EXISTENCE OF FINITE BASES FOR QUASI-EQUATIONS OF UNARY ALGEBRAS WITH 0

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Abstract. A finite unary algebra of finite type with a constant function 0 that is a one-element subalgebra, and whose operations have range $\{0,1\}$, is called a $\{0,1\}$-valued unary algebra with 0. Such an algebra has a finite basis for its quasi-equations if and only if the relation defined by the rows of the non-trivial functions in the clone form an order ideal.

1. Introduction

An ongoing question is to determine which finite algebras have a finite basis for their equations or quasi-equations. G. Birkhoff's early work [2] shows that a finite unary algebra with finite type has a finite basis for its equations. More recently, V. K. Kartashov considered commutative unary algebras, that is, unary algebras where every pair of basic operations commute. He showed that every variety of commutative unary algebras of finite type has a finite basis of equations [9]. Note that Kartashov's result includes non-finitely-generated varieties.

Mal'cev [10] proved that every variety of mono-unary algebras (unars) has a one-element basis of equations. Kartashov has also shown (see [8]) that every finite mono-unary algebra has a finite basis of quasi-equations.

Non-existence of finite basis results include that of I. P. Bestsennyi, who showed in [1] that a three-element unary algebra of finite type does not have a finite basis for its quasi-equations if and only if it has as a term reduct one of three bad algebras. Since then, Hyndman [6] showed that any finite unary algebra of finite type with a pp-acyclic relation does not have a finite basis for its quasi-equations. The connection between these results is that the three bad algebras of Bestsennyi all have a pp-acyclic relation.

Continuing with the flavour of non-existence of a finite basis, Casperson and Hyndman in [5] show that if the graph of a group operation can be defined using positive primitive formulas, then a finite unary algebra of finite type does not have a finite basis for its quasi-equations.

When working with a finite unary algebra, the clone of non-trivial operations can be presented as a table of elements. Properties of the rows of this table can be used to determine if the algebra has a finite basis for its quasi-equations. For particular finite unary algebras that we call $\{0,1\}$-valued unary algebra with 0, and define in the next section, we show that the rows form an order ideal if and only if the algebra has a finite basis for its quasi-equations.

2. $\{0,1\}$-Valued Unary Algebras with 0: Definitions and Main Result

Consider a finite unary algebra of finite type, $\mathbf{M}$, with constant 0 that is a one-element subalgebra and such that the range of all basic operations is included in

$\{0,1\}$. Assume that the clone of functions of $\mathbf{M}$ is $\{f_0, f_1, \ldots, f_s, f_{s+1}\}$ where f_0 is the constant 0 function and f_{s+1} is the identity function. We call such an algebra a **$\{0,1\}$-valued unary algebra with** 0. These algebras are by definition finite and of finite type, and hence generate locally finite varieties. See Figure 1.

	f_0	f_1	$\ldots$	f_s	f_{s+1}
0	0	0	$\ldots$	0	0
1	0	0 or 1	$\ldots$	0 or 1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	0	0 or 1	$\ldots$	0 or 1	m
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

FIGURE 1. A generic $\{0,1\}$-valued unary algebra with 0.

Let $\mathbf{M}$ be a $\{0,1\}$-valued unary algebra with 0. For $c \in M$ define $\mathbf{row}(c) \in \{0,1\}^s$ to be the tuple $\langle f_1(c), \ldots, f_s(c)\rangle$. Let $\mathbf{Rows}(\mathbf{M})$ be the s-ary relation

$$\mathbf{Rows}(\mathbf{M}) = \{\mathbf{row}(c) \mid c \in M\}.$$

This relation is referred to as the **rows of $\mathbf{M}$**. For $\zeta \in \mathbf{Rows}(\mathbf{M})$ and for c in M, when $\zeta = \mathbf{row}(c)$ we say that ζ is **witnessed** by c. Note that there may be multiple witnesses for ζ.

A partial order on $\mathbf{Rows}(\mathbf{M})$ is induced by the order on $\{0,1\}$ with $0 < 1$. $\mathbf{Rows}(\mathbf{M})$ is an **order ideal** if for every ζ in $\mathbf{Rows}(\mathbf{M})$ and every $\sigma \in \{0,1\}^s$ with $\sigma \leq \zeta$, the row σ is also in $\mathbf{Rows}(\mathbf{M})$. This order is important, and indeed the main result of this article is the following.

Theorem 30. *Let $\mathbf{M}$ be a finite $\{0,1\}$-valued unary algebra with 0. Then $\mathbf{M}$ has a finite basis for its quasi-equations if and only if the rows of $\mathbf{M}$ form an order ideal.*

3. Quasicritical Algebras

One direction of the proof of Theorem 29 consists of showing that algebras whose rows form an order ideal have finite bases for their quasi-equations. To show that a finite algebra has a finite basis for its quasi-equations we use the concept of quasicritical algebras. Let $Q(\mathbf{M}) = \mathbb{ISP}(\mathbf{M})$ be the quasivariety generated by $\mathbf{M}$, and let $Q_n(\mathbf{M})$ be the quasivariety of all algebras in $\mathbb{HSP}(\mathbf{M})$ that satisfy the at most n-variable quasi-equations of $\mathbf{M}$. For $\mathbf{M}$ finite, of finite type, and in a locally finite variety, for a given N there are only finitely many quasi-equations with at most N variables. To show that such an $\mathbf{M}$ has a finite basis for its quasi-equations, it suffices to show that there exists an N such that for all algebras $\mathbf{E}$ we have $\mathbf{E} \in Q_N(\mathbf{M})$ implies $\mathbf{E} \in \mathbb{ISP}(\mathbf{M})$. In fact, we can restrict our attention to finite algebras $\mathbf{E}$ that are quasicritical. We first define quasicriticality and explore the concept generally, and then show how it applies to $\{0,1\}$-valued unary algebras with 0.

