1. Solution of problems to decide whether a given statement in lattice theory (Gruppenkalkul) is provable or not

The logical Gruppenkalkul has to do with certain things that are usually called gruppen. Between these sometimes stands a binary relation that is normally called inclusion; one says that an element a is contained in another one b. In the following I will write the holding of this relation as simply \( a \leq b \). Further among the elements are two ternary relations, in which an element is the common constituent of two others, the greatest element of those contained in both, or an element that is the smallest of those that contain two others. The holding of these relations between three elements I will denote in the following by \((a, b, c) \in \wedge\) and \((a, b, c) \in \vee\), respectively.

The axioms that form the basis for lattice theory are then the following:

\begin{itemize}
  \item[I.] \( x \leq x \) for each \( x \).
  \item[II.] From \( x \leq y \) and \( y \leq z \) follows \( x \leq z \).
  \item[III\_\times.] From \((x, y, z) \in \wedge\) follows \( z \leq x \) and \( z \leq y \).
  \item[III\_+] From \((x, y, z) \in \vee\) follows \( x \leq z \) and \( y \leq z \).
  \item[IV\_\times.] From \((x, y, z) \in \wedge\) together with \( u \leq x \) and \( u \leq y \) follows \( u \leq z \).
  \item[IV\_+] From \((x, y, z) \in \vee\) together with \( x \leq u \) and \( y \leq u \) follows \( z \leq u \).
  \item[V\_\times.] From \((x, y, z) \in \wedge\) together with \( x \leq x' \), \( x' \leq x \), \( y \leq y' \), \( y' \leq y \), \( z \leq z' \) and \( z' \leq z \) follows \((x', y', z') \in \wedge\).
  \item[V\_+] From \((x, y, z) \in \vee\) together with \( x \leq x' \), \( x' \leq x \), \( y \leq y' \), \( y' \leq y \), \( z \leq z' \) and \( z' \leq z \) follows \((x', y', z') \in \vee\).
  \item[VI\_\times.] For each \( x \) and \( y \) there is a \( z \) such that \((x, y, z) \in \wedge\).
  \item[VI\_+] For each \( x \) and \( y \) there is a \( z \) such that \((x, y, z) \in \vee\).
\end{itemize}

As in Schröder’s Algebra der Logik, the notation \( a \leq b \), \( ab = c \) and \( a + b = c \) will also be written in place of \( a \leq b \), \((a, b, c) \in \wedge\) and \((a, b, c) \in \vee\).

\begin{itemize}
  \item[Date:] March 12, 2014.
  \item[1] See E. Schröder: Algebra der Logik, B. 1, p. 628-632.
  \item[2] Skolem’s notation is different, and makes it clear that he regards meet and join as relations. Inclusion is denoted by simply a pair \((ab)\). Meet is denoted by a triple with a curved line above, join by a triple with a curved line below. We have adopted this variant to emphasize the relational point of view. Note that you could have for example \((a, b, c) \in \vee\) and \((a, b, c') \in \vee\). The lattice theory axioms would then give \( c \leq c' \) and \( c' \leq c \), but not \( c = c' \) as they are distinct symbols. - JBN
\end{itemize}
The validity of a statement in the algebra that can be derived from these axioms, is now contained as a simple case when the following can be demonstrated: if various pairs and triples \( x \leq y, (x, y, z) \in \Lambda \), and so forth are given, the statement can eventually be generated by repeated and combined applications of the axioms to these and further pairs and triples. Indeed, the listed axioms have the sense of a generating principle, whereby from axioms I–V new pairs and triples in the original symbols (letters) are generated, while from axiom VI new triples are created, in which however new symbols also appear. This is a purely combinatorial interpretation of deduction, which I want to emphasize, has shown itself to be especially useful in logical investigations. Indeed, each proof is nothing other than the generation of new pairs and triples, as I will now illustrate by examples.

