LOGIC ON OTHER PLANETS

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No one doubts the educational value of travel. Exposure to different languages and cultures gives one an appreciation one's own culture, as well as a taste for the exotic. The traveller sees everything in a new light.

Travel to the other planets of our solar system remains difficult and prohibitively expensive. Nonetheless, recent unmanned space exploration has significantly increased our understanding of the neighborhood and its inhabitants. In this note, we take the mathematics student on a tour of the solar system, focusing on the logic used on different planets.

1. Logic on Earth

We begin with our own planet. The logic of us Earthlings, as taught in mathematics courses globally, is expressed formally in the algebra of logic developed by George Boole (and others) in the nineteenth century.[1]

The essence of an elementary logic[2] is that we have certain well-defined statements, called *propositions*, to which we can assign truth values. On Earth, the truth values used are T (which is interpreted as meaning *true*) and F (*false*). Propositions can be combined using various connectives: $\wedge$ (*and*), $\vee$ (*or*), and $\neg$ (*not*). The truth values of compound propositions are determined by the operations of a certain small algebra, on Earth the familiar two-element Boolean algebra $\mathbf{B}_2 = \langle \{T, F\}; \vee, \wedge, \neg, T, F \rangle$ whose operation tables are given below.

By means of "truth tables" we establish the laws (identities) valid in $\mathbf{B}_2$, which are the familiar axioms of Boolean algebra. Indeed, the following laws, along with their duals obtained by interchanging $\wedge$ and $\vee$, T and F, form a basis from which all the identities of Boolean algebra can be derived.[3]

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[1]Stan Burris' web page on the history of mathematical logic [5] contains a history with additional references. Boole's algebra was not modern Boolean algebra, but from somewhere in the asteroid belt between Earth (Boolean algebra) and Mars (Boolean rings). For a discussion of this interesting topic, see Burris [5], Gasser [7] and Hailperin [8].

[2]Here we consider only propositional logic. The predicate logic of other planets, including thier quantifiers and rules of inference, also deserve study. For a very general approach, see the second (technical) section of the note [12]. All this begs the question of to what extent the formalization of the laws of thought reflects, or even should reflect, the genuine article.

[3]Elementary Boolean algebra is presented in most textbooks on discrete mathematics or logic. To dig deeper, one can do worse than Halmos and Givant [10].

$\wedge$	T	F
T	T	F
F	F	F

$\vee$	T	F
T	T	T
F	T	F

$$\neg T = F \qquad \neg F = T$$

TABLE 1. Earth

For all x, y and z,

$$\begin{aligned} x \wedge x &= x \\ x \wedge y &= y \wedge x \\ x \wedge (y \wedge z) &= (x \wedge y) \wedge z \\ x \wedge (x \vee y) &= x \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ x \wedge T &= x \\ x \wedge \neg x &= F \\ \neg(\neg x) &= x \\ \neg(x \wedge y) &= \neg x \vee \neg y. \end{aligned}$$

We want to compare this algebra, and its laws, with those of other planets.

For example, suppose we run into a Martian who uses a different type of logic. Let us assume that he has a notion of a *proposition*, and that he assigns some sort of (what we would call) *truth values* to propositions, but that these notions are possibly different from ours (*true* and *false*) and that they are combined using the rules of a different algebra. The question then becomes whether his logic is stronger or weaker than ours, or equivalent, or incomparable. That is, we want to know when one logic can be interpreted in another, so that the second being can make sense of the first.

2. Logic on Mars

One of the more amazing discoveries of the Mars rovers is that the inhabitants of that planet use the integers modulo 2 as the basis for their logic. That is, Martians assign propositions a truth value of either 0 or 1, and combine them using the operations $+$ and $\cdot$ of Table 2.

Again, we don't know at this point in time what meaning a Martian has in mind when he says that a proposition has the binary value 1 or 0 (and probably won't know until we visit them). Maybe he means *true* or *false*; maybe not. But we do know that Martians combine propositions

$+$	1	0
1	0	1
0	1	0

$\cdot$	1	0
1	1	0
0	0	0

TABLE 2. Mars

using the laws of binary arithmetic, which are reflected in the algebra $\mathbf{Z}_2 = \langle \{1, 0\}; +, \cdot, 1, 0\rangle$.