A finite algebra $\mathbf{E}$ is **quasicritical** if it is not isomorphic to any subdirect product of its proper subalgebras. V. A. Gorbunov shows in [3, 4] that the lattice of subquasivarieties of a locally finite quasivariety $\mathcal{V}$ is finite if and only if the number of quasicritical algebras in $\mathcal{V}$ is finite. In fact, given two quasivarieties $\mathcal{U}$ and $\mathcal{V}$ of the same type, if $\mathcal{U}$ is a proper sub-quasivariety of $\mathcal{V}$, then there is a quasicritical algebra in $\mathcal{V}$ that is not in $\mathcal{U}$. This allows a proof technique to show that two quasivarieties are equal by showing that all quasicritical algebras in one are already in the other, and vice versa.

Thus counting (as in [1]) or classifying the quasicritical algebras can assist with determining the existence of a finite basis. Here we develop techniques that give explicit embeddings of appropriate quasicritical algebras into powers of $\mathbf{M}$. The next lemma indicates that, for locally finite varieties, looking at the finite algebras that are quasicritical is sufficient. This lemma underpins the proof of Theorem 6.

Lemma 1. *Let $\mathbf{M}$ be a finite algebra in a locally finite variety $\mathcal{K}$ of finite type. If every finite algebra in $\mathcal{K}$ that is quasicritical and satisfies the n-variable quasi-equations of $\mathbf{M}$ is in the quasivariety generated by $\mathbf{M}$, then the n-variable quasi-equations of $\mathbf{M}$ form a basis of the quasi-equations of $\mathbf{M}$ relative to $\mathcal{K}$.*

Proof. We prove the contrapositive. Assume that the n-variable quasi-equations do not form a basis of the quasi-equations of $\mathbf{M}$ relative to $\mathcal{K}$. Set $Q(\mathbf{M}) = \mathbb{ISP}(\mathbf{M})$. Let $Q_n(\mathbf{M})$ be the subquasivariety of $\mathcal{K}$ defined by the n-variable quasi-equations of $\mathbf{M}$. We have

$$\mathcal{K} \supseteq Q_1(\mathbf{M}) \supseteq Q_2(\mathbf{M}) \ldots \supseteq Q_n(\mathbf{M}) \ldots \supseteq Q(\mathbf{M})$$

and $Q(\mathbf{M}) = \bigcap_{n\geq 1} Q_n(\mathbf{M})$.

There is an algebra $\mathbf{E}_1$ in $Q_n(\mathbf{M})$ that is not in $Q(\mathbf{M})$. This algebra fails a quasi-equation Υ of $\mathbf{M}$ in s variables for some $s > n$. There are elements $a_1, a_2, \ldots, a_s$ of $\mathbf{E}_1$ which invalidate Υ. Set $\mathbf{E}_2$ to be the subalgebra of $\mathbf{E}_1$ generated by $a_1, a_2, \ldots a_s$. This algebra also fails Υ but satisfies the n-variable quasi-equations. Either $\mathbf{E}_2$ is quasicritical or $\mathbf{E}_2$ can be written as a subdirect product of quasicritical subalgebras. Each of these quasicritical algebras satisfy the n-variable quasi-equations. At least one must fail Υ. Thus there is a quasicritical algebra that satisfies the n-variable quasi-equations of $\mathbf{M}$ but not all quasi-equations of $\mathbf{M}$. □

Note that for a finite unary algebra $\mathbf{M}$ of finite type, the variety generated by $\mathbf{M}$ is finitely based by Birkhoff's result. Thus, when Lemma 1 can be applied, we produce a finite basis for the quasi-equations.

3.1. **Quasicritical Unary Algebras.** For finite unary algebras with a one-element subalgebra $\{0\}$, there are restrictions on what the structure of a quasicritical algebra can be.

Lemma 2. *Assume $\mathbf{E}$ is a unary algebra that has a one-element subalgebra $\{0\}$. If $E = A \cup B$ where $A \cap B = \{0\}$ and $\mathbf{A}$ and $\mathbf{B}$ are proper subalgebras of $\mathbf{E}$, then $\mathbf{E}$ is not quasicritical.*

Proof. Let $\alpha\colon \mathbf{E} \to \mathbf{A} \times \mathbf{B}$ be given by

$$\alpha(x) = \begin{cases} (x, 0) & \text{if } x \in A, \\ (0, x) & \text{if } x \in B. \end{cases}$$

Then α is a subdirect embedding. □

An **irredundant generating set** of an algebra $\mathbf{E}$ is a subset D such that D generates $\mathbf{E}$ but no proper subset of D does. For A any subset of a unary algebra

$\mathbf{E}$, the subalgebra generated by A is $\mathbf{Sg}^{\mathbf{E}}(A) = \bigcup_{a\in A} \mathbf{Sg}^{\mathbf{E}}(\{a\})$. This implies that when D is a generating set of $\mathbf{E}$ for $d \in D$ we have

$$\mathbf{E} = \mathbf{Sg}^{\mathbf{E}}(D) = \mathbf{Sg}^{\mathbf{E}}(D \setminus \{d\}) \cup \mathbf{Sg}^{\mathbf{E}}(\{d\}).$$

Thus, when D is an irredundant generating set and $d \neq f(d)$ for all non-identity functions f in the clone, the set $E \setminus \{d\}$ is a subalgebra of $\mathbf{E}$.