We take for example the statement \( ab + ac \leq a(b + c) \) in Schröder’s terminology. It is established as follows:

From \((a, b, d) \in \Lambda \) in conjunction with \((a, c, e) \in \Lambda, (b, c, f) \in \bigvee, (d, e, g) \in \bigvee\), and \((a, f, h) \in \Lambda\) follows the pair \( g \leq h \). Proof: From \((b, c, f) \in \bigvee\) it follows by \( \text{III}_+ \) that \( b \leq f \). From \((a, b, d) \in \Lambda\) it follows likewise by \( \text{III}_x \) that \( d \leq a \) and \( d \leq b \). From \( d \leq b \) and \( b \leq f \) it follows by \( \text{II} \) that \( d \leq f \). From \( d \leq a \) and \( d \leq f \) together with \((a, f, h) \in \Lambda\) it follows by \( \text{VI}_x \) that \( d \leq h \). From \((b, c, f) \in \bigvee\) it follows by \( \text{III}_+ \) that \( c \leq f \). From \((a, c, e) \in \bigvee\) it follows by \( \text{III}_x \) that \( e \leq a \) and \( e \leq c \). From \( e \leq c \) and \( c \leq f \) it follows by \( \text{II} \) that \( e \leq f \). From \( e \leq a \) in conjunction with \( e \leq f \) and \((a, f, h) \in \Lambda\) it follows by \( \text{VI}_x \) that \( e \leq h \). From \( d \leq h \) together with \( e \leq h \) and \((d, e, g) \in \bigvee\) it follows by \( \text{IV}_+ \) finally that \( g \leq h \).

We consider also the statement \((a + b) + c \leq a + (b + c)\) in Schröder’s notation. That means, that the pair \( d \leq e \) follows from the four triples \((a, b, \alpha) \in \bigvee, (\alpha, c, d) \in \bigvee, (b, c, \beta) \in \bigvee, (a, \beta, e) \in \bigvee\).

Proof: From \((b, c, \beta) \in \bigvee\) it follows by \( \text{III}_+ \) that \( b \leq \beta \), and in the same way from \((a, \beta, e) \in \bigvee\) comes \( \beta \leq e \). By \( \text{II} \) we obtain from \( b \leq \beta \) and \( \beta \leq e \) that \( b \leq e \). But by \( \text{III}_+ \) from \((a, \beta, e) \in \bigvee\) we also obtain \( a \leq e \). From \( a \leq e, b \leq e \) and \((a, b, \alpha) \in \bigvee\) by \( \text{IV}_+ \) we obtain \( \alpha \leq e \). From \((b, c, \beta) \in \bigvee\) one can conclude by \( \text{III}_+ \) that \( c \leq \beta \), which with \( \beta \leq e \) gives \( c \leq e \) by \( \text{II} \). From \( \alpha \leq e, c \leq e \) and \((\alpha, c, d) \in \bigvee\) together by \( \text{IV}_+ \) we obtain finally \( d \leq e \).

In these proofs only axioms I–V are used. Axiom VI always comes into use when one wants to prove an existence statement. Thus for example the following very simple theorem:

If \( a, b, c \) are given, there is always a \( d \) such that \( a \leq d \) and \( b \leq d \) and \( c \leq d \).

Proof: Using \( \text{VI}_+ \) there is an \( \alpha \) with the property that \((a, b, \alpha) \in \bigvee\), and from this \( d \) such that \( \alpha \land c = d \) holds. By \( \text{III}_+ \) from \((a, b, \alpha) \in \bigvee\) the pairs \( a \leq \alpha \) and \( b \leq \alpha \) can be formed, and likewise from \((\alpha, c, d) \in \bigvee\) the pairs \( \alpha \leq d \) and \( c \leq d \). From \( a \leq \alpha \) and \( \alpha \leq d \) by \( \text{II} \) we can form \( \alpha \leq d \).

\[\text{Our version has } e \leq f, \text{ a typo. - JBN}\]
Likewise $b \leq d$ follows from $b \leq \alpha$ together with $\alpha \leq d$. Thus we have $a \leq d$ and $b \leq d$ and $c \leq d$, whence the theorem is proved.

