Appendix 3: Formal Concept Analysis

Exercise 13 of Chapter 2 is to show that a binary relation $R \subseteq A \times B$ induces a pair of closure operators, described as follows. For $X \subseteq A$, let

$$\sigma(X) = \{b \in B : x \; R \; b \text{ for all } x \in X\}.$$

Similarly, for $Y \subseteq B$, let

$$\pi(Y) = \{a \in A : a \; R \; y \text{ for all } y \in Y\}.$$

Then the composition $\pi\sigma : \mathfrak{P}(A) \to \mathfrak{P}(A)$ is a closure operator on A, given by

$$\pi\sigma(X) = \{a \in A : a \; R \; b \text{ whenever } x \; R \; b \text{ for all } x \in X\}.$$

Likewise, $\sigma\pi$ is a closure operator on B, and for $Y \subseteq B$,

$$\sigma\pi(Y) = \{b \in B : a \; R \; b \text{ whenever } a \; R \; y \text{ for all } y \in Y\}.$$

In this situation, the lattice of closed sets $\mathcal{C}_{\pi\sigma} \subseteq \mathfrak{P}(A)$ is dually isomorphic to $\mathcal{C}_{\sigma\pi} \subseteq \mathfrak{P}(B)$, and we say that R establishes a *Galois connection* between the $\pi\sigma$-closed subsets of A and the $\sigma\pi$-closed subsets of B.

Of course, $\mathcal{C}_{\pi\sigma}$ is a complete lattice. Moreover, every complete lattice can be represented *via* a Galois connection.

Theorem. *Let $\mathcal{L}$ be a complete lattice, A a join dense subset of L and B a meet dense subset of L. Define $R \subseteq A \times B$ by $a \; R \; b$ if and only if $a \leq b$. Then, with σ and π defined as above, $\mathcal{L} \cong \mathcal{C}_{\pi\sigma}$ (and $\mathcal{L}$ is dually isomorphic to $\mathcal{C}_{\sigma\pi}$).*

In particular, for an arbitrary complete lattice, we can always take $A = B = L$. If $\mathcal{L}$ is algebraic, a more natural choice is $A = L^c$ and $B = M^*(\mathcal{L})$ (compact elements and completely meet irreducibles). If $\mathcal{L}$ is finite, the most natural choice is $A = J(\mathcal{L})$ and $B = M(\mathcal{L})$. Again the proof of this theorem is elementary.

Formal Concept Analysis is a method developed by Rudolf Wille and his colleagues in Darmstadt (Germany), whereby the philosophical Galois connection between objects and their properties is used to provide a systematic analysis of certain very general situations. Abstractly, it goes like this. Let G be a set of "objects" (*Gegenstände*) and M a set of relevant "attributes" (*Merkmale*). The relation $I \subseteq G \times M$ consists of all those pairs $\langle g, m \rangle$ such that g has the property m. A *concept* is a pair $\langle X, Y \rangle$ with $X \subseteq G$, $Y \subseteq M$, $X = \pi(Y)$ and $Y = \sigma(X)$. Thus

$\langle X, Y \rangle$ is a concept if X is the set of all elements with the properties of Y, and Y is exactly the set of properties shared by the elements of X. It follows (as in exercise 12, Chapter 2) that $X \in \mathcal{C}_{\pi\sigma}$ and $Y \in \mathcal{C}_{\sigma\pi}$. Thus if we order concepts by $\langle X, Y \rangle \leq \langle U, V \rangle$ iff $X \subseteq U$ (which is equivalent to $Y \supseteq V$), then we obtain a lattice $\mathfrak{B}(G, M, I)$ isomorphic to $\mathcal{C}_{\pi\sigma}$.

A small example will illustrate how this works. The rows of Table A1 correspond to seven fine musicians, and the columns to eight possible attributes (chosen by a musically trained sociologist). An $\times$ in the table indicates that the musician has that attribute.[1] The corresponding concept lattice is given in Figure A2, where the musicians are abbreviated by lower case letters and their attributes by capitals.

	Instrument	Classical	Jazz	Country	Black	White	Male	Female
J. S. Bach	$\times$	$\times$				$\times$	$\times$	
Rachmaninoff	$\times$	$\times$				$\times$	$\times$	
King Oliver	$\times$		$\times$		$\times$		$\times$	
W. Marsalis	$\times$	$\times$	$\times$		$\times$		$\times$	
B. Holiday			$\times$		$\times$			$\times$
Emmylou H.				$\times$		$\times$		$\times$
Chet Atkins	$\times$		$\times$	$\times$		$\times$	$\times$	

Table A1.

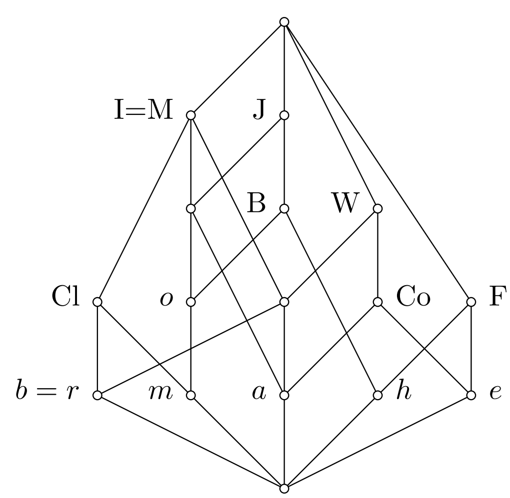

Figure A2

[1] To avoid confusion, androgynous rock stars were not included.