Lemma 3. *Assume* $\mathbf{E}$ *is a unary algebra that has an irredundant generating set that contains distinct* a*,* b*, and* c*. If* $f(a) = f(b) = f(c)$ *for all terms* f *that are not the identity map, then* $\mathbf{E}$ *is not quasicritical.*

Proof. Let D be the irredundant generating set containing a, b, and c. Let $A = E \setminus \{c\}$. If $c = f(c)$ for some non-identity term f then $c = f(a)$ which contradicts the minimality of the generating set. Thus $\mathbf{A}$ is a proper subalgebra of $\mathbf{E}$. Embed $\mathbf{E}$ into $\mathbf{A} \times \mathbf{A}$ via

$$\alpha(x) = \begin{cases} (x, x) & \text{if } x \in A, \\ (a, b) & \text{if } x = c. \end{cases}$$

As $f(\alpha(c)) = f((a, b)) = (f(a), f(b)) = (f(c), f(c)) = \alpha(f(c))$ for any non-identity term f, the map α is a subdirect embedding. □

3.2. **Quasicritical: $\{0, 1\}$-Valued Unary Algebras with 0.** We now turn our attention to the nature of quasicritical algebras in the variety generated by $\mathbf{M}$, a $\{0, 1\}$-valued unary algebra with 0.

The following observations are used frequently. Because 0 forms a one-element subalgebra, the constant 1 is not a function in the clone, while the constant 0 function is f_0. Thus the range of each f_i is $\{0, 1\}$ for $1 \leq i \leq s$. For $i \leq s$ and $j \leq s$, we have on $\mathbf{M}$ that

$$f_i \circ f_j = \begin{cases} f_0 & \text{if } f_i(1) = 0, \\ f_j & \text{if } f_i(1) = 1. \end{cases} \tag{1}$$

This holds as 0 forms a one-element subalgebra, so that $f_i(0) = 0$. Equation (1) implies that the non-trivial functions in the clone must correspond to basic operations of the algebra.

As we continually are concerned with whether $f_i(1) = 1$ or not, we partition $\{0, 1, 2 \ldots, s\}$ as $I_0 \cup I_1$ such that for a in $\{0, 1\}$ and i in I_a we have $f_i(1) = a$. Thus for $i_0 \in I_0$ and $i_1 \in I_1$ the following equations hold in $\mathbf{M}$ for $0 \leq j \leq s$:

$$f_{i_0}(f_j(x)) \approx f_0(x) \approx 0 \quad \text{and} \quad f_{i_1}(f_j(x)) \approx f_j(x). \tag{2}$$

Note that

$$\mathbf{row}(1)(i) = \begin{cases} 1 & \text{if } i \in I_1, \\ 0 & \text{if } i \in I_0. \end{cases} \tag{3}$$

Lemma 4. *Let* $\mathbf{M}$ *be a* $\{0, 1\}$*-valued unary algebra with* 0*. Let* $\mathbf{E}$ *be a finite algebra in the variety generated by* $\mathbf{M}$*. Assume* $\mathbf{E}$ *is quasicritical and has at least three elements. Let* D *be an irredundant set of generators of* $\mathbf{E}$ *and set* $C = E \setminus (D \cup \{0^{\mathbf{E}}\})$*. For every* $c \in C$ *and every* $d \in D$*, we have*

- $\{f_j(d) \mid j \leq s\} \subseteq C \cup \{0^{\mathbf{E}}\}$ *and*
- $f_i(c) = \begin{cases} c & \textit{if } i \in I_1, \\ 0^{\mathbf{E}} & \textit{if } i \in I_0. \end{cases}$

Moreover, for each $d \in D$ there exists an $i \leq s$ with $f_i(d) \in C$, and consequently C is non-empty.

Proof. Suppose that for some $d \in D$ and some $j \leq s$, we have $d = f_j(d)$. Then for all $i \leq s$ we have $f_i(d) = f_i(f_j(d)) \in \{0^{\mathbf{E}}, f_j(d)\} = \{0^{\mathbf{E}}, d\}$ by Equation (2). Thus $\{0^{\mathbf{E}}, d\}$ is a subalgebra of $\mathbf{E}$, and, as $\mathbf{E}$ has at least three elements, $\{0^{\mathbf{E}}, d\}$ is a proper subalgebra. As D is irredundant $d \notin \mathbf{Sg}^{\mathbf{E}}(D \setminus \{d\})$. This fact and Lemma 2 imply that $\mathbf{E}$ is not quasicritical, which is a contradiction. Thus we have $d \neq f_j(d)$ for every $d \in D$ and every $j \leq s$. Together with irredundancy of D this implies $D \cap f_j(D) = \emptyset$ for $j \leq s$. Hence $f_j(D) \subseteq C \cup \{0^{\mathbf{E}}\}$ for each $j \leq s$.

However, if for some $d \in D$ we have $f_i(d) = 0^{\mathbf{E}}$ for every $i \leq s$ then $\mathbf{Sg}^{\mathbf{E}}(d) = \{0^{\mathbf{E}}, d\}$ and, by Lemma 2, $\mathbf{E}$ is not quasicritical. Thus we may assume for each $d \in D$ there is an $i \leq s$ with $f_i(d) \in C$.