One can now ask the question whether the Axioms $VI_x$ and $VI_+$ will ever be used in the proof of statements of the kind, that do not deal with the existence of elements. *A priori* this seems to be very thinkable. One can show the following: If known pairs and triples formed from the original symbols $a_1, \ldots, a_n$ are given, and new triples with new symbols $b_1, b_2, \ldots$ are brought forth, then only those further pairs and triples formed with the original symbols $a_1, \ldots, a_n$ are provable, as could be proved solely by applications of Axioms I–V. In most other axiom systems conclusions of a higher degree would also take place; it is remarkable that this is not true here, where the provability of *pairs* is concerned. Indeed, the following theorem holds, which I shall prove.

**Theorem.** Let a system $S$ of pairs and triples be given, formed out of the symbols $a_1, \ldots, a_n$. Let $\Sigma$ be the system that contains all those pairs and triples that can be formed from $S$ with the help of Axioms I–V. One now includes new triples by repeated applications of VI, in which new symbols $b_1, b_2, \ldots$ occur, and then let $\Sigma'$ be the system of all pairs and triples, that can then be formed with the help of Axioms I–V. Then $\Sigma'$ contains no other pairs in the symbols $a_1, \ldots, a_n$ than those that already occur in $\Sigma$.

I will not prove the theorem immediately, but first make a preparation. Let us assume that we have a system $S$ of pairs and triples for which the Axioms I–V, but not VI, are satisfied. I now adjoin by means of VI an $\alpha$ for two symbols $a_1$ and $a_2$, so that $(a_1, a_2, \alpha) \in \wedge$ takes place. From this, the following pairs will be adjoined to $S$:

1. The pair $\alpha \leq \alpha$.
2. All pairs $\alpha \leq a_r$ where $a_r$ has the property that, whenever $a_s \leq a_1$ and $a_s \leq a_2$ occur at least once in $S$, then $a_s \leq a_r$ occurs.\footnote{Skolem is ambiguous here. What he wants is that if $a_1$ and $a_2$ already have a greatest lower bound $a_r$ in $S$, then $\alpha \leq a_r$. The way it is written, if $a_1$ and $a_2$ have no common lower bound, then $\alpha \leq a_r$ would hold for all $a_r$, making $\alpha$ a least element. That is safe, in that his proof still works, but unnecessary. If $a_1$ and $a_2$ have no common lower bound, this step can be omitted, and that seems to be what Skolem intended anyway. - JBN}
3. All pairs $a_n \leq \alpha$ where $a_n \leq a_1$ and $a_n \leq a_2$ occur in $S$.

**Lemma 1.** Let $S'$ be the system obtained from $S$ by adjoining the triple $(a_1, a_2, \alpha) \in \wedge$ and the pairs mentioned in 1), 2) and 3). Then Axioms I–IV hold for $S'$.

Proof: One sees immediately that Axiom I holds for $S'$; for each symbol that occurs in one of the pairs or triples of $S'$ is either one of the symbols $a_1, \ldots, a_n$ or the new symbol $\alpha$. Now all the pairs $a_r \leq a_r$ ($r = 1, 2, \ldots, n$) occur by assumption in $S$, and therefore also in $S'$. Besides $\alpha \leq \alpha$ occurs in $S'$ by 1).
One sees that Axiom II is true for $S'$ as follows. Two pairs in $S$ that are of the form $a_r \leq a_s$ and $a_s \leq a_t$ give the pair $a_r \leq a_t$ in $S$, which satisfies Axiom II, and thus it also appears in $S'$. – If we take a pair $a_r \leq a_r'$ in $S$ and a pair of the form $\alpha \leq a_r$ in $S'$, then $a_s \leq a_r$ should always occur in $S$, whenever $a_s \leq a_1$ and $a_s \leq a_2$ occur in $S$, and it follows from that by II, that if $a_s \leq a_1$ and $a_s \leq a_2$ both occur in $S$, then it must be that $(a_s \leq a_r) \in S$. Then also $(a_s \leq a_r') \in S$, whenever $(a_s \leq a_1) \in S$ and $(a_s \leq a_2) \in S$, so by 2) $\alpha \leq a_r'$ should also be in $S'$. – If we take a pair $a_r \leq a_s$ in $S$ and a pair $a_s \leq \alpha$ in $S'$, then $a_s \leq a_1$ and $a_s \leq a_2$ must both occur in $S$, from which it follows that $(a_r \leq a_1) \in S$ and $(a_r \leq a_2) \in S$, and therefore by 3) $(a_r \leq \alpha) \in S'$. – Finally, we consider two pairs in $S'$ of the form $a_r \leq \alpha$ and $\alpha \leq a_s$. When $(a_r \leq \alpha) \in S'$, then by 3) both $a_r \leq a_1$ and $a_r \leq a_2$ are in $S'$, and since $\alpha \leq a_s$ should be in $S'$, it follows from that by 2) that $(a_r \leq a_s) \in S'$, and hence also $(a_r \leq a_s) \in S'$. – The case, when one or both pairs are of the form $\alpha \leq \alpha$, we clearly do not need to consider, since then it is trivial that II is satisfied. – Therefore II also holds for the system $S'$.