Moreover, we know how to translate between the two logical systems. In the first case, interpreting $\mathbf{Z}_2$ into $\mathbf{B}_2$, we have

$$\begin{aligned} 0 &\mapsto F \\ 1 &\mapsto T \\ x + y &\mapsto x \oplus y \\ x \cdot y &\mapsto x \wedge y \end{aligned}$$

where the *exclusive-or* function $x \oplus y = (x \vee y) \wedge \neg(x \wedge y)$. In the reverse direction, we have

$$\begin{aligned} F &\mapsto 0 \\ T &\mapsto 1 \\ x \vee y &\mapsto x + y + x \cdot y \\ x \wedge y &\mapsto x \cdot y \\ \neg x &\mapsto 1 + x. \end{aligned}$$

Moreover, it is straightforward to extend this correspondence between $\mathbf{B}_2$ and $\mathbf{Z}_2$ to an equivalence between Boolean algebras and Boolean rings.[4]

In other words, Martians think just like Earthlings, except that they use the exclusive disjunction $\oplus$ rather than the inclusive $\vee$. Furthermore, they have no concept of "not", although the function $1 + x$ plays the role of $\neg x$.

The laws of Martian logic are just the identities of commutative ring theory, with the additional axiom $x + x = 0$.

3. Logic on the Inner Planets

The logic used on Mercury and Venus is based on what are known on Earth as Post algebras. Mercury uses the algebra $\mathbf{P}_3$, while Venus (being somewhat larger) uses $\mathbf{P}_4$. These algebras are given by $\mathbf{P}_n = \langle n, \wedge, ' \rangle$ where $n = \{0, 1, \ldots, n - 1\}$. The operation $x \wedge y$ is the smaller of the two, while

[4] That is, equivalence between algebras extends to an equivalence between the varieties they generate. For an introduction to interpretation and equivalence between varieties, see McKenzie, McNulty and Taylor [11].

$x' = x + 1 \mod n$. Post algebras are generalizations of Boolean algebras: $\mathbf{P}_2$ is naturally equivalent to $\mathbf{B}_2$.[5]

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$0' = 1 \qquad 1' = 2 \qquad 2' = 0$

Table 3. Mercury

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$0' = 1 \qquad 1' = 2 \qquad 2' = 3 \qquad 3' = 0$

Table 4. Venus

These algebras are familiar to the student as some of the first examples of multi-valued logics. However, they were in use on the inner planets long before the 1930's.

At this point, we have to be precise in defining interpretability.[6] Let us say that $\mathbf{A} = \langle A, \mathcal{F} \rangle$ *can be interpreted in* $\mathbf{B} = \langle B, \mathcal{G} \rangle$ if $\mathbf{A}$ is polynomially equivalent to a subset of $\mathbf{B}$, i.e., if there exist

(1) a subset $S \subseteq B$, and
(2) for each $f \in \mathcal{F}$, a polynomial $\gamma(f)$ of $\mathbf{B}$,

such that $\mathbf{A}$ is isomorphic to $\langle S, \gamma(\mathcal{F}) \rangle$.

It is well known that the two-element Boolean algebra $\mathbf{B}_2$ can be interpreted in $\mathbf{P}_n$ for any $n \geq 2$. For example, we can take $S = \{0, 1\}$ and the $\wedge$ operation the same. As an exercise, the reader can find an operation $n(x)$ on $\mathbf{P}_n$ such that $n(0) = 1$ and $n(1) = 0$. The problem is the other way around: you can't interpret an algebra with three or more elements in a two-element algebra. Nor will passing to varieties help: you can't represent a simple algebra with three or more elements in any Boolean algebra. In a sense, *the*

[5] That is, $\mathbf{P}_2$ is equivalent to $\mathbf{B}_2$ with the operations $\wedge$ and $\neg$, from which one can obtain the operation $\vee$ by DeMorgan's law.

[6] The notion of two algebras being *term equivalent* or *polynomially equivalent*, or of one variety (equational class) being *interpreted* in another, are standard. See, e.g., [9] or [11]. But we have some leeway in defining when one algebra can be interpreted in another.

$\star$	R	P	S
R	R	P	R
P	P	P	S
S	R	S	S

Table 5. *Jupiter*

$\circ$	R	P	S
R	S	P	R
P	P	R	S
S	R	S	P

Table 6. *Saturn*

logics of Mercury and Venus are more powerful than that of Earth! We will encounter this same phenomenon on some of the outer planets.[7]

A finite set of laws for each $\mathbf{P}_n$ has been found by T. Traczyk [14] (see [2] or [6]). These identities involve extra operations, and perhaps a more natural set of laws awaits discovery.

