LATTICES THAT ARE THE JOIN OF TWO PROPER SUBLATTICES

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Every lattice is the complete join of all its one-element sublattices. In this paper we address the question: *Which lattices* $\mathbf{L}$ *have the property that* L *is finitely join reducible in* $\operatorname{Sub}\mathbf{L}$*?* That is, when do there exist proper sublattices $\mathbf{A}$, $\mathbf{B}$ such that $\mathbf{L} = \mathbf{A} \vee \mathbf{B}$? In particular, could it be that every nontrivial lattice has this property, in which case every element of $\operatorname{Sub}\mathbf{L}$ would be finitely join reducible?

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Let us mention a related problem. Recall the following result of Tom Whaley [4].

Theorem 0.1. *If* $\mathbf{L}$ *is a lattice and* $\kappa = |\,\mathbf{L}\,|$ *is a regular infinite cardinal, then* $\mathbf{L}$ *has a proper sublattice of cardinality* κ*.*

The question is: *Is this true when* $|\,\mathbf{L}\,|$ *is singular?* Note that if $|\,\mathbf{L}\,|$ is uncountable and $\mathbf{L} = \mathbf{A} \vee \mathbf{B}$, then either $|\,\mathbf{A}\,| = |\,\mathbf{L}\,|$ or $|\,\mathbf{B}\,| = |\,\mathbf{L}\,|$. So if there is a lattice which has no proper sublattice of the same cardinality, then it must be join irreducible in $\operatorname{Sub}\mathbf{L}$.

If a lattice contains a maximal proper sublattice, then it is join reducible. However, we have the following theorem of M. E. Adams [1].

Theorem 0.2. *There exist distributive lattices with no maximal sublattices.*

But, as we shall see below, every distributive lattice is join reducible.

1. Elementary results

Let us begin with some elementary but useful observations.

Theorem 1.1. *If* $\mathbf{L}$ *has a nontrivial finite homomorphic image, then* $\mathbf{L}$ *is the disjoint union of finitely many proper sublattices.*

Proof. If $h\colon \mathbf{L} \to \mathbf{K}$ where $\mathbf{K}$ is finite, then $\mathbf{L} = \bigcup\{h^{-1}(a) : a \in K\}$. $\square$

Corollary 1.2. *If* $\mathbf{L}$ *has a prime ideal, then* $\mathbf{L}$ *is the disjoint union of two proper sublattices.*

A lattice $\mathbf{L}$ is said to be **join semidistributive** if whenever $a \vee b = a \vee c$ then $a \vee b = a \vee (b \wedge c)$.

Theorem 1.3. *If* $\mathbf{L}$ *is join semidistributive and has a greatest element* 1*, then* $\mathbf{L}$ *has a prime ideal.*

Proof. If 1 is join irreducible, then $\mathbf{L} = \{1\} \vee (\mathbf{L} \setminus \{1\})$. So we may assume that 1 is join reducible, and fix elements $a \neq 1$ and $b \neq 1$ such that $1 = a \vee b$. Let I be an ideal maximal with respect to $b \in I$ and $1 \notin I$. Let $F = \{x \in L : \exists i \in I \text{ with } x \vee i = 1\}$. Assume $u, v \in F$. Then there exist $i_0, i_1 \in I$ with $1 = u \vee i_0 = v \vee i_1$. Thus $1 = u \vee i = v \vee i$ where $i = i_0 \vee i_1$. As $\mathbf{L}$ is join semidistributive, we have $1 = (u \wedge v) \vee i$, so $u \wedge v \in F$. Thus F is a filter. To see that F^c is I, pick $x \notin F$. For all $i \in I$, we have $x \vee i \neq 1$. The ideal generated by x and I is $\{y \in L : y \leq x \vee i \text{ for some } i \in I\}$, which does not contain 1. Since I is maximal, x is in I. Thus $F^c \subseteq I$. If $t \in F \cap I$, then there exists $i \in I$ such that $t \vee i = 1$, whence $1 \in I$. As this is false, $F^c = I$. Hence F and I are both prime. □

What we actually prove in Theorem 1.3 is that if a lattice is join semidistributive at its largest element 1, then the lattice is a proper union of two sublattices.

Finally, given a lattice $\mathbf{L} = \langle L; \wedge, \vee \rangle$ we can form a new algebra $\mathbf{L}^\infty = \langle L \cup \{\infty\}; \wedge, \vee, \infty \rangle$ by adding a new greatest element to $\mathbf{L}$ as a constant. The obvious natural correspondence between sublattices of $\mathbf{L}$ and subalgebras of $\mathbf{L}^\infty$ is an isomorphism, and it is often a convenience to work with the subalgebras of $\mathbf{L}^\infty$, which are always nonempty. The subalgebra of $\mathbf{L}^\infty$ corresponding to $\mathbf{S} \leq \mathbf{L}$ will be denoted $\mathbf{S}^\infty$. We will refer to an algebra $\mathbf{L}^\infty$ as an **augmented lattice**.

2. LATTICES WITH DCC

In this section we will consider sublattices of lattices satisfying the descending chain condition. We need to recall some terminology and basic results.

Let A, B, C be finite subsets of a lattice $\mathbf{L}$, and let $d \in L$. We say that A **refines** B, written $A \ll B$, if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. We say that C is a **join cover** of d if $d \leq \bigvee C$. A join cover C of d is **nontrivial** if $d \nleq c$ for all $c \in C$. It is a **minimal** join cover if $A \ll C$ and $C \nsubseteq A$ implies $d \nleq \bigvee A$.