If Axiom III were applied to triples $(a_m, a_n, a_p) \in \bigwedge$ or $(a_m, a_n, a_p) \in \bigvee$ in $S$, then we have by assumption those pairs that should further belong to $S$. If $III_\chi$ were applied to the new triple $(a_1, a_2, \alpha) \in \bigwedge$, then we obtain the pairs $\alpha \leq a_1$ and $\alpha \leq a_2$. These are however of the form mentioned in 2) and therefore belong to $S'$. Therefore III is satisfied by $S'$.

If we have $(a_q \leq a_m) \in S$, $(a_q \leq a_n) \in S$ and $(a_m, a_n, a_p) \in \bigwedge$ in $S$, then so must $a_q \leq a_p$ be in $S$, since $IV_\chi$ holds for $S$. Now it could be that in $S'$ the pairs $\alpha \leq a_m$ and $\alpha \leq a_n$ occur; then $(a_s \leq a_1) \in S$ and $(a_s \leq a_2) \in S$ would imply $(a_s \leq a_m) \in S$ and $(a_s \leq a_n) \in S$. Since $IV_\chi$ holds for $S$, it follows that $(a_s \leq a_p) \in S$, whenever $(a_s \leq a_1) \in S$ and $(a_s \leq a_2) \in S$, from which it further follows by 2) that $(\alpha \leq a_p) \in S'$. – Now let the pairs $a_r \leq a_1$ and $a_r \leq a_2$ in $S$ and the triple $(a_1, a_2, \alpha) \in \bigwedge$ in $S'$ be given. Then by 3) $(a_r \leq \alpha) \in S'$. – From the presence of $\alpha \leq a_1$, $\alpha \leq a_2$ and $(a_1, a_2, \alpha) \in \bigwedge$ in $S'$ that $\alpha \leq \alpha$ follows is certain; for by 1) this condition always occurs in $S'$. – Therefore $IV_\chi$ holds for $S'$.

If $(a_m, a_n, a_p) \in \bigvee$, $a_m \leq a_q$ and $a_n \leq a_q$ are all in $S$, then by Axiom $IV_+$ which holds for $S$ we have $a_p \leq a_q$ in $S$, and thus a fortiori in $S'$. If $(a_m, a_n, a_p) \in \bigvee$ is in $S$ and $a_m \leq \alpha$ and $a_n \leq \alpha$ are both in $S'$, then we have $a_m \leq a_1$, $a_m \leq a_2$, $a_n \leq a_1$ and $a_n \leq a_2$ all in $S$, so that $a_p \leq a_1$ and $(a_p \leq a_2) \in S$, since $IV_+$ holds for $S$. Then however by 3) $(a_p \leq \alpha) \in S'$ should hold. – Thus also $IV_+$ holds for $S'$.

Hereby the satisfaction of Axioms I–IV for $S'$ is proved, and we can suitably conclude, that $S'$ is closed with respect to these Axioms (generating principles).

Further, the following theorem can be proved:

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5I write from now on often $\alpha \in S$ as a short expression for $\alpha$ (which here is always a pair or a triple) occurs in $S$ ($\alpha$ is an element of $S$).
Lemma 2. If $\Sigma$ is a system of pairs and triples that is closed with respect to Axioms I–IV, then by adjoining all triples that are deducible by using Axiom V (without repeating), one obtains a system $\Sigma'$ that is closed with respect to Axioms I–V.

Proof: Since $\Sigma'$ contains no pairs besides those already contained in $\Sigma$, so first of all all Axioms I and II are satisfied.