For $c \in C$ there exist $j \leq s$ and $d \in D$ with $c = f_j(d)$. For $i \leq s$ we have $f_i(c) = f_i(f_j(d)) \in \{0^{\mathbf{E}}, f_j(d)\} = \{0^{\mathbf{E}}, c\}$ with $f_i(c) = c$ when $i \in I_1$. □

Consider the quasivariety generated by a $\{0,1\}$-valued unary algebra with 0 when there is at most one non-trivial function in the clone. The next lemma shows that such a quasivariety has a finite quasi-equational basis. Thus, after this lemma we freely assume that there are at least two non-trivial functions.

Lemma 5. *Let $\mathbf{M}$ be a $\{0,1\}$-valued unary algebra with 0 with at most one non-zero, non-identity function in its clone. Then $\mathbb{HSP}(\mathbf{M})$ has finitely many quasicritical algebras, and therefore $\mathbb{ISP}(\mathbf{M})$ has a finite basis for its quasi-equations.*

Proof. If $\mathbf{M}$ has no non-trivial functions in its clone, then the only quasicritical algebra in $\mathbb{HSP}(\mathbf{M})$ is the two-element algebra. Thus we may assume that there is a non-trivial function.

Let f be the non-zero, non-identity function in the clone of $\mathbf{M}$. Either $f^2 = f$ or $f^2 = f_0$, the constant 0-valued function. Assume $\mathbf{E}$ is a finite algebra in the variety generated by $\mathbf{M}$ and that $\mathbf{E}$ is quasicritical with at least three elements. Fix D an irredundant set of generators of $\mathbf{E}$, and set $C = E \setminus (D \cup \{0^{\mathbf{E}}\})$. By Lemma 4, C is non-empty; and for each d in D there is some non-trivial function that maps d into C. As the only such function is f, we have that $f(d)$ is in C. In addition, for each $c \in C$ there exists $d \in D$ with $f(d) = c$. Suppose there are at least two elements in C. Pick one, say c, and let $D_c = \{d \in D \colon f(d) = c\}$. Then $D_c \cup \{0^{\mathbf{E}}, c\}$ is a subalgebra of $\mathbf{E}$ as is $\mathbf{E} \setminus (D_c \cup \{c\})$. Their intersection is $\{0^{\mathbf{E}}\}$, so $\mathbf{E}$ is not quasicritical by Lemma 2. Thus C is a singleton. By Lemma 3, D has at most two elements, and hence $\mathbf{E}$ has at most four elements.

Thus every quasicritical algebras in the variety generated by $\mathbf{M}$ satisfies the four-variable quasi-equations of $\mathbf{M}$. By Lemma 1, $\mathbf{M}$ has a finite basis for its quasi-equations. □

4. When the Rows Form an Order Ideal

Consider $\mathbf{M}_1$ whose operations are those shown in Figure 2. This $\{0,1\}$-valued unary algebra with 0 has rows which form an order ideal. The next theorem applies to algebras like $\mathbf{M}_1$ to show that they have a finite basis for their quasi-equations.

Theorem 6. *Let $\mathbf{M}$ be a finite $\{0,1\}$-valued unary algebra with 0 whose rows form an order ideal. Then $\mathbf{M}$ has a finite basis for its quasi-equations.*

	f_0	f_1	f_2	f_3
0	0	0	0	0
1	0	0	0	1
2	0	1	1	0
3	0	1	0	0
4	0	0	1	0

FIGURE 2. $\mathbf{Rows}(\mathbf{M}_1)$ form an order ideal.

The remainder of this section is the proof of Theorem 6. Recall that the clone of functions of $\mathbf{M}$ is $\{f_0, f_1, \ldots, f_s, f_{s+1}\}$ where f_0 is the constant 0 and f_{s+1} is the identity function. By Lemma 5 we may assume $s \geq 2$. Set $N = 2 + |M|\,(|M| - 1)$. As $\mathbf{M}$ has at least two elements, we have $N \geq 4$. We wish to show that the N-variable quasi-equations form a basis. Let $\mathbf{E}$ be an arbitrary finite algebra that satisfies the N-variable quasi-equations of $\mathbf{M}$. By Lemma 1 it suffices to show that if $\mathbf{E}$ is quasicritical then $\mathbf{E}$ is in $\mathbb{ISP}(\mathbf{M})$. If $\mathbf{E}$ has at most N elements and satisfies the N-variable quasi-equations, then $\mathbf{E}$ satisfies all quasi-equations of $\mathbf{M}$ and hence is in $\mathbb{ISP}(\mathbf{M})$. Thus we also assume that $\mathbf{E}$ has at least $N + 1 \geq 5$ elements.

For the next subsection we do not need the assumption of quasicriticality of $\mathbf{E}$.

4.1. **The Quasi-equational Order Ideal Property.** In this subsection we develop the concept of the quasi-equational order ideal property and show that it is equivalent to $\mathbf{Rows}(\mathbf{M})$ being an order ideal.

