2) (Zermelo well-ordering principle) *Every nonempty set admits a well-ordering (a total order satisfying the DCC).*
(3) (Hausdorff maximality principle) *Every chain in an ordered set $\mathcal{P}$ can be embedded in a maximal chain.*
(4) (Zorn's Lemma) *If every chain in an ordered set $\mathcal{P}$ has an upper bound in $\mathcal{P}$, then $\mathcal{P}$ contains a maximal element.*
(5) *If every chain in an ordered set $\mathcal{P}$ has a least upper bound in $\mathcal{P}$, then $\mathcal{P}$ contains a maximal element.*

Let us start by proving the equivalence of (1), (2) and (4).

(4) $\implies$ (2): Given a nonempty set X, let $\mathcal{Q}$ be the collection of all pairs (Y, R) such that $Y \subseteq X$ and R is a well ordering of Y, i.e., $R \subseteq Y \times Y$ is a total order satisfying the DCC. Order $\mathcal{Q}$ by $(Y, R) \sqsubseteq (Z, S)$ if Y is an initial segment of Z and R is the restriction of S to Y. In order to apply Zorn's Lemma, check that if $\{(Y_\alpha, R_\alpha) : \alpha \in A\}$ is a chain in $\mathcal{Q}$, then $(\overline{Y}, \overline{R}) = (\bigcup Y_\alpha, \bigcup R_\alpha) \in \mathcal{Q}$ and $(Y_\alpha, R_\alpha) \sqsubseteq (\overline{Y}, \overline{R})$ for every $\alpha \in A$, and so $(\overline{Y}, \overline{R})$ is an upper bound for $\{(Y_\alpha, R_\alpha) : \alpha \in A\}$. Thus $\mathcal{Q}$ contains a maximal element (U, T). Moreover, we must have $U = X$. For otherwise we could choose an element $z \in X - U$, and then the pair (U', T') with $U' = U \cup \{z\}$ and $T' = T \cup \{(u, z) : u \in U\}$ would satisfy $(U, T) \sqsubset (U', T')$, a contradiction. Therefore T is a well ordering of $U = X$, as desired.

(2) $\implies$ (1): Given a well ordering $\leq$ of X, we can define a choice function ϕ on the nonempty subsets of X by letting $\phi(A)$ be the least element of A under the ordering $\leq$.

(1) $\implies$ (4): For a subset S of an ordered set $\mathcal{P}$, let S^u denote the set of all upper bounds of S, i.e., $S^u = \{x \in P : x \geq s \text{ for all } s \in S\}$.

Let $\mathcal{P}$ be an ordered set in which every chain has an upper bound. By the Axiom of Choice there is a function ϕ on the subsets of P such that $\phi(S) \in S$ for every nonempty $S \subseteq P$. We use the choice function ϕ to construct a function that assigns a strict upper bound to every subset of P that has one as follows: if $S \subseteq P$ and $S^u - S = \{x \in P : x > s \text{ for all } s \in S\}$ is nonempty, define $\gamma(S) = \phi(S^u - S)$.

Fix an element $x_0 \in P$. Let $\mathfrak{B}$ be the collection of all subsets $B \subseteq P$ satisfying the following properties.

(1) B is a chain.
(2) $x_0 \in B$.
(3) $x_0 \leq y$ for all $y \in B$.
(4) If A is a nonempty order ideal of B and $z \in B \cap (A^u - A)$, then $\gamma(A) \in B \cap z/0$.

The last condition says that if A is a proper ideal of B, then $\gamma(A)$ is in B, and moreover it is the least element of B strictly above every member of A.

Note that $\mathfrak{B}$ is nonempty, since $\{x_0\} \in \mathfrak{B}$.

Next, we claim that *if B and C are both in $\mathfrak{B}$, then either B is an order ideal of C or C is an order ideal of B.* Suppose not, and let $A = \{t \in B \cap C : t/0 \cap B = t/0 \cap C\}$. Thus A is the largest common ideal of B and C; it contains x_0, and by assumption is a proper ideal of both B and C. Let $b \in B - A$ and $c \in C - A$. Now B is a chain and A is an ideal of B, so $b \notin A$ implies $b > a$ for all $a \in A$, whence $b \in B \cap (A^u - A)$. Likewise $c \in C \cap (A^u - A)$. Hence by (4), $\gamma(A) \in B \cap C$. Moreover, since b was arbitrary in $B - A$, again by (4) we have $\gamma(A) \leq b$ for all $b \in B - A$, and similarly $\gamma(A) \leq c$ for all $c \in C - A$. Therefore

$$\gamma(A)/0 \cap B = A \cup \{\gamma(A)\} = \gamma(A)/0 \cap C$$

whence $\gamma(A) \in A$, contrary to the definition of γ.

It follows, that if B and C are in $\mathfrak{B}$, $b \in B$ and $c \in C$, and $b \leq c$, then $b \in C$.

Also, you can easily check that if $B \in \mathfrak{B}$ and $B^u - B$ is nonempty, then $B \cup \{\gamma(B)\} \in \mathfrak{B}$.

Now let $U = \bigcup_{B \in \mathfrak{B}} B$. We claim that $U \in \mathfrak{B}$. It is a chain because for any two elements b, $c \in U$ there exist B, $C \in \mathfrak{B}$ with $b \in B$ and $c \in C$; one of B and C is an ideal of the other, so both are contained in the larger set and hence comparable. Conditions (2) and (3) are immediate. If a nonempty ideal A of U has a strict upper bound $z \in U$, then $z \in C$ for some $C \in \mathfrak{B}$. By the observation above, A is an ideal of C, and hence the conclusion of (4) holds.

Now U is a chain in $\mathcal{P}$, and hence by hypothesis U has an upper bound x. On the other hand, $U^u - U$ must be empty, for otherwise $U \cup \{\gamma(U)\} \in \mathfrak{B}$, whence $\gamma(U) \in U$, a contradiction. Therefore $x \in U$ and x is maximal in $\mathcal{P}$. In particular, $\mathcal{P}$ has a maximal element, as desired.

Now we prove the equivalence of (3), (4) and (5).

(4) $\implies$ (5): This is obvious, since the hypothesis of (5) is stronger.

(5) $\implies$ (3): Given an ordered set $\mathcal{P}$, let $\mathcal{Q}$ be the set of all chains in $\mathcal{P}$, ordered by set containment. If $\{C_\alpha : \alpha \in A\}$ is a chain in $\mathcal{Q}$, then $\bigcup C_\alpha$ is a chain in $\mathcal{P}$ that is the least upper bound of $\{C_\alpha : \alpha \in A\}$. Thus $\mathcal{Q}$ satisfies the hypothesis of (5), and hence it contains a maximal element C, which is a maximal chain in $\mathcal{P}$.

(3) $\implies$ (4): Let $\mathcal{P}$ be an ordered set such that every chain in $\mathcal{P}$ has an upper bound in P. By (3), there is a maximal chain C in $\mathcal{P}$. If b is an upper bound for C, then in fact $b \in C$ (by maximality), and b is a maximal element of $\mathcal{P}$.

There are many variations of the proof of Theorem 1.5, but it can always be arranged so that there is only one hard step, and the rest easy. The above version seems fairly natural.