Now one takes a triple $(a, b, c) \in \Lambda$ or $(a, b, c) \in \mathcal{V}$ that belongs to $\Sigma$, so that $c \leq a$ and $c \leq b$, or respectively $a \leq c$ and $b \leq c$, already belong to $\Sigma$. If $(\alpha, \beta, \gamma) \in \Lambda$, or respectively $(\alpha, \beta, \gamma) \in \mathcal{V}$, is one of the new triples, then there should be three symbols $a$, $b$, $c$ such that $(a, b, c) \in \Lambda$, or respectively $(a, b, c) \in \mathcal{V}$, is in $\Sigma$, while besides $a \leq \alpha$, $\alpha \leq a$, $b \leq \beta$, $\beta \leq b$, $c \leq \gamma$, $\gamma \leq c$ all occur in $\Sigma$. Then the pairs $c \leq a$ and $c \leq b$, or respectively $a \leq c$ and $b \leq c$, occur in $\Sigma$. Since $\Sigma$ is closed with respect to II, certainly the pairs $\gamma \leq \alpha$ and $\gamma \leq \beta$, or respectively $\alpha \leq \gamma$ and $\beta \leq \gamma$ occur in $\Sigma$. Thus Axiom III holds for $\Sigma'$.

If $(a, b, c) \in \Lambda$ or $(a, b, c) \in \mathcal{V}$ is a triple in $\Sigma$, while $d \leq a$ and $d \leq b$, or respectively $a \leq d$ and $b \leq d$, are pairs in $\Sigma'$, and therefore also in $\Sigma$, then $(d \leq c) \in \Sigma$, or respectively $(c \leq d) \in \Sigma$. If $(\alpha, \beta, \gamma) \in \Lambda$, or respectively $(\alpha, \beta, \gamma) \in \mathcal{V}$ is a triple that occurs in $\Sigma'$ but not in $\Sigma$, then there are in this case three symbols $a$, $b$, $c$ such that the pairs $a \leq \alpha$, $\alpha \leq a$, $b \leq \beta$, $\beta \leq b$, $c \leq \gamma$, $\gamma \leq c$ all occur in $\Sigma$, while $(a, b, c) \in \Lambda$, or respectively $(a, b, c) \in \mathcal{V}$, belongs to $\Sigma$. If now $d \leq a$ and $d \leq b$, or respectively $a \leq d$ and $\beta \leq d$, are pairs in $\Sigma'$, and consequently also in $\Sigma$, then $d \leq c$ and $d \leq b$, or respectively $a \leq d$ and $b \leq d$, belong to $\Sigma$, thus $d \leq c$, or respectively $c \leq d$, is in $\Sigma$, whence further $d \leq \gamma$, or respectively $\gamma \leq d$, is in $\Sigma$. Therefore $\Sigma'$ is closed with respect to Axiom IV.

If $(a, b, c) \in \Lambda$ or $(a, b, c) \in \mathcal{V}$ is in $\Sigma$ and further $a \leq \alpha$, $\alpha \leq a$, $b \leq \beta$, $\beta \leq b$, $c \leq \gamma$, $\gamma \leq c$ are all in $\Sigma$, then $(\alpha, \beta, \gamma) \in \Lambda$ is in $\Sigma'$, or respectively $(\alpha, \beta, \gamma) \in \mathcal{V}$ is in $\Sigma'$. $(a, b, c) \in \Lambda$ or respectively $(a, b, c) \in \mathcal{V}$ is in $\Sigma$. If $(\alpha, \beta, \gamma) \in \Lambda$ or $(\alpha, \beta, \gamma) \in \mathcal{V}$ belong to $\Sigma'$ but not to $\Sigma$, then there exist three symbols $a$, $b$, $c$ such that the pairs $a \leq \alpha$, $\alpha \leq a$, $b \leq \beta$, $\beta \leq b$, $c \leq \gamma$, $\gamma \leq c$ all belong to $\Sigma$, while besides $(a, b, c) \in \Lambda$ or respectively $(a, b, c) \in \mathcal{V}$ is in $\Sigma$. Now if also the pairs $\alpha \leq \alpha'$, $\alpha' \leq \alpha$, $\beta \leq \beta'$, $\beta' \leq \beta$, $\gamma \leq \gamma'$, $\gamma' \leq \gamma$ are in $\Sigma'$, and hence also in $\Sigma$, then the pairs $a \leq \alpha'$, $\alpha' \leq a$, $b \leq \beta'$, $\beta' \leq b$, $c \leq \gamma'$, $\gamma' \leq c$ all belong to $\Sigma$, and hence also in $\Sigma$, and therefore $(\alpha', \beta', \gamma') \in \Lambda$, or respectively $(\alpha', \beta', \gamma') \in \mathcal{V}$ must occur in $\Sigma'$. Therefore Axiom V also holds for $\Sigma'$.