Lemma 2.1. *Let* $\mathbf{L}$ *be a lattice satisfying the DCC.*

1. *Every element of* $\mathbf{L}$ *is a join of join irreducible elements.*
2. *Every join cover of an element of* $\mathbf{L}$ *refines to a minimal one.*
3. *The elements of a minimal join cover are an antichain of join irreducible elements.*

For $\mathbf{L}$ a lattice satisfying the DCC, let $\mathrm{J}^*(\mathbf{L})$ denote the set of (finitely) join irreducible elements of $\mathbf{L}$, including the least element 0 of $\mathbf{L}$. For an augmented lattice, by convention the new greatest element ∞ is not in $\mathrm{J}^*(\mathbf{L}^\infty)$.

Let $\mathbf{S} \leq \mathbf{L}$ be a sublattice. Define $\alpha_S : L \to S^\infty$ by, for $x \in L$,

$$\alpha_S(x) = \bigwedge \{s \in S^\infty : x \leq s\}.$$

As $\mathbf{L}^\infty$ satisfies the DCC and $\infty \in \mathbf{S}^\infty$, this meet is defined, and in fact $\alpha_S(x)$ is the least element a in S^∞ such that $x \leq a$. It is clear that for all $x \in L$ we have $x \leq \alpha_S(x)$, and $x = \alpha_S(x)$ if and only if $x \in S$.

Lemma 2.2. *The function α_S preserves joins.*

Proof. Let x, y be in L. As $u \leq v$ in $\mathbf{L}$ implies $\alpha_S(u) \leq \alpha_S(v)$, we have $\alpha_S(x) \vee \alpha_S(y) \leq \alpha_S(x \vee y)$. On the other hand, $u \leq \alpha_S(u) \in S$ for all $u \in L$, so $x \vee y \leq \alpha_S(x) \vee \alpha_S(y) \in S$. By the definition of α_S it follows that $\alpha_S(x \vee y) \leq \alpha_S(x) \vee \alpha_S(y)$. □

For $x, y \in L$ define the set $T_{x,y} = \{t \in L : x \leq t \text{ implies } y \leq t\}$. The set $T_{x,y}$ is not normally a sublattice of $\mathbf{L}$, but by Lemma 2.3 it is meet closed. Lemma 2.4 shows that the intersection of appropriately chosen $T_{x,y}$ is a sublattice of $\mathbf{L}$.

Lemma 2.3. *For fixed x and y, the set $T_{x,y}$ is meet closed.*

Proof. If $t_1, t_2 \in T_{x,y}$ and $x \leq t_1 \wedge t_2$, then we have that $x \leq t_i$ implies $y \leq t_i$ for $i = 1, 2$. Thus $y \leq t_1 \wedge t_2$. □

Although it is true that $S = \bigcap_{x \in L} T_{x,\alpha_S(x)}$, we use and prove the following stronger fact.

Lemma 2.4. *For $\mathbf{S}$ a sublattice of $\mathbf{L}$, we have* $S = \displaystyle\bigcap_{p \in \mathrm{J}^*(\mathbf{L})} T_{p,\alpha_S(p)}$.

Proof. Let $p \in \mathrm{J}^*(\mathbf{L})$. If $x \in S$ and $p \leq x$, then by definition $\alpha_S(p) \leq x$. Thus $S \subseteq \bigcap_{p \in \mathrm{J}^*(\mathbf{L})} T_{p,\alpha_S(p)}$.

Suppose $x \notin S$. If $x \in \mathrm{J}^*(\mathbf{L})$ then $x \notin T_{x,\alpha_S(x)}$. Otherwise $x < \alpha_S(x)$ and we may write $x = \bigvee Q$ for some finite $Q \subseteq \mathrm{J}^*(\mathbf{L})$. By Lemma 2.2, $x < \alpha_S(x) = \bigvee_{q \in Q} \alpha_S(q)$. For some $q_0 \in Q$ we have $q_0 \leq x$ but $\alpha_S(q_0) \nleq x$, so $x \notin T_{q_0,\alpha_S(q_0)}$. □

Lemma 2.5 and Theorem 2.6 show that sublattices of $\mathbf{L}$ correspond to maps from $\mathrm{J}^*(\mathbf{L})$ to L satisfying a certain nice set of properties.

Lemma 2.5. *The restriction of α_S to $\mathrm{J}^*(\mathbf{L})$ satisfies the following:*

1. *$p \leq \alpha_S(p)$ for all $p \in \mathrm{J}^*(\mathbf{L})$;*
2. *$p \leq \alpha_S(q)$ implies $\alpha_S(p) \leq \alpha_S(q)$ for all $p, q \in \mathrm{J}^*(\mathbf{L})$;*
3. *if*
   - *$p \in \mathrm{J}^*(\mathbf{L})$ and $T \subseteq L$ with T finite,*
   - *$p \leq \bigvee T$ in $\mathbf{L}$, and*
   - *$\alpha_S(p) \nleq \bigvee T$,*

   then there exists q such that
   - *$q \in \mathrm{J}^*(\mathbf{L})$,*
   - *$q \leq t$ for some $t \in T$, and*
   - *$\alpha_S(q) \nleq \bigvee T$.*

Proof. Suppose $p \leq \bigvee T$ and $\alpha_S(p) \nleq \bigvee T$. As $\mathbf{L}$ has the descending chain condition, we may refine T to a minimal join cover Q of p. Thus $Q \ll T$ and $Q \subseteq \mathrm{J}^*(\mathbf{L})$ and $p \leq \bigvee Q \leq \bigvee T$. Now $p \leq \bigvee Q$ implies $\alpha_S(p) \leq \bigvee_{q \in Q} \alpha_S(q)$ while $\alpha_S(p) \nleq \bigvee T$. Hence $\bigvee_{q \in Q} \alpha_S(q) \nleq \bigvee T$, and for some $q \in Q$ we have $\alpha_S(q) \nleq \bigvee T$. □