For i_0 and T chosen such that $i_0 \in T \subseteq \{1, 2, \ldots, s\}$, let $\Sigma^T_{i_0}$ be the one-variable quasi-equation

$$\Big[\mathop{\&}_{i,j \in T} f_i(x) \approx f_j(x)\Big] \to f_{i_0}(x) \approx 0,$$

and let $\Gamma^T_{i_0}$ be the associated (at most) s-variable quasi-equation

$$\Big[\mathop{\&}_{i,j \in T} z_i \approx z_j\Big] \to z_{i_0} \approx 0.$$

Notice that for $a \in M$ we have that $\mathbf{row}(a) = \langle f_1(a), f_2(a), \ldots, f_s(a)\rangle$ satisfies $\Gamma^T_{i_0}$ if and only if a satisfies $\Sigma^T_{i_0}$. For $i_0 \in T$ the operation f_{i_0} is not the constant 0 map, so if $\mathbf{M}$ satisfies some $\Sigma^T_{i_0}$ then $|T| \geq 2$.

We say that $\mathbf{M}$ satisfies the **quasi-equational order ideal property** when $\mathbf{M}$ is a $\{0, 1\}$-valued unary algebra and for every tuple σ in $\{0, 1\}^s \setminus \mathbf{Rows}(\mathbf{M})$, there exist $T \subseteq \{1, 2, \ldots, s\}$ and $i_0 \in T$ such that $\mathbf{M}$ satisfies $\Sigma^T_{i_0}$ but σ does not satisfy $\Gamma^T_{i_0}$. Within algebras that satisfy this property we look for failure of $\Gamma^T_{i_0}$ rather than prove directly that $\Sigma^T_{i_0}$ holds. In Lemma 9 we will see that the quasi-equational order ideal property is equivalent to $\mathbf{Rows}(\mathbf{M})$ being an order ideal.

The next lemma and corollary guarantee the existence of particular rows in $\mathbf{Rows}(\mathbf{M})$ when $\mathbf{M}$ satisfies the quasi-equational order ideal property and allows us to prove one half of Lemma 9. It is the existence of witnesses to these rows that allow us to prove Theorem 6.

Lemma 7. *Suppose that* $\mathbf{M}$ *satisfies the quasi-equational order ideal property. Let* $\mathbf{E}$ *be an algebra in* $\mathbb{HSP}(\mathbf{M})$ *satisfying the one-variable quasi-equations of* $\mathbf{M}$*. For* *all* $\zeta \in \{0,1\}^s$ *if there exists* $e \in \mathbf{E}$*, and* $c \in \mathbf{E}$ *with* $c \neq 0^{\mathbf{E}}$*, such that*

$$\zeta(i) = 1 \text{ implies } f_i(e) = c,$$

then we have $\zeta \in \mathbf{Rows}(\mathbf{M})$*.*

Proof. Let $\zeta \in \{0,1\}^s$ and assume that c, $e \in \mathbf{E}$ have the properties that $c \neq 0^{\mathbf{E}}$ and that $\zeta(i) = 1$ implies $f_i(e) = c$. If for every $\Sigma_{i_0}^T$ that holds in $\mathbf{M}$ we obtain $\Gamma_{i_0}^T$ holding for ζ, then we must have ζ in $\mathbf{Rows}(\mathbf{M})$, as otherwise we would have a contradiction to the quasi-equational order ideal property.

Suppose that $\Sigma_{i_0}^T$ holds in $\mathbf{M}$ and hence in $\mathbf{E}$. To show that $\Gamma_{i_0}^T$ holds for ζ, we show that there is an element j_0 in T with $\zeta(j_0) = 0$. When this happens, either $\zeta(i) = 0$ for all i in T so that $\Gamma_{i_0}^T$ holds; or $\zeta(j_1) = 1$ for some j_1 in T in which case the hypothesis of $\Gamma_{i_0}^T$ fails so that $\Gamma_{i_0}^T$ holds.

Consider what happens when we set $x = e$ in $\Sigma_{i_0}^T$. Either the hypothesis of $\Sigma_{i_0}^T$ holds, in which case the result holds, to wit, $\{f_i(e) \mid i \in T\} = \{0^{\mathbf{E}}\}$; or the hypothesis of $\Sigma_{i_0}^T$ fails and $\{f_i(e) \mid i \in T\}$ is not a singleton. In the first case $f_{i_0}(e) \neq c$. Since $\zeta(i) = 1$ implies $f_i(e) = c$ it follows that $\zeta(i_0) = 0$. In the latter case, not all $f_i(e) = c$. Pick j_0 with $f_{j_0}(e) \neq c$ and it follows that $\zeta(j_0) = 0$. By the previous paragraph $\Gamma_{i_0}^T$ is satisfied by ζ. By the first paragraph, ζ is in $\mathbf{Rows}(\mathbf{M})$. □

Corollary 8. *For* $\mathbf{M}$ *and* $\mathbf{E}$ *satisfying the assumptions of Lemma 7, and for* $c, e \in \mathbf{E}$*, with* $c \neq 0^{\mathbf{E}}$*, the tuple* ζ_c^e *defined by*

$$\zeta_c^e(i) = \begin{cases} 1 & \text{if } f_i(e) = c, \\ 0 & \text{otherwise} \end{cases}$$

is in $\mathbf{Rows}(\mathbf{M})$*.*

Lemma 9. *For* $\mathbf{M}$ *a* $\{0,1\}$*-valued unary algebra with* 0*, we have that* $\mathbf{Rows}(\mathbf{M})$ *is an order ideal if and only if* $\mathbf{M}$ *satisfies the quasi-equational order ideal property.*