If one now applies Lemma 2 to the system $S'$ from the preceding lemma (i.e., one sets $\Sigma = S'$), then one obtains the following statement:

If one has a system $S$, for which Axioms I–V are satisfied, and one in accordance with Axiom IV$_x$ adjoins a triple $a_1 \land a_2 = \alpha$, where $\alpha$ is a new symbol, then by applying the Axioms I–V extending sufficiently (finitely) many times one obtains a system $S_1$, for which Axioms I–V still hold, and
Since the exact same statement is provable when one takes Axiom \( VI_+ \) in place of \( VI_x \), this follows further:

If one has a system that is closed with respect to Axioms I–V, and one obtains through repeated applications of Axiom VI, followed by extensions through I–V, a system \( S' \), which is still closed with respect to I–V, then \( S' \) contains no other pairs involving the letters occurring in the pairs and triples of \( S \), besides those pairs already occurring in \( S \).

Thereby the assertion on page 18 is entirely proved.

If one wants to find whether a pair or certain pairs follow from certain given pairs and triples by means of Axioms I–VI, then one can leave Axiom VI out of the consideration; one needs only to search whether it is (they are) provable with Axioms I–V alone. Since there are only a finite number of pairs and triples, that are provable from a given finite number of such by Axioms I–V, thus is found a method involving only a finite amount of work to decide, whether a given statement in the symbols of the Gruppenkalkul holds in general or not, if this statement only acts as follows: when this and the former pairs and triples are given, then this and the former pairs involve the same original symbols. That also the question is decidable, whether one or more triples follow from certain given pairs and triples, is entirely clear, since this question can be rewritten into a fully equivalent one about the provability of one or more pairs. Namely, to see whether a triple \((a, b, c)\) is provable from a system \( S \) of pairs and triples, we can adjoin a symbol \( \alpha \) and the triple \((a, b, \alpha)\), and let \( S' \) be the system that results by adjoining \((a, b, \alpha)\) to \( S \). It is then necessary and sufficient for the provability of the triple \((a, b, c)\) from the system \( S \) by means of I–VI, that the pairs \( c \leq \alpha \) and \( \alpha \leq c \) can be formed with the same axioms from the system \( S' \).

I give here an example.

Example 1: That the inclusion \( b \leq a \) does not in general follow from the reverse inclusion \( a \leq b \), can be seen as follows. That \( a \leq b \) is given, means in our terminology that the pair \((a, b)\) is given. Since here two original symbols \( a \) and \( b \) occur, the pairs \( a \leq a \) and \( b \leq b \) can be formed. But then the three pairs \( a \leq a \), \( a \leq b \), \( b \leq b \) together form a closed system with respect to Axioms I–V. Since the pair \( b \leq a \) does not occur in this system, so is the above statement proved, that \( b \leq a \) does not follow from \( a \leq b \), i.e., the statement “if \( a \leq b \) then \( b \leq a \)” does not hold in the Gruppenkalkul; such a statement does not hold, or in other words, is not provable.