Theorem 2.6. *Suppose* $\gamma \colon \mathrm{J}^*(\mathbf{L}) \to L^\infty$ *satisfies properties (1), (2) and (3). Then* $S^\gamma = \displaystyle\bigcap_{p \in \mathrm{J}^*(\mathbf{L})} T_{p,\gamma(p)}$ *is a sublattice of* $\mathbf{L}$*, and* γ *is the restriction of* α_{S^γ} *to* $\mathrm{J}^*(\mathbf{L})$*.*

Proof. Let $S = S^\gamma$. By Lemma 2.3, for each $p \in \mathrm{J}^*(\mathbf{L})$ the set $T_{p,\gamma(p)}$ is meet closed, and hence S is also. To see that S is join closed, suppose that $t_1 \vee t_2 \notin S$. Then there exists $p \in \mathrm{J}^*(\mathbf{L})$ such that $t_1 \vee t_2 \notin T_{p,\gamma(p)}$. That is, $p \leq t_1 \vee t_2$ and $\gamma(p) \nleq t_1 \vee t_2$. By (3) there exists a $q \in \mathrm{J}^*(\mathbf{L})$ and $i \in \{1, 2\}$ with $q \leq t_i$ and $\gamma(q) \nleq t_1 \vee t_2$. *A fortiori*, $\gamma(q) \nleq t_i$. Thus $t_i \notin T_{q,\gamma(q)}$ whence $t_i \notin S$. Therefore S is join closed.

If $s \in S$, then for all $p \in \mathrm{J}^*(\mathbf{L})$ we have $s \in T_{p,\gamma(p)}$, so $p \leq s$ implies $\gamma(p) \leq s$. Hence $\alpha_S \geq \gamma$ on $\mathrm{J}^*(\mathbf{L})$. For the reverse inclusion, fix $p \in \mathrm{J}^*(\mathbf{L})$. For all $q \in \mathrm{J}^*(\mathbf{L})$, by (2) we have $\gamma(q) \in T_{p,\gamma(p)}$. Thus $\gamma(q) \in S$. In particular, $\gamma(p) \in S$ and $p \leq \gamma(p)$, so $\alpha_S(p) \leq \gamma(p)$. Thus $\alpha_S{\restriction}_{\mathrm{J}^*(\mathbf{L})} \leq \gamma$. □

At this point, it is useful to formulate some equivalent versions of condition (3) of Lemma 2.5.

Lemma 2.7. *Suppose* $\gamma \colon \mathrm{J}^*(\mathbf{L}) \to L^\infty$ *satisfies properties (1) and (2). Then the following are equivalent.*

1. *If* $p \in \mathrm{J}^*(\mathbf{L})$ *and* $Q \subseteq \mathrm{J}^*(\mathbf{L})$*, then* $p \leq \bigvee Q$ *implies* $\gamma(p) \leq \bigvee_{q \in Q} \gamma(q)$*.*
2. *If* $p \in \mathrm{J}^*(\mathbf{L})$ *and* Q *is a minimal nontrivial join cover of* p*, then* $\gamma(p) \leq \bigvee_{q \in Q} \gamma(q)$*.*
3. *If* $p \in \mathrm{J}^*(\mathbf{L})$ *and* $T \subseteq L$ *is finite, with* $p \leq \bigvee T$ *and* $\gamma(p) \nleq \bigvee T$*, then there exists* $q \in \mathrm{J}^*(\mathbf{L})$ *such that* $q \leq t$ *for some* $t \in T$ *and* $\gamma(q) \nleq \bigvee T$*.*
4. *If* $p \in \mathrm{J}^*(\mathbf{L})$ *and* $T \subseteq L$ *is finite, with* $p \leq \bigvee T$ *and* $\gamma(p) \nleq \bigvee T$*, then there exist* $q \in \mathrm{J}^*(\mathbf{L})$ *and* $t \in T$ *such that* $q \leq t$ *and* $\gamma(q) \nleq t$*.*
5. γ *is the restriction of* α_{S^γ} *to* $\mathrm{J}^*(\mathbf{L})$*.*

The nontrivial steps in proving the equivalence cyclically are embedded in what we have done so far.

A map $\gamma \colon \mathrm{J}^*(\mathbf{L}) \to L^\infty$ that satisfies properties (1), (2) and (3) of Lemma 2.5 (and hence the equivalent conditions of Lemma 2.7) will be called an **admissible map**. The set of all admissible maps on $\mathrm{J}^*(\mathbf{L})$ will be denoted by $\mathcal{A}(\mathbf{L})$.

We now turn to examples and applications of Theorem 2.6. By way of notation, we will use $x/0$ and $1/x$ to denote principal ideals and filters, respectively, even for lattices which may have no least or greatest element.

1. The identity map $i_{\mathrm{J}^*(\mathbf{L})}$ corresponds to the largest sublattice $\mathbf{L}$.
2. The constant map ∞ on $\mathrm{J}^*(\mathbf{L})$ corresponds to the empty sublattice.

(3) An ideal I of $\mathbf{L}$ is represented by a map η with $\eta(p) = p$ for $p \in I$, and $\eta(p) = \infty$ when $p \notin I$.
(4) A filter F of $\mathbf{L}$ is represented by the map φ with $\varphi(p) = \bigwedge\{x \in F : x \geq p\}$. In the presence of the DCC, every filter is principal, and the filter $1/a$ is represented by the map $\varphi(p) = p \vee a$.