Formal concept analysis has been applied to hundreds of real situations outside of mathematics (e.g., law, medicine, psychology), and has proved to be a useful tool for understanding the relation between the concepts involved. Typically, these applications involve large numbers of objects and attributes, and computer programs have been developed to navigate through the concept lattice. A good brief introduction to concept analysis may be found in Wille [2] or [3], and the whole business is explained thoroughly in Ganter and Wille [1]. For online introductions, see the website of Uta Priss,

$$/www.upriss.org/fca/fca.html$$

Likewise, the representation of a finite lattice as the concept lattice induced by the order relation between join and meet irreducible elements (i.e., $\leq$ restricted to $J(\mathcal{L}) \times M(\mathcal{L})$) provides and effective and tractable encoding of its structure. As an example of the method, let us show how one can extract the ordered set $\mathcal{Q}_{\mathcal{L}}$ such that $\mathbf{Con}\, \mathcal{L} \cong \mathcal{O}(\mathcal{Q}(\mathcal{L}))$ from the table.

Given a finite lattice $\mathcal{L}$, for $g \in J(\mathcal{L})$ and $m \in M(\mathcal{L})$, define

$$\begin{aligned} g \nearrow m \quad & \text{if } g \nleq m \text{ but } g \leq m^*, \text{ i.e., } g \leq n \text{ for all } n > m, \\ m \searrow g \quad & \text{if } m \ngeq g \text{ but } m \geq g_*, \text{ i.e., } m \geq h \text{ for all } h < g, \\ g \updownarrow m \quad & \text{if } g \nearrow m \text{ and } m \searrow g. \end{aligned}$$

Note that these relations can easily be added to the table of $J(\mathcal{L}) \times M(\mathcal{L})$.

These relations connect with the relation $\underline{D}$ of Chapter 10 as follows.

Lemma. *Let $\mathcal{L}$ be a finite lattice and $g,\, h \in J(\mathcal{L})$. Then $g \,\underline{D}\, h$ if and only if there exists $m \in M(\mathcal{L})$ such that $g \nearrow m \searrow h$.*

Proof. If $g \,\underline{D}\, h$, then there exists $x \in L$ such that $g \leq h \vee x$ but $g \nleq h_* \vee x$. Let m be maximal such that $m \geq h_* \vee x$ but $m \ngeq g$. Then $m \in M(\mathcal{L})$, $g \leq m^*$, $m \geq h_*$ but $m \ngeq h$. Thus $g \nearrow m \searrow h$.

Conversely, suppose $g \nearrow m \searrow h$. Then $g \leq m^* \leq h \vee m$ while $g \nleq m = h_* \vee m$. Therefore $g \,\underline{D}\, h$. □

As an example, the table for the lattice in Figure A2 is given in Table A3. This is a reduction of the original Table A1: $J(\mathcal{L})$ is a subset of the original set of objects, and likewise $M(\mathcal{L})$ is contained in the original attributes. Arrows indicating the relations $\nearrow$, $\searrow$ and $\updownarrow$ have been added. The Lemma allows us to calculate $\underline{D}$ quickly, and we find that $|\mathcal{Q}_{\mathcal{L}}| = 1$, whence $\mathcal{L}$ is simple.

	I=M	Cl	J	Co	B	W	F
b=r	$\times$	$\times$	$\updownarrow$	$\updownarrow$	$\searrow$	$\times$	$\updownarrow$
o	$\times$	$\updownarrow$	$\times$		$\times$	$\nearrow$	$\nearrow$
m	$\times$	$\times$	$\times$	$\searrow$	$\times$	$\updownarrow$	$\updownarrow$
h	$\updownarrow$	$\searrow$	$\times$	$\searrow$	$\times$	$\updownarrow$	$\times$
e	$\updownarrow$	$\searrow$	$\updownarrow$	$\times$	$\updownarrow$	$\times$	$\times$
a	$\times$	$\updownarrow$	$\times$	$\times$	$\updownarrow$	$\times$	$\updownarrow$

Table A3.

References

1. B. Ganter and R. Wille, *Formale Begriffsanalyse: Mathematische Grundlagen*, Springer-Verlag, Berlin-Heidelberg, 1996. Translated by C. Franzke as Formal Concept Analysis: Mathematical Foundations, Springer, 1998.
2. R. Wille, *Restructuring lattice theory: an approach based on hierarchies of concepts*, Ordered Sets, I. Rival, ed., Reidel, Dordrecht-Boston, 1982, pp. 445–470.
3. R. Wille, *Concept lattices and conceptual knowledge systems*, Computers and Mathematics with Applications **23** (1992), 493–515.