4. Logic on the Four Large Planets

The citizens of Jupiter, Saturn, Uranus and Neptune all use a form of logic based on *jan-ken-po* (the rock-paper-scissors game), using the relational structure $\mathbf{J} = \{\{R, P, S\}, <\}$ with the relation $R < P < S < R$. Each proposition is assigned a value of R (*rock*), P (*paper*), S (*scissors*), and on Uranus and Neptune possibly T (*tie*). The combination of two propositions with distinct values from the set $\{R, P, S\}$ always takes that larger value (the "winner"), but ties are dealt with differently on each of the planets, resulting in different logics. These are represented by the four algebras given in Tables 3–6.

Because our knowledge of these planets is obtained from fly-by missions, we have no idea what a denizen of one of these planets means by assigning a truth value of say *rock* to a proposition. Because of the similarity of the logics, there is considerable speculation as to whether the inhabitants have evolved from a common ancestor, or whether there is communication between the large planets. Resolving these questions may have to await a visit, one way or the other.

In each case, note that the operation is commutative but not associative, since $R*(P*S) = R$ while $(R*P)*S = S$, where $*$ represents the operation on any of the four algebras. So the algebra of logic on these planets is different from Boolean algebra, but that doesn't mean that their logic is necessarily

[7]Of course, multi-valued logics are used on Earth, precisely because of their capability of expressing more complicated situations. Fuzzy logic and the logic of quantum computing are prime examples.

$\diamond$	R	P	S	T
R	T	P	R	T
P	P	T	S	T
S	R	S	T	T
T	T	T	T	T

TABLE 7. *Uranus*

$\bullet$	R	P	S	T
R	T	P	R	R
P	P	T	S	P
S	R	S	T	S
T	R	P	S	T

TABLE 8. *Neptune*

weaker. For that, we need to know which of these logics is strong enough to model Boolean algebra.

Thus we attempt to find, in each of the four algebras, elements w, z and polynomials $c(x)$, $m(x,y)$ and $j(x,y)$ acting as complements ($\neg$), meet ($\wedge$) and join ($\vee$) on $\{w,z\}$. By DeMorgan's law, $x \vee y = \neg(\neg x \wedge \neg y)$ and its dual, we only have to find one of meet and join. It would be swifter to look for the Sheffer stroke "not-and", or its dual "not-or", but perhaps not as instructive. We take advantage of the symmetry,[8] and allow constants as well as variables in the expressions.

Unless we are exceptionally lucky or clever, we need to construct a portion of the truth table for the algebra in question. For concreteness, let us consider the subset $\{R,P\}$ of the algebra $\mathbf{J}$ of Jupiter. To find a complement operation, consider the vectors in $\mathbf{J}^2$:

$$\begin{aligned} x &= \langle R,P\rangle \\ r &= \langle R,R\rangle \\ p &= \langle P,P\rangle \\ s &= \langle S,S\rangle. \end{aligned}$$

We want to know if the subalgebra generated by these four elements contains $c(x) = \langle P,R\rangle$, and if so, how it is obtained. It does, and the simplest expression appears to be $c(x) = r \star (p \star (s \star x)))$.

Finding the join operation is too easy, since $R < P < S < R$ and $\star$ is idempotent, making $x \star y = x \vee y$ on any two-element subset of $\mathbf{J}$. We could now find the meet operation on $\{R,P\}$ using DeMorgan's law, but if we wanted to do so directly we would look at the subalgebra of $\mathbf{J}^4$ generated

[8]Each of these algebras has an automorphism with $R \mapsto P \mapsto S \mapsto R$.

Algebra	set	complement	join/meet
J	$\{R, P\}$	$R(P(Sx))$	$j(x, y) = xy$
S	$\{R, P\}$	$R(P(Sx))$	NONE
U	$\{R, P\}$	$R(P(Sx))$	NONE
U	$\{R, T\}$	NONE	NONE
N	$\{R, P\}$	$R(P(Sx))$	$j(x, y) = x(R[(Ry)(Sx)])$
N	$\{R, T\}$	Rx	$m(x, y) = x(S(y(P(Sx))))$

TABLE 9. Interpreting $\mathbf{B}_2$ into jan-ken-po algebras

by

$$\begin{aligned} x &= \langle R, R, P, P\rangle \\ y &= \langle R, P, R, P\rangle \\ r &= \langle R, R, R, R\rangle \\ p &= \langle P, P, P, P\rangle \\ s &= \langle S, S, S, S\rangle \end{aligned}$$

and find the vector $m(x, y) = \langle R, R, R, P\rangle$.