Lemma 2.8. *Assume* $\gamma, \mu \in \mathcal{A}(\mathbf{L})$. *Then* $\gamma \leq \mu$ *if and only if* $S^\gamma \geq S^\mu$.

Proof. Indeed, if $\gamma(p) \leq \mu(p)$ then $\mu(p) \leq x$ implies $\gamma(p) \leq x$, whence $T_{p,\mu(p)} \subseteq T_{p,\gamma(p)}$. If this holds for all $p \in \mathrm{J}^*(\mathbf{L})$, then $S^\mu \leq S^\gamma$. Conversely, assume $\gamma \nleq \mu$, so that $\gamma(p) \nleq \mu(p)$ for some $p \in \mathrm{J}^*(\mathbf{L})$. Then $\mu(p) \in S^\mu$, but $\mu(p) \notin T_{p,\gamma(p)}$, whence $\mu(p) \notin S^\gamma$. Thus $S^\mu \nleq S^\gamma$. □

The pointwise meet or join of admissible maps need not be admissible. Nonetheless, the next result is an immediate consequence of Lemma 2.8.

Theorem 2.9. *Let* $\gamma, \delta \in \mathcal{A}(\mathbf{L})$. *Then* $S^\gamma \vee S^\delta = \mathbf{L}$ *if and only if there is no* $\mu \in \mathcal{A}(\mathbf{L})$ *such that* $i_{\mathrm{J}^*(\mathbf{L})} < \mu \leq \gamma \wedge \delta$.

In particular, if $\gamma \wedge \delta = i_{\mathrm{J}^*(\mathbf{L})}$ then $S^\gamma \vee S^\delta = \mathbf{L}$.

Recall that a lattice is **semimodular** if $a \succ a \wedge b$ implies $a \vee b \succ b$. We will say that a lattice is **finitely atomistic** if every element is a join of finitely many atoms. The term **atomistic** will be used in the context of complete lattices to mean that every element is a complete join of atoms. In a finitely atomistic semimodular lattice, every element has finite height, and hence the DCC holds.

Theorem 2.10. *If* $\mathbf{L}$ *is a finitely atomistic semimodular lattice, then* $\mathbf{L}$ *is join reducible in* $\operatorname{Sub}\mathbf{L}$.

Proof. We may assume that $\mathbf{L}$ is not isomorphic to any $\mathbf{M}_\kappa$, for that case is trivial. Thus we can find atoms a, b, c with $c \nleq a \vee b$. Let $\alpha(p) = p \vee a$, $\beta(p) = p \vee b$ and $\gamma(p) = p \vee c$ for all atoms p.

Let p be an atom distinct from a, b, c. Note that, by semimodularity, the join of two distinct atoms has height 2. If $\alpha(p) \neq \beta(p)$, then $\alpha(p) \wedge \beta(p) = p$. On the other hand, if $\alpha(p) = \beta(p)$, then $p \vee a = p \vee b = a \vee b$. In that case, $p \vee a \neq p \vee c$, so that $\alpha(p) \wedge \gamma(p) = p$. Hence $\alpha \wedge \beta \wedge \gamma = i_{\mathrm{J}^*(\mathbf{L})}$, and by Theorem 2.9, $\mathbf{L}$ is join reducible. □

Temporarily leaving the class of lattices satisfying the DCC, we can modify the preceding argument to show that the lattice of subspaces of any vector space is join reducible.

Theorem 2.11. *If* $\mathbf{L}$ *is an atomistic modular complete lattice, then* $\mathbf{L}$ *is join reducible in* $\operatorname{Sub}\mathbf{L}$.

Proof. Recall that any lattice satisfying the hypotheses of the theorem is relatively complemented. Again bypassing the case $\mathbf{M}_\kappa$, we can choose atoms a, b, c with

$c \nleq a \vee b$. We want to show that $\mathbf{L} = 1/a \vee 1/b \vee 1/c$. As in the proof of Theorem 2.10 all the atoms of $\mathbf{L}$ are in this join.

So let $x \in L$. Without loss of generality, we may assume that x is not an atom, $x \ngeq a$ and $x \ngeq b$. If $x \vee a \neq x \vee b$, then $x = (x \vee a) \wedge (x \vee b) \in 1/a \vee 1/b$. So let us assume that $x \vee a = x \vee b$, in which case the element $d = x \wedge (a \vee b)$ is an atom, and as in the preceding theorem $d \in 1/a \vee 1/c$. Let y be a relative complement of d in $x/0$. If $y \vee a = y \vee b$, then using $d \leq a \vee b$ and $y \vee d = x$ we see that $y \vee a = y \vee a \vee b = x \vee a \vee b = x \vee a$, which contradicts even semimodularity since $y < x < x \vee a$. Hence $y \vee a \neq y \vee b$, and $y \in 1/a \vee 1/b$. Therefore $x = y \vee d \in 1/a \vee 1/b \vee 1/c$. □

3. An example

In this section we will construct a lattice which is finitely atomistic, satisfies the DCC, and is join semidistributive, but which has no maximal sublattice and no prime ideal.

The atoms of our lattice $\mathbf{T}$ will be certain finite sequences of integers. Let $\mathbb{Z}^*$ denote the nonzero integers, and let $\mathbb{Z}^+$ denote the positive integers. For $k \geq 0$, let S_k be the set of all sequences $\langle 0, s_1, \ldots, s_k \rangle$ with $s_i \in \mathbb{Z}^*$ for all $i > 0$. Note that S_0 consists of the single finite sequence $\langle 0 \rangle$. Let $S = \bigcup_{k \geq 0} S_k$.