Proof. Assume that $\mathbf{Rows}(\mathbf{M})$ is an order ideal. Suppose that $\sigma \in \{0,1\}^s$ but σ is not in $\mathbf{Rows}(\mathbf{M})$. Let $T = \{i \colon \sigma(i) = 1\}$. As $\sigma \neq \mathbf{row}(0)$ the set T is non-empty. For any i_0 in T the quasi-equation $\Sigma_{i_0}^T$, which is

$$\Big[\mathop{\&}_{i,j \in T} f_i(w) \approx f_j(w) \Big] \rightarrow f_{i_0}(w) \approx 0,$$

is satisfied by $\mathbf{M}$. To see this, note that the failure of this quasi-equation to hold in $\mathbf{M}$ implies the existence of an element m in M with $f_i(m) = 1$ for $i \in T$. Since $\sigma \leq \mathbf{row}(m)$ for any such m and $\mathbf{Rows}(\mathbf{M})$ forms an order ideal, we have σ in $\mathbf{Rows}(\mathbf{M})$, a contradiction. However, the corresponding $\Gamma_{i_0}^T$ fails for any $\tau \geq \sigma$, in particular for σ itself. Thus $\mathbf{M}$ satisfies the quasi-equational order ideal property.

Now assume that $\mathbf{M}$ satisfies the quasi-equational order ideal property. Suppose that $m \in \mathbf{M}$, and $\zeta \in \{0,1\}^s$ is such that $\zeta \leq \mathbf{row}(m)$. Set $\mathbf{E} = \mathbf{M}$, $e = m$, and $c = 1$, so that $\zeta(i) = 1$ implies $f_i(m) = 1$, that is, $f_i(e) = c$. By Lemma 7, we have that ζ is in $\mathbf{Rows}(\mathbf{M})$. Thus $\mathbf{Rows}(\mathbf{M})$ is downward closed, that is, an order ideal. □

In view of Lemma 9, the assumption of Theorem 6 that $\mathbf{Rows}(\mathbf{M})$ forms an order ideal allows us to use Lemma 7 freely.

4.2. **Homomorphisms.** To show that an algebra $\mathbf{E}$ is in $\mathbb{ISP}(\mathbf{M})$, it is sufficient to show that for any pair of distinct elements in $\mathbf{E}$ there is a homomorphism separating them, that is show that for every pair $(e_1, e_2) \in E^2 \setminus \Delta_E$, there is a homomorphism $h\colon \mathbf{E} \to \mathbf{M}$ such that $h(e_1) \neq h(e_2)$. That $\mathbf{E} \in \mathbb{ISP}(\mathbf{M})$ follows, as then the map $\mathbf{E} \to \mathbf{M}^{\mathrm{Hom}(\mathbf{E},\mathbf{M})}$ given by $e \mapsto \langle h(e) : h \in \mathrm{Hom}(\mathbf{E}, \mathbf{M}) \rangle$ is an embedding.

Under the hypotheses of Theorem 6 and the additional assumption that $\mathbf{E}$ is quasicritical, it is sufficient to consider three types of homomorphisms. These homomorphisms are indexed by the sets C, J_1, and J_2 which we now define. We start by partitioning the elements of $\mathbf{E}$. Let D be an irredundant set of generators of $\mathbf{E}$. Set $C = E \setminus (D \cup \{0^{\mathbf{E}}\})$. By Lemma 4, C is non-empty and for every $c \in C$ and every $d \in D$ we have

- $\{f_j(d) \mid j \leq s\} \subseteq C \cup \{0^{\mathbf{E}}\}$,
- $\{f_i(c) \mid i \leq s\} \subseteq \{0^{\mathbf{E}}, c\}$,

and for every $d \in D$ there is some $j \leq s$ with $f_j(d) \in C$. To simplify notation in various situations we let $\overline{f}(x)$ denote the tuple $\langle f_1(x), \ldots, f_s(x) \rangle$.

In what follows, let

$$J_1 = \{(a, b) \in D^2 \colon a \neq b \text{ and } \overline{f}(a) = \overline{f}(b)\}$$

and

$$J_2 = \{(c, d) \in C \times D \colon \overline{f}(c) = \overline{f}(d)\}.$$

The overview for the remainder of this section is as follows. Lemma 10 shows that $J_1 \cup J_2$ has at most one element. Corollary 12 shows that for $c \in C$, we can construct a homomorphism $h_c\colon \mathbf{E} \to \mathbf{M}$ such that $h_c(c) \neq h_c(c')$ for $c' \in E \setminus D$. For $(a, b) \in J_1$, Lemma 15 provides a homomorphism $h\colon \mathbf{E} \to \mathbf{M}$ such that $h(a) \neq h(b)$. For $(c, d) \in J_2$, Lemma 20 provides a homomorphism $h\colon \mathbf{E} \to \mathbf{M}$ such that $h(c) \neq h(d)$. Lemma 21 assembles this information to show that there is a homomorphism separating any pair of distinct elements of $\mathbf{E}$, thereby completing the proof of Theorem 6.

Lemma 11 and Corollary 12 construct homomorphisms from $\mathbf{E}$ to $\mathbf{M}$ that separate elements of C. Note that Lemma 11 does *not* assume that $\mathbf{M}$ has the quasi-equational order ideal property, but Corollary 12 does.

The next lemma demonstrates that, in a quasicritical algebra, there are very few pairs of elements (u, v) with $\overline{f}(u) = \overline{f}(v)$. We would like to thank the anonymous referee for this observation.