It is straightforward and instructive to program a computer to do these calculations, which are going to get more complicated in a minute. The results can be summarized as follows (see Table 9).[9]

For the algebra **J** of Jupiter, using the two-element subset $\{R, P\}$, we take $c(x) = R(P(Sx)))$ and $j(x, y) = xy$.

For the algebra **S** of Saturn, also using $\{R, P\}$, we can use the same $c(x)$, but there is no choice of polynomial for join or meet. By symmetry, the same situation holds for any two-element subset. How to explain this?

For the algebra **U** of Uranus, using say $\{R, P\}$, we can again use $c(x)$, but there is no join or meet.

Again for the algebra of Uranus, but this time using $\{R, T\}$, there is no way to find complement nor meet, since T is a zero for the operation $\diamond$.

For the algebra **N** of Neptune, using $\{R, P\}$, we can again use $c(x)$, and this time $j(x, y) = x(R[(Ry)(Sx)])$.

Also for the algebra of Neptune, using the set $\{R, T\}$ and regarding say $R < T$, we can take $c(x) = Rx$ and $m(x, y) = x(S(y(P(Sx))))$.

5. Functional Completeness

There is another way to measure the strength of a logic. A well-known property of Boolean algebras, sometimes called the *fundamental law of switching circuits*, is that every truth table represents some Boolean expression. We can compare that with the situation on the largest asteroid,

[9]As some of the expressions are starting to become complicated, we use concatenation and omit the operation symbols $\star$, $\circ$, $\diamond$ and $\bullet$.

Ceres, where the two-element distributive lattice $\langle \{T, F\}, \wedge, \vee \rangle$ is used for logic. Lacking negation, folks there cannot generate all possible truth tables using their logical functions.

An algebra $\mathbf{A}$ is said to be *primal* if, for every $n \geq 1$, every function $f : \mathbf{A}^n \to \mathbf{A}$ can be represented as a term function (an expression in the operations of $\mathbf{A}$ and variables, not involving constants from $\mathbf{A}$). Similarly, $\mathbf{A}$ is said to be *functionally complete* if, for every $n \geq 1$, every function $f : \mathbf{A}^n \to \mathbf{A}$ can be represented as a polynomial function (an expression in the operations of $\mathbf{A}$ and variables and constants from $\mathbf{A}$). There is a wonderfully clear discussion of these concepts in Section 3.1 of the book by Kaarli and Pixley [9].

So let us turn to the question of whether the algebras which form the basis for the propositional logics of the various planets are primal or functionally complete. (Surely this is a good property to have.) For the inner planets this is well-known: Post algebras, Boolean algebras and Boolean rings are primal. Actually, for the outer planets Jupiter through Neptune, primality is out of the question, as none of these algebras is rigid.[10] Moreover, we already know that $\mathbf{S}$ and $\mathbf{U}$ are not functionally complete, as we have exhibited (partial) functions that cannot be represented by any polynomial. In a manner of speaking, people who live on Saturn and Uranus are handicapped in their thinking by the type of logic they use.

That leaves us with the task of determining whether Jupiter and Neptune are functionally complete. Thumbing through Kaarli and Pixley [9], we see that we could try to show that say $\mathbf{J}$ is simple and has a Pixley polynomial, or that the ternary discriminator is a polynomial. (See Theorem 3.1.4.) These involve polynomials in three variables, which means working in at least $\mathbf{J}^{3^3} = \mathbf{J}^{27}$. That is a finite algebra to be sure, but a bit too large for comfort.

Plan B is to try to represent the Post algebras $\mathbf{P}_3$ in $\mathbf{J}$, and $\mathbf{P}_4$ in $\mathbf{N}$, respectively. This works for Jupiter. We use the map $R \mapsto 0$, $P \mapsto 1$, and $S \mapsto 2$ and to establish the order; the choice is arbitrary. Again writing programs to work in the direct product $\mathbf{J}^{3^2} = \mathbf{J}^9$, which is equivalent to truth tables, we find that $x' = (Px)(S(Rx))$ and

$$x \wedge y = [S(R(P(S(y(Rx)))))][R(P(S(R(P(x(Sy))))))]$$

work. These are actually simple-as-possible expressions, i.e., they involve the minimum number of operation symbols. Thus $\mathbf{J}$ is functionally complete.