The lattice $\mathbf{T}$ will be the join semilattice with 0 generated by S subject to the set R of all relations of the form

$$\langle \mathbf{s} \rangle \leq \langle \mathbf{s}, i \rangle \vee \langle \mathbf{s}, -i \rangle$$

with $\mathbf{s} \in S_k$ for some $k \geq 0$ and $i \in \mathbb{Z}^+$. We claim that this lattice has the desired properties.

It is convenient to let $T_k = \mathbf{Sg}(S_0 \cup \cdots \cup S_k)$. Note that each T_k is an ideal in T_{k+1}.

Lemma 3.1. *For every $x \in T$, the ideal $x/0$ is finite. Hence $\mathbf{T}$ satisfies the DCC.*

Proof. Suppose $x = \bigvee A$, where A is a finite set of atoms. If p is an atom and $p \leq x$, then p is an initial subsequence of some atom $a \in A$. □

Corollary 3.2. *The lattice $\mathbf{T}$ has no prime ideal.*

Proof. If $\mathbf{T}$ contained a prime ideal, then its complement would be a prime filter, which would necessarily be principal of the form $1/p$ with p a join prime element. But $\mathbf{T}$ contains no join prime element. □

Lemma 3.3. *Every element $x \in T$ has a unique irredundant join decomposition. Hence $\mathbf{T}$ is join semidistributive.*

Proof. Again let $x = \bigvee A$, and let k be minimal such that $x \in T_k$. Let $B_k = \emptyset$ and, given B_{j+1}, let $\mathbf{s} \in B_j$ if there is an integer i such that $\langle \mathbf{s}, i \rangle$ and $\langle \mathbf{s}, -i \rangle$ are both in $A \cup B_{j+1}$. Set $B = \bigcup_{0 \leq j \leq k} B_j$. An atom p is below x if and only if $p \in A \cup B$, and $x = \bigvee (A \setminus B)$ is the unique irredundant decomposition of x. □

Lemma 3.4. *Let* φ *be an admissible map on* $\mathbf{T}$*. If* $p \in S_k$ *and* $\varphi(p) > p$*, then there exist infinitely many* $q \in S_{k+1}$ *with* $\varphi(q) > q$*.*

Proof. Let $p = \langle \mathbf{s} \rangle$, and let $B = \{b \in S : b \leq \varphi(p)\}$. For every $i \in \mathbb{Z}^+$ we have $\langle \mathbf{s} \rangle \leq \langle \mathbf{s}, i \rangle \vee \langle \mathbf{s}, -i \rangle$, and hence $\varphi(\langle \mathbf{s} \rangle) \leq \varphi(\langle \mathbf{s}, i \rangle) \vee \varphi(\langle \mathbf{s}, -i \rangle)$. Now $\langle \mathbf{s}, i \rangle \vee \langle \mathbf{s}, -i \rangle$ contains only three atoms, *viz.*, p and the two joinands. So unless $B \subseteq \{\langle \mathbf{s} \rangle, \langle \mathbf{s}, i \rangle, \langle \mathbf{s}, -i \rangle\}$, which can happen for at most one i, then either $\varphi(\langle \mathbf{s}, i \rangle) > \langle \mathbf{s}, i \rangle$ or $\varphi(\langle \mathbf{s}, -i \rangle) > \langle \mathbf{s}, -i \rangle$. □

Lemma 3.5. *Let* φ *be an admissible map on* $\mathbf{T}$*, and let* $k \geq 0$*. Define* $\varphi_k : \mathrm{J}^*(\mathbf{T}) \to T^\infty$ *by*

$$\varphi_k(p) = \begin{cases} p & \text{if } p \in S_0 \cup \cdots \cup S_k, \\ \varphi(p) & \text{otherwise.} \end{cases}$$

Then φ_k *is an admissible map.*

This is straightforward to verify.

Theorem 3.6. *The lattice* $\mathbf{T}$ *has no maximal sublattice.*

Proof. Let $\varphi \in \mathcal{A}(\mathbf{T})$ with $\varphi > i_{\mathrm{J}^*(\mathbf{T})}$. Let k be such that $\varphi(p) > p$ for some $p \in S_k$. By Lemmas 3.4 and 3.5 we have $\varphi > \varphi_k > i_{\mathrm{J}^*(\mathbf{T})}$, whence S^φ is not maximal. □

Note, however, that $\mathbf{T}$ is join reducible. If $\mathbf{A} = \mathbf{Sg}(\{\langle 0 \rangle\} \cup \{\langle \mathbf{s}, i \rangle : i > 0\})$ and $\mathbf{B} = \mathbf{Sg}(\{\langle \mathbf{s}, i \rangle : i < 0\})$, then $\mathbf{T} = \mathbf{A} \vee \mathbf{B}$.

4. Lattices without DCC

If we drop the descending chain condition and consider arbitrary lattices, then we must replace join irreducible elements with join irreducible filters. With this change, we can prove analogues of the results in the preceding section. The modifications are straightforward, but not as easy to use. We will sketch the more general theory in this section.

Let $\mathcal{F}(\mathbf{L})$ denote the filter lattice of a lattice $\mathbf{L}$. We order $\mathcal{F}(\mathbf{L})$ by reverse set inclusion. For $X \subseteq L$, let $\mathbf{Fg}(X)$ denote the filter generated by X.