Lemma 10. *The set $J_1 \cup J_2$ has at most one element.*

Proof. Suppose that (a, b) and (u, v) are in $J_1 \cup J_2$. Thus b and v are in D. The sets $A = E \setminus \{b\}$ and $B = E \setminus \{v\}$ are proper subuniverses of E as D is a minimal generating set. Embed $\mathbf{E}$ into $\mathbf{A} \times \mathbf{B}$ via

$$h(x) = \begin{cases} (x, x) & \text{if } x \in A \cap B, \\ (a, b) & \text{if } x = b, \\ (v, u) & \text{if } x = v. \end{cases}$$

As $f_i(h(b)) = f_i((a, b)) = (f_i(a), f_i(b)) = (f_i(b), f_i(b)) = h(f_i(b))$ and $f_i(h(v)) = f_i((v, u)) = (f_i(v), f_i(u)) = (f_i(v), f_i(v)) = h(f_i(v))$ for $0 \leq i \leq s$, the map h is a subdirect embedding, contradicting the quasicriticality of $\mathbf{E}$. $\square$

Lemma 11. *Let* $\mathbf{M}$ *be a* $\{0,1\}$*-valued unary algebra with* 0*, and let* $\mathbf{E} \in \mathbb{HSP}(\mathbf{M})$ *be quasicritical, have at least three elements, and satisfy the one-variable quasi-equations of* $\mathbf{M}$*. Let* D *be an irredundant set of generators of* $\mathbf{E}$*. Set* $C = E \setminus (D \cup \{0^{\mathbf{E}}\})$*, and fix an element* $c \in C$*.*

For each $d \in D$*, let* $\rho_d \in \{0,1\}^s$ *be defined by*

$$\rho_d(j) = 1 \quad \text{if and only if} \quad f_j(d) = c.$$

Suppose that $h\colon D \to M$ *is a map such that* $h(d)$ *witnesses* ρ_d *for each* $d \in D$*, that is,* $\mathbf{row}(h(d)) = \rho_d$*. Then*

(1) h *extends uniquely to a homomorphism* $\hat{h}\colon \mathbf{E} \to \mathbf{M}$*;*
(2) $\hat{h}(c) = 1$*; and*
(3) $\hat{h}(c') = 0$ *for* $c' \notin D \cup \{c\}$*.*

Proof. As D is a generating set, any homomorphism that extends h is unique. To show that such a homomorphism exists, set

$$\hat{h}(e) = \begin{cases} h(e) & \text{when } e \in D; \\ 1 & \text{when } e = c; \\ 0 & \text{otherwise.} \end{cases}$$

We now show for $0 \le j \le s+1$ and $d \in D$ that

$$\hat{h}(f_j(d)) = f_j(h(d)). \tag{4}$$

Consider any d in D. For $j = 0$ we have $\hat{h}(f_0(d)) = \hat{h}(0^{\mathbf{E}}) = 0 = f_0(h(d))$. For $j = s+1$ we have $\hat{h}(f_{s+1}(d)) = \hat{h}(d) = h(d) = f_{s+1}(h(d))$.

We now consider the cases where $1 \le j \le s$. Since $\mathbf{row}(h(d)) = \rho_d$, we have $f_i(h(d)) = \rho_d(i)$ for $1 \le i \le s$. First, suppose that $f_j(d) = c$. Then $\hat{h}(f_j(d)) = \hat{h}(c) = 1$, and $f_j(h(d)) = \rho_d(j)$. But $\rho_d(j) = 1$ exactly when $f_j(d) = c$. Now suppose $f_j(d) \ne c$, so that $f_j(d) \notin \{c\} \cup D$ and $\hat{h}(f_j(d)) = 0$. On the other side we have $f_j(h(d)) = \rho_d(j) = 0$. The last equality holds because $f_j(d) \ne c$. Thus (4) holds and we use it to show that $\hat{h}$ is a homomorphism.

We wish to compute $\hat{h}(f_i(e))$ for any $e \in E$ and i with $0 \le i \le s+1$. Pick any $d \in D$, and any j with $0 \le j \le s+1$ such that $e = f_j(d)$. There is an r with $0 \le r \le s+1$ such that $f_i \circ f_j = f_r$. Now we compute $\hat{h}(f_i(e)) = \hat{h}(f_i(f_j(d))) = \hat{h}(f_r(d)) = f_r(h(d)) = f_i(f_j(h(d))) = f_i(\hat{h}(f_j(d))) = f_i(\hat{h}(e))$, showing that $\hat{h}$ is a homomorphism. □

Corollary 12. *Let* $\mathbf{M}$ *be a* $\{0,1\}$*-valued unary algebra with* 0 *satisfying the quasi-equational order ideal property, and let* $\mathbf{E} \in \mathbb{HSP}(\mathbf{M})$ *be quasicritical, have at least three elements, and satisfy the one-variable quasi-equations of* $\mathbf{M}$*. Let* D *be an irredundant set of generators of* $\mathbf{E}$*. Set* $C = E \setminus (D \cup \{0^{\mathbf{E}}\})$*.*