For Neptune's algebra $\mathbf{N}$ we find that $x' = x(x((Rx)(P(Sx))))$ and then the roof caves in: 4^{16} is just too big an algebra to handle efficiently with a computer. So we continue reading in Kaarli and Pixley, and come to Theorem 3.1.5.

[10] That is, they all have a nontrivial automorphism unless we regard the elements of the algebra itself as operational constants.

Theorem. *A finite algebra* $\mathbf{A}$ *is functionally complete if and only if the following conditions are satisfied:*

(i) *There are elements* 0, 1 *of* $\mathbf{A}$ *and binary polynomial operations* $+$ *and* $\cdot$ *satisfying*

$$x + 0 = 0 + x = x, \qquad x \cdot 1 = x, \qquad x \cdot 0 = 0$$

for all $x \in \mathbf{A}$.

(ii) *For each* $a \in \mathbf{A}$, *the characteristic function* χ_a *is a polynomial of* $\mathbf{A}$.

Now this result is naturally set up for life on Neptune. The element T is an identity for the operation $\bullet$ of $\mathbf{N}$, so we can take $0 \mapsto T$ and $+ \mapsto \bullet$. Arbitrarily, we choose $1 \mapsto R$. Finding the characteristic polynomials involves calculations in $\mathbf{N}^4$, a relatively small algebra. We find that (suppressing the $\bullet$'s with concatenation)

$$\begin{aligned}
\chi_R(x) &= x((Px)P) \\
\chi_P(x) &= ((Rx)x)R \\
\chi_S(x) &= S(S((Px)(Rx))) \\
\chi_T(x) &= ((Sx)(S((Px)R)))S.
\end{aligned}$$

Finally, we must find an expression $\cdot$ such that in $\mathbf{N}^8$ we have

$$\langle R, P, S, T, R, P, S, T\rangle \cdot \langle R, R, R, R, T, T, T, T\rangle = \langle R, P, S, T, T, T, T, T\rangle.$$

We program that, and find that

$$x \cdot y = [(((P(yS))x)S)(((P(yS))(xS))(yx))][((S((xS)(x(x(yS)))))((P(yS))x))x]$$

works. Thus Neptune's algebra $\mathbf{N}$ is also functionally complete!

6. Logic in the Kuiper Belt

Very little is known about logic on Pluto, Sedna, and the other large objects in the Kuiper Belt.[11] There is considerable debate as to which, if any, of these objects should be regarded as planets. There is less debate that, due to the extreme cold, their logic is not commutative. Bjarni Jónsson has even suggested that their arithmetic is not commutative: you can count a deck of cards, shuffle it, count again and get a different answer.[12] Clearly this is a frontier for further exploration.

[11] See Mike Brown's web page [3] for their astronomical properties.

[12] Private communication.

7. Laws of Thought

The efficacy of elementary algebra is manifest. Familiarity with the identities and rules of algebra enables young students to perform easily and surely computations that would have challenged the best scholars five hundred years ago on Earth. Part of the advantage gained is in the symbolic notation, and part in the fact that a few simple axioms characterize the algebra of fields and their polynomial rings.[13] A major motivation for Boole and other nineteenth-century logicians was to provide a similar algebra for logical calculations. (See again Burris [5].)

We have two problems here. The first is to determine *all* identities valid in a given algebra $\mathbf{A}$ with some fixed number of variables. As long as $\mathbf{A}$ is finite, this can be accomplished by constructing truth tables over the given algebra.[14] If there are n variables, then this is a computation in the direct power $\mathbf{A}^{|\mathbf{A}|^n}$. Only in the smallest cases is this feasible by hand. As we have seen, the computations are easily programmed, but size quickly becomes a major limitation as either $|\mathbf{A}|$ or n increases. Nonetheless, in principle this solves the problem. For terms s and t, the identity $s = t$ holds in $\mathbf{A}$ if and only if s and t have the same truth table.