Let $\mathrm{J}^*(\mathbf{L})$ denote the set of (finitely) join irreducible nonempty filters of $\mathbf{L}$. By convention, the least element of $\mathcal{F}(\mathbf{L})$, which is L, is in $\mathrm{J}^*(\mathbf{L})$. Because $\mathcal{F}(\mathbf{L})$ always has a greatest element, the empty filter, we do not need to consider augmented lattices in this setting. This definition agrees, *via* the natural correspondence, with the definition of $\mathrm{J}^*(\mathbf{L})$ for lattices satisfying the DCC.

Let $\mathbf{S} \leq \mathbf{L}$ be a sublattice. For $F \in \mathcal{F}(\mathbf{L})$, define $\alpha_S : \mathcal{F}(\mathbf{L}) \to \mathcal{F}(\mathbf{L})$ by $\alpha_S(F) = \mathbf{Fg}(F \cap S)$. It is clear that for all $F \in \mathcal{F}(\mathbf{L})$ we have $F \leq \alpha_S(F)$, and $F = \alpha_S(F)$ if and only if S is cofinal downwards in F, i.e., for every $f \in F$ there exists $s \in S$ with $s \leq f$.

Lemma 4.1. *The function* α_S *preserves joins.*

Proof. We want to show that $\alpha_S(F \vee G) = \alpha_S(F) \vee \alpha_S(G)$, i.e., that $\alpha_S(F \cap G) = \alpha_S(F) \cap \alpha_S(G)$. Clearly α_S is order preserving, so $\alpha_S(F \cap G) \subseteq \alpha_S(F) \cap \alpha_S(G)$. For the reverse inclusion, let $x \in \alpha_S(F) \cap \alpha_S(G)$. Then there exist elements s, t such that $x \geq s \in S \cap F$ and $x \geq t \in S \cap G$. Thus $x \geq s \vee t \in S \cap F \cap G$, whence $x \in \alpha_S(F \cap G)$, as desired. □

For $F, G \in \mathcal{F}(\mathbf{L})$ define the set $T_{F,G} = \{t \in L : t \in F \text{ implies } t \in G\} = F^c \cup G$. The next lemma records two elementary facts about these sets.

Lemma 4.2. *Let F, $G \in \mathcal{F}(\mathbf{L})$, and let $\mathbf{S}$ be a sublattice of $\mathbf{L}$. Then*

1. *$T_{F,G}$ is meet closed,*
2. *$S \subseteq T_{F,\alpha_S(F)}$.*

Again, a sublattice $\mathbf{S}$ is determined by the restriction of α_S to join irreducible filters.

Lemma 4.3. *For $\mathbf{S}$ a sublattice of $\mathbf{L}$, we have* $S = \bigcap_{F \in \mathrm{J}^*(\mathbf{L})} T_{F,\alpha_S(F)}$.

Proof. Suppose $x \notin S$. Let F be a filter which is maximal (w.r.t. set inclusion) such that $x \in F$ and $F \cap S \cap x/0 = \emptyset$. Then $F \in \mathrm{J}^*(\mathbf{L})$: for if $F = G \cap H$ properly, then there would exist elements $s \in G \cap S \cap x/0$ and $t \in H \cap S \cap x/0$, whence $s \vee t \in G \cap H \cap S \cap x/0 = F \cap S \cap x/0$, a contradiction. Now $x \in F$, but $x \notin \alpha_S(F)$ as there is no element $s \in S \cap F$ with $x \geq s$. Thus $x \notin T_{F,\alpha_S(F)}$. □

Lemma 4.4. *Let $\mathbf{S}$ be a sublattice of $\mathbf{L}$, and set* $U_{\alpha_S} = \bigcup_{G \in \mathrm{J}^*(\mathbf{L})} (G \setminus \alpha_S(G))$. *The restriction $\alpha_S \colon \mathrm{J}^*(\mathbf{L}) \to \mathcal{F}(\mathbf{L})$ satisfies the following:*

1. *$F \leq \alpha_S(F)$ for all $F \in \mathrm{J}^*(\mathbf{L})$;*
2. *if $x \in L$, $F \in \mathrm{J}^*(\mathbf{L})$ and $x/0 \cap F \subseteq U_{\alpha_S}$, then $x \notin \alpha_S(F)$;*
3. *if*
   - *$F \in \mathrm{J}^*(\mathbf{L})$ and $\{G_1, \dots, G_m\} \subseteq \mathcal{F}(\mathbf{L})$,*
   - *$F \leq \bigvee_{1 \leq i \leq m} G_i$ in $\mathcal{F}(\mathbf{L})$, and*
   - *$\alpha_S(F) \nleq \bigvee_{1 \leq i \leq m} G_i$,*

   then there exists H such that
   - *$H \in \mathrm{J}^*(\mathbf{L})$,*
   - *$H \leq G_{i_0}$ for some i_0, and*
   - *$\alpha_S(H) \nleq \bigvee_{1 \leq i \leq m} G_i$.*

The non-comatose reader will observe that condition (2) of Lemma 2.5 has been replaced by a more complicated condition. It is still true that $F \leq \alpha_S(G)$ implies

$\alpha_S(F) \leq \alpha_S(G)$, but that does not appear to be sufficient to prove Theorem 4.5 below.

Proof. Property (1) just states that $F \supseteq \mathbf{Fg}(F \cap S) = \alpha_S(F)$.

To prove (2), assume $x \in \alpha_S(F)$. Then there is an element s such that $x \geq s \in F \cap S$. For every filter G, if $s \in G$ then $s \in \alpha_S(G)$. Hence $s \notin \bigcup_{G \in \mathcal{F}(\mathbf{L})} (G \setminus \alpha_S(G))$. *A fortiori*, $s \notin U_{\alpha_S}$.