Example 2. Let us prove that the inclusion \( a(b+c) \leq ab+ac \) (in Schröder’s notation), which in set theory (Klassenkalkul) is generally satisfied, is not generally satisfied in lattice theory (Gruppenkalkul). In our terminology this task means the following: the triples \((a, b, d)\) \( \in \Lambda \), \((a, c, e)\) \( \in \Lambda \), \((b, c, f)\) \( \in \Lambda \), \((d, e, g)\) \( \in \Lambda \), and \((a, f, h)\) \( \in \Lambda \) are given. Does the pair \( h \leq g \) follow from this and Axioms I–VI? To investigate this we have merely to apply Axioms I–V so long, until the system is closed with respect to these axioms. Axiom I
gives the pairs $a \leq a, b \leq b, c \leq c, d \leq d, e \leq e, f \leq f, g \leq g, h \leq h$. Axiom III gives us $d \leq a, d \leq b, e \leq a, e \leq c, b \leq f, c \leq f, d \leq g, e \leq g, h \leq a, h \leq f$, and then we obtain by II further $d \leq f$ and $e \leq f$, and by means of $IV_+$ also $g \leq f$. But then no more pairs or triples can be formed from the 5 given triples by means of I–V. The pair $h \leq g$ is not among the pairs contained. Thus is the non-provability of $h \leq g$ from the 5 given triples by means of I–VI proved, or in other words, the non-provability of the so-called distributive law $a(b + c) \leq ab + ac$ in lattice theory. (See Schröder, Algebra der Logik, B. 1, p. 642 and 686.)

Example 3. Let us show, that the inclusion $(a + b)(a + c)(b + c) \leq ab + ac + bc$, which holds in set theory, is not provable in lattice theory. Then we have to show: If the triples

\[
\begin{align*}
(a, b, \gamma) & \in \bigvee, \quad (a, c, \beta) \in \bigvee, \quad (b, c, \alpha) \in \bigvee \\
(a, b, \gamma') & \in \bigwedge, \quad (a, c, \beta') \in \bigwedge, \quad (b, c, \alpha') \in \bigwedge \\
(\alpha, \beta, d) & \in \bigwedge, \quad (d, \gamma, e) \in \bigwedge \\
(\alpha', \beta', d') & \in \bigvee, \quad (d', \gamma', e') \in \bigvee
\end{align*}
\]

then from that we cannot deduce the pair $e \leq e'$ by means of Axioms I–V. For the given method, we need merely to show that $e \leq e'$ is not derivable by applications of I–V alone. Therefore we apply I–V until we obtain a system that is closed with respect to I–V, where we start from the given 10 triples. Now first we obtain with the help of I from the 13 symbols $a, b, \ldots$ the "self" pairs $a \leq a, b \leq b, \ldots$. By means of III we obtain the pairs

\[
\begin{align*}
a & \leq \gamma & b & \leq \gamma & a & \leq \beta & c & \leq \beta \\
b & \leq \alpha & c & \leq \alpha & \gamma' & \leq a & \gamma' & \leq b \\
\beta' & \leq a & \beta' & \leq c & \alpha' & \leq b & \alpha' & \leq c \\
d & \leq \alpha & d & \leq \beta & e & \leq \gamma & e & \leq d \\
\alpha' & \leq d' & \beta' & \leq d' & \gamma' & \leq e' & d' & \leq e'
\end{align*}
\]

and by applications of II

\[
\begin{align*}
e & \leq \alpha & e & \leq \beta & \alpha' & \leq \gamma & \beta' & \leq \gamma \\
\alpha' & \leq \alpha & \alpha' & \leq \beta & \alpha' & \leq \gamma & \beta' & \leq \alpha \\
\beta' & \leq \beta & \beta' & \leq \gamma & \gamma' & \leq \alpha & \gamma' & \leq \beta \\
\gamma' & \leq \gamma
\end{align*}
\]
and further through IV

\[
\begin{align*}
& c \leq d & d' \leq c & \alpha' \leq d & \beta' \leq d \\
& \gamma' \leq d & \alpha' \leq e & \beta' \leq e & \gamma' \leq e \\
& d' \leq \alpha & d' \leq \beta & d' \leq \gamma & d' \leq d \\
& d' \leq e & e' \leq \alpha & e' \leq \beta & e' \leq \gamma \\
& e' \leq d & e' \leq e
\end{align*}
\]

whereby the system is closed with respect to I–V. Since the pair \( e \leq e' \) does not occur, so therefore the non-provability in lattice theory of the statement 

\[
(a + b)(a + c)(b + c) \leq ab + ac + bc
\]

is proved.