The real objective, however, is (if possible) to find a *finite* set of identities for the algebra $\mathbf{A}$ from which the rest can be derived. This is called an *equational basis* for $\mathbf{A}$, and finite bases don't always exist. It is not hard to see that the real problem, with respect to the existence of a finite basis, is to bound the number of variables involved. For the n-variable identities of an algebra $\mathbf{A}$ are all encoded in the operation tables of a subalgebra of $\mathbf{A}^{|\mathbf{A}|^n}$, which is in fact the n-generated free algebra in the variety generated by $\mathbf{A}$. This may be large, but it is finite. In practice, one would like to then reduce these laws to a more manageable set.

A technical difficulty arises: *Do we want to name some (or all) of the elements of* $\mathbf{A}$ *as constants?* Do we want to allow laws such as $x \wedge T = x$ or $x \wedge \neg x = F$? While in some cases this can be avoided,[15] whether or not elements are named as constants can make a difference in whether the laws of an algebra are finitely based. (There is a wonderful example due to Roger Bryant [4], where naming an element of a finite group as a constant can make its variety not have a finite basis.) Following the lead of Boolean algebras and Boolean rings, let us agree in this case that we will consider all the elements of the algebra $\mathbf{A}$ as constants.

[13]Somehow the wonder of this is grasped by many students in spite of bad schools and puberty.

[14]The sophisticated reader will recognize this process as the construction of the free algebra in the variety generated by $\mathbf{A}$ over a given set of variables.

[15]For example, in Boolean algebras one can replace T by $x \vee \neg x$ and F by $x \wedge \neg x$, adding the identities $x \vee \neg x = y \vee \neg y$ and $x \wedge \neg x = y \wedge \neg y$, thus defining T and F implicitly instead of explicitly.

If an algebra $\mathbf{A}$ is functionally complete and its elements are treated as constants, then it becomes *primal*. In that case, a well-known theorem of Kirby Baker [1] ensures that there is a finite basis: *The laws of a finite primal algebra are finitely based.*[16] Thus Mercury, Venus, Earth, Mars, Jupiter and Neptune all have a finite basis for their logics. For Boolean algebras and Boolean rings (Earth and Mars), these laws are quite familiar. For the Post algebras of Mercury and Venus, we have a reasonable set of identities, though perhaps a simpler basis could be found. For Jupiter and Neptune, all we have so far is an existence theorem. One can only speculate how much more is known on those planets by their inhabitants.

The algebra $\mathbf{S}$ of Saturn turns out to be equivalent to the integers modulo 3 under addition. That is, treating the element R of $\mathbf{S}$ as a constant, the algebras $\mathbf{S} = \langle\{R, P, S\}, \circ, R\rangle$ and $\mathbf{Z}_3 = \langle\{0, 1, 2\}, +, 0\rangle$ can be interpreted in each other *via* the correspondence

$$R \leftrightarrow 0 \qquad P \leftrightarrow 1 \qquad S \leftrightarrow 2$$

$$x \circ y \leftrightarrow 2x + 2y + 2 \qquad (x \circ y) \circ R \leftrightarrow x + y.$$

Furthermore, the obvious basis for abelian groups of exponent 3,

$$x + (y + z) = (x + y) + z \qquad x + y = y + x \qquad x + 0 = x \qquad x + (x + x) = 0$$

translates into a finite basis for the laws of $\mathbf{S}$, with one additional identity. For example, the associative law becomes

$$(((x \circ y) \circ R) \circ z) \circ R = (x \circ ((y \circ z) \circ R)) \circ R.$$

In addition, we need an identity that says that if you go from $\circ$ to $+$ and back again, you get $\circ$:

$$(((((x \circ x) \circ R) \circ ((y \circ y) \circ R)) \circ R) \circ S) \circ R = x \circ y.$$

Needless to say, this may not be the most natural basis for $\mathbf{S}$.[17]

The algebra $\mathbf{U}$ of Uranus is another matter. We don't know whether the variety generated by $\mathbf{U}$ is finitely based. Perhaps the Uranians do. It is interesting to think of the possibility of living on a planet where the rules of logic are not finitely based - just as we have to live with the Gödel Incompleteness Theorem for our number theory.

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[16] Baker's theorem applies more generally to finite algebras in congruence distributive varieties.

[17] Note that, in order to write these equations, we need to regard the elements R and S as constants of $\mathbf{S}$, and hence the elements 0 and 2 as constants of $\mathbf{Z}_3$. But the technical problems associated with treating non-identity elements of groups as constants, as in Bryant's non-finite basis example to which we alluded earlier [4], don't arise in this case.

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