Assume that the hypotheses of (3) hold, and let $x \in \bigcap G_i \setminus \alpha_S(F)$. Note that $x \notin \alpha_S(F)$ means that $F \cap S \cap x/0 = \emptyset$. Now $\alpha_S(F) \supseteq \bigcap \alpha_S(G_i)$, so there is an index i_0 such that $x \notin \alpha_S(G_{i_0})$, i.e., $G_{i_0} \cap S \cap x/0 = \emptyset$. Extend G_{i_0} to a filter H that is maximal (w.r.t. set inclusion) such that $H \cap S \cap x/0 = \emptyset$. As before, H is join irreducible: if $H = A \cap B$ properly, then there are elements $s \in A \cap S \cap x/0$ and $t \in B \cap S \cap x/0$, whence $s \vee t \in H \cap S \cap x/0$, a contradiction. Moreover, $x \in G_{i_0} \subseteq H$ but $x \notin \alpha_S(H)$, so $G_{i_0} \nsubseteq \alpha_S(H)$, as desired. □

Theorem 4.5. *Suppose* $\gamma\colon \mathrm{J}^*(\mathbf{L}) \to \mathcal{F}(\mathbf{L})$ *satisfies properties (1), (2) and (3) of Lemma 4.4. Then* $S^\gamma = \bigcap_{F \in \mathrm{J}^*(\mathbf{L})} T_{F,\gamma(F)}$ *is a sublattice of* $\mathbf{L}$*, and* γ *is the restriction of* α_{S^γ} *to* $\mathrm{J}^*(\mathbf{L})$.

Proof. S^γ is meet closed by Lemma 4.2, so it remains to show that it is join closed. Assume $t_1 \vee t_2 \notin S^\gamma$. Then $t_1 \vee t_2 \in F \setminus \gamma(F)$ for some $F \in \mathrm{J}^*(\mathbf{L})$. Hence $F \leq 1/t_1 \vee 1/t_2$ but $\gamma(F) \nleq 1/t_1 \vee 1/t_2$. By property (3), there exists $H \in \mathrm{J}^*(\mathbf{L})$ such that $H \leq 1/t_1$, say, and $\gamma(H) \nleq 1/t_1 \vee 1/t_2$. It follows that $\gamma(H) \nleq 1/t_1$, so $t_1 \in H \setminus \gamma(H)$, $t_1 \notin T_{H,\gamma(H)}$ and $t_1 \notin S^\gamma$. Thus S^γ is join closed.

The statement $\gamma = \alpha_{S^\gamma}\restriction_{\mathrm{J}^*(\mathbf{L})}$ means that for all $x \in L$ and all $F \in \mathrm{J}^*(\mathbf{L})$,

$$F \cap S^\gamma \cap x/0 = \emptyset \quad \text{if and only if} \quad x \in \gamma(F).$$

If $x \geq s$ for some $s \in F \cap S^\gamma$, then $s \in F \cap T_{F,\gamma(F)}$, whence $s \in \gamma(F)$ and thus $x \in \gamma(F)$. On the other hand, if $F \cap S^\gamma \cap x/0 = \emptyset$, then

$$x/0 \cap F \subseteq \Big(\bigcap_{G \in \mathrm{J}^*(\mathbf{L})} T_{G,\gamma(G)}\Big)^c = \bigcup_{G \in \mathrm{J}^*(\mathbf{L})} (G \setminus \gamma(G)) = U_\gamma.$$

By property 2, $x \notin \gamma(F)$, as desired. □

A map $\gamma\colon \mathrm{J}^*(\mathbf{L}) \to \mathcal{F}(\mathbf{L})$ that satisfies properties (1), (2) and (3) of Lemma 4.4 will be called an **admissible map**. The set of all admissible maps on $\mathrm{J}^*(\mathbf{L})$ will be denoted by $\mathcal{A}(\mathbf{L})$.

For the sake of completeness, we include the following.

Lemma 4.6. *Let* $\gamma \in \mathcal{A}(\mathbf{L})$ *and* $F, G \in \mathcal{F}(\mathbf{L})$*. If* $F \leq \gamma(G)$*, then* $\gamma(F) \leq \gamma(G)$.

Proof. First, let us rewrite property (2). Note that for an admissible map

$$U_\gamma = \bigcup_{G \in \mathrm{J}^*(\mathbf{L})} (G \setminus \gamma(G)) = (\bigcap_{G \in \mathrm{J}^*(\mathbf{L})} (G^c \cup \gamma(G)))^c = (S^\gamma)^c.$$

Thus (2) says that

$$x/0 \cap F \cap S^\gamma = \emptyset \quad \text{implies} \quad x \notin \gamma(F).$$

Assume $\gamma(G) \subseteq F$. If $x \in \gamma(G)$, then $x/0 \cap \gamma(G) \cap S^\gamma \neq \emptyset$, whence $x/0 \cap F \cap S^\gamma \neq \emptyset$. So let s be an element in $x/0 \cap F \cap S^\gamma$. As $s \in F \cap T_{F,\gamma(F)}$ we have $s \in \gamma(F)$, whence $x \in \gamma(F)$. Thus $\gamma(G) \subseteq \gamma(F)$. □

The generalization of Lemma 2.8 is direct.