That we really are able to decide, with the help of the method given in these paragraphs, whether a certain given statement in lattice theory holds or not, when this statement is formed from elementary statements, i.e., relative coefficients or in other words pairs and triples \( x \leq y, (x, y, z) \in \land, (x, y, z) \in \lor \) by finitely many applications of conjunction, disjunction, negation and production - therefore under the exclusion of summation\(^6\) - can be shown in the following way.

Each such statement can be written in the form\(^7\)

\[
(1) \quad \forall x_1 \forall x_2 \ldots \forall x_n U(x_1, \ldots, x_n)
\]

where \( U(x_1, \ldots, x_n) \) is a statement that is constructed from pairs and triples of the form \( x \leq y, (x, y, z) \in \land, (x, y, z) \in \lor \) by finitely many applications of conjunction, disjunction and negation. Each such statement can further be written as a product (conjunction) of finitely many statements \( E_1, E_2, \ldots \), where each statement \( E_r \) can be formed from elementary statements by means of negation and disjunction only. The given statement (1) will then be equivalent to the simultaneous satisfaction of statements all of the form

\[
(2) \quad \forall x_1 \forall x_2 \ldots \forall x_n E_r(x_1, \ldots, x_n)
\]

In order to determine whether (1) holds in general, one therefore needs only to determine whether (2) holds for each value of \( r \). Now let

\[
(3) \quad \forall x_1 \ldots \forall x_n E(x_1, \ldots, x_n)
\]

be one of these statements (2). \( E \) may then be written as a sum (disjunction, alternative) of finitely many segments, where each segment is a negated or non-negated elementary statement. Let \( e_1, e_2, \ldots, e_p \) be the non-negated elementary statements that occur in \( E \), while \( \bar{e}_{p+1}, \bar{e}_{p+2}, \ldots, \bar{e}_{p+q} \) are the negated ones, so that \( e_{p+1}, e_{p+2}, \ldots, e_{p+q} \) are also non-negated elementary statements. Then the whole sentence \( E \) means that either \( e_1 \) or \( e_2 \) or \( \ldots \) or \( e_p \) holds, whenever \( e_{p+1}, e_{p+2}, \ldots \) through \( e_{p+q} \) take place. \( E \) has therefore

\(^6\)Here \textit{summation} means existential quantifiers \( \exists x \). Skolem is dealing with the universal theory of lattices. - JBN

\(^7\)Skolem uses \( \prod_x \) as the universal quantifier \( \forall x \). - JBN

\(^8\)These pairs and triples I call \textit{elementary statements}. - JBN
the meaning of an implication; it means, in our combinatorial terminology, that when these pairs and triples are present, at least one of certain pairs and triples are present, and that this should hold for any choice assigned to the original symbols in the pairs and triples. Further, this means only that at least one of the pairs and triples \(e_1, \ldots, e_p\) should be derivable by means of Axioms I–VI from the pairs and triples \(e_{p+1}, \ldots, e_{p+q}\). That can always be decided by the method given above. One needs only to discover, whether \(e_1\) follows from \(e_{p+1}, e_{p+2}, \ldots, e_{p+q}\), then whether \(e_2\) follows from \(e_{p+1}, \ldots, e_{p+q}\), etc. through whether \(e_p\) follows from \(e_{p+1}, \ldots, e_{p+q}\).

There is however one special case that we must deal with separately, namely when all the elementary statements occurring in \(E\) are negated, therefore when the number \(p = 0\). It is clear though that such a statement can never be a generally satisfied formula of lattice theory. This follows from the fact that the given axioms are all of a positive sense, that they all say that certain pairs or triples should exist, or should exist when others do, while they never say that pairs or triples should not exist, or that the existence of certain pairs or triples should follow from the non-existence of others. One can thus always form pairs and triples at will out of the original symbols, and then apply Axioms I–VI arbitrarily often, without thereby ever coming to a contradiction. Thus one can never prove in lattice theory a statement of the form

\[
\forall x_1 \ldots \forall x_n (\overline{e_1} + \overline{e_2} + \ldots + \overline{e_q})
\]

where \(e_1, e_2, \ldots, e_q\) are non-negative elementary statements in \(x_1, \ldots, x_n\).