Lemma 4.7. *Assume $\gamma, \mu \in \mathcal{A}(\mathbf{L})$. Then $\gamma \leq \mu$ if and only if $S^\gamma \geq S^\mu$.*

Proof. If $\gamma \leq \mu$, then $\mu(F) \subseteq \gamma(F)$ for every $F \in \mathrm{J}^*(\mathbf{L})$. Hence $T_{F,\mu(F)} \subseteq T_{F,\gamma(F)}$ for all F, whence $S^\mu \leq S^\gamma$. Conversely, assume $\gamma \nleq \mu$, so that $\mu(F) \nsubseteq \gamma(F)$ for some $F \in \mathrm{J}^*(\mathbf{L})$. Let $x \in \mu(F) \setminus \gamma(F)$. As is the proof of Lemma 4.6, $x \in \mu(F)$ implies that there is an element $s \in x/0 \cap F \cap S^\mu$. But $s \in F$ while $s \notin \gamma(F)$ as $s \leq x$, so $x \notin T_{F,\gamma(F)}$ and $x \notin S^\gamma$. Hence $S^\mu \nleq S^\gamma$. □

5. More Stuff

This last section contains some ideas that did not seem to lead anywhere, and will likely be deleted later.

An admissible map σ is *special at a* if $\sigma(x) \in \{x, x \vee a\}$ for all $x \in \mathrm{J}^*(\mathbf{L})$. The corresponding sublattice will also be called *special*.

The model for this is that for any ideal I, the sublattice $I \cup 1/a$ is special at a. More generally, define a *pseudo-ideal* to be a subset $U \subseteq L$ such that

(1) $x \leq u < 1$ and $u \in U$ implies $x \in U$,
(2) u, $v \in U$ and $u \vee v < 1$ implies $u \vee v \in U$.

Then the union of $1/a$ and a pseudo-ideal U is also a special sublattice. An example of a pseudo-ideal would be 0 and any subset of the atoms of $\mathbf{M}_n$, with or without 1. Also note that pseudo-ideals form a closure system, i.e., the intersection of pseudo-ideals is again one, so we can talk about the pseudo-ideal generated by a set.

Given a special map admissible σ, define

$$F = \{x \in \mathrm{J}^*(\mathbf{L}) : \sigma(x) = x\}$$
$$R = \{x \in \mathrm{J}^*(\mathbf{L}) : \sigma(x) = x \vee a\}$$

noting that $F \cap R = 1/a \cap \mathrm{J}^*(\mathbf{L})$.

Theorem 5.1. *If σ is a special admissible map, then σ satisfies*

(2′) *R is an order filter,*

(3′) *if* $p \in R$, $p \leq \bigvee Q$ *and* $a \nleq \bigvee Q$, *then* $Q \cap R \neq \emptyset$.

Conversely, given a pair (R, a) *satisfying* (2′) *and* (3′), *for* $x \in \mathrm{J}^*(\mathbf{L})$ *let*

$$\sigma(x) = \begin{cases} x \vee a & \textit{if } x \in R \\ x & \textit{if } x \notin R. \end{cases}$$

Then σ *is a special admissible map.*

Proof. Let σ be a special admissible map. If $p \leq q$, then $p \leq \sigma(q)$ by Lemma 2.5(1), so $\sigma(p) \leq \sigma(q)$ by Lemma 2.5(2). So if $p \in R$ and $p \leq q$, then $p \vee a = \sigma(p) \leq \sigma(q)$ and $a \leq \sigma(q)$, i.e., since σ is special, $\sigma(q) = q \vee a$. This proves (2′).

We use Lemma 2.7(1), which is equivalent to Lemma 2.5(3). If $p \in R$ and $p \leq \bigvee Q$, then $p \vee a = \sigma(p) \leq \bigvee_{q \in Q} \sigma(q)$. If also $a \nleq \bigvee Q$, then we cannot have $\sigma(q) = q$ for all $q \in Q$, whence some $q \in Q$ is also in R. This proves (3′).

Conversely, given σ defined as in the Theorem, we check the properties of Lemmas 2.5(1), 2.5(2) and 2.7(1). That $p \leq \sigma(p)$ and $\sigma(p) \in \{p, p \vee a\}$ are clear.

Suppose $p \leq \sigma(q)$. If $\sigma(p) = p$ then $\sigma(p) \leq \sigma(q)$, so assume that $\sigma(p) = p \vee a$, i.e., $p \in R$. Then by (2′) we have $q \in R$, whence $\sigma(q) = q \vee a \geq p \vee a = \sigma(p)$, as desired.

Assume $p \leq \bigvee Q$. If $\sigma(p) = p$ then $\sigma(p) \leq \bigvee Q \leq \bigvee_{q \in Q} \sigma(q)$, so w.l.o.g. suppose $\sigma(p) = p \vee a$. If $a \leq \bigvee Q$ then $\sigma(p) \leq \bigvee Q \leq \bigvee \sigma(q)$, while if not by (3′) we have $a \leq q_0 \vee a = \sigma(q_0)$ for some $q_0 \in Q$, so that again $\sigma(p) = p \vee a \leq \bigvee \sigma(q)$. □

Given a special admissible map σ, let

$$\begin{aligned} C &= \mathrm{J}^*(\mathbf{L}) - R \\ &= \{x \in \mathrm{J}^*(\mathbf{L}) : x \ngeq a \text{ and } \sigma(x) = x\}. \end{aligned}$$

The previous theorem translates thusly.

Theorem 5.2. *A special map* σ *is admissible if and only if*

(2″) C *is an order ideal,*

(3″) *if* $Q \subseteq C$, $p \leq \bigvee Q$ *and* $a \nleq \bigvee Q$, *then* $p \in C$.

The conditions in the previous theorem amount to saying that C is a pseudo-ideal in the lattice $\mathbf{L}/\uparrow a$ obtained by collapsing $1/a$.

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