Then, for $c \in C$*, there exists a homomorphism* $h_c\colon \mathbf{E} \to \mathbf{M}$ *such that for* $c' \in E \setminus D$

$$h_c(c) = h_c(c') \quad \text{if and only if} \quad c = c'.$$

In particular, $h_c(c) \ne 0$*.*

Proof. Fix $c \in C$. For $d \in D$, consider $\rho_d \in \{0,1\}^s$ defined by

$$\rho_d(j) = 1 \quad \text{if and only if} \quad f_j(d) = c.$$

By Corollary 8, ρ_d is in $\mathbf{Rows}(\mathbf{M})$. Define $h\colon D \to \mathbf{M}$ by setting $h(d)$ to be any witnesses that ρ_d is in $\mathbf{Rows}(\mathbf{M})$. Then by Lemma 11 there is a homomorphism

$h_c : \mathbf{E} \to \mathbf{M}$ extending h such that $h_c(c) = 1$ and for $c' \neq c$, with $c' \notin D$, we have $h(c') = 0$. □

Under certain circumstances Corollary 12 suffices to prove Theorem 6. If $\mathbf{M}$ satisfies the two-variable quasi-equation $\overline{f}(x) \approx \overline{f}(y) \to x \approx y$, then so does $\mathbf{E}$. The quasi-equation $\overline{f}(x) \approx \overline{f}(y) \to x \approx y$ holding in M says that $\mathbf{row}(c_1) = \mathbf{row}(c_2)$ implies that $c_1 = c_2$. In this case we say that the relation $\mathbf{Rows}(\mathbf{M})$ is **uniquely witnessed**. When $\mathbf{Rows}(\mathbf{M})$ is uniquely witnessed, we have $J_1 \cup J_2 = \emptyset$ and we shall see in Lemma 21 that $\{h_c \mid c \in C\}$ separates points in $\mathbf{E}$. In general we need more homomorphisms.

Recall that $J_1 = \{(a, b) \in D^2 \colon a \neq b \text{ and } \overline{f}(a) = \overline{f}(b)\}$. Lemma 15 states that if (a, b) in J_1, then there is a homomorphism $h \colon \mathbf{E} \to \mathbf{M}$ such that $h(a) \neq h(b)$, that is, there are homomorphisms to separate the elements in a pair in J_1. From now through the proof of Lemma 15 we assume that $(a, b) \in J_1$ and use the following algebras and formulas. Let $\mathbf{E}_0 = \mathbf{Sg}^{\mathbf{E}}(\{a, b\})$, and let $\mathbf{E}_\omega$ be the algebra whose universe is given by $E_\omega = \{e \in E \colon \mathbf{Sg}^{\mathbf{E}}(\{e\}) \cap E_0 = \{0^{\mathbf{E}}\}\}$. By Lemma 4, $C \subseteq E_0 \cup E_\omega$.

We now define various formulas that relate to the existence of appropriate homomorphisms. Let $\psi_0(w_a, w_b)$ be the two-variable formula, and for $e \in E$ let $\psi_e(w_a, w_b, w_e)$ be the three-variable formula, defined by:

$$\psi_0(w_a, w_b): \qquad \overline{f}(w_a) \approx \overline{f}(w_b) \ \ \& \bigotimes_{f_i(a) = f_j(a)} f_i(w_a) \approx f_j(w_a);$$

$$\psi_e(w_a, w_b, w_e): \qquad \psi_0(w_a, w_b) \ \ \& \bigotimes_{f_i(e) = f_j(a)} f_i(w_e) \approx f_j(w_a).$$

The formula ψ_0 identifies elements that behave like a and b, while ψ_e captures how e interacts with a and b. Note that for $(a, b) \in J_1$ and $e \in E$, the formulas $\psi_0(a, b)$ and $\psi_e(a, b, e)$ hold in the algebra $\mathbf{E}$.

Let H_1 be the set of homomorphisms

$$H_1 = \{h \in \mathrm{Hom}(\mathbf{E}_0 \cup \mathbf{E}_\omega, \mathbf{M}) \colon h(a) \neq h(b) \text{ and } h|_{\mathbf{E}_\omega} \equiv 0\}.$$

We now prove two technical lemmas about homomorphisms before proceeding to the proof of Lemma 15.

Lemma 13. *The set H_1 is non-empty. Moreover, for every pair (m_a, m_b) of distinct elements in M^2 such that $\psi_0(m_a, m_b)$ holds in $\mathbf{M}$, there is exactly one $h \in H_1$ such that $h(a) = m_a$ and $h(b) = m_b$.*

Proof. The quasi-equation $\psi_0(w_a, w_b) \to w_a \approx w_b$ fails in $\mathbf{E}_0$ (*i.e,* in $\mathbf{E}$) with $w_a = a$ and $w_b = b$, so it must also fail in $\mathbf{M}$. This means that there exist distinct $m_a, m_b \in M$ with $\psi_0(m_a, m_b)$ holding.

We demonstrate one-to-one correspondences between H_1, $\{h|_{E_0} \mid h \in H_1\}$, and $\{(h(a), h(b)) \mid h \in H_1\}$. Let $H_0 = \{h \in \mathrm{Hom}(\mathbf{E}_0, \mathbf{M}) \colon h(a) \neq h(b)\}$. The map $\mathrm{ext} : H_0 \to H_1$ given by

$$[\mathrm{ext}\, h_0](x) = \begin{cases} h_0(x) & x \in E_0, \\ 0 & x \in E_\omega; \end{cases}$$

is the inverse of the restriction map $|_{E_0} \colon H_1 \to H_0$, so H_0 and H_1 are in one-to-one correspondence. For $h \in H_0$, as $\{a, b\}$ generates $\mathbf{E}_0$ it is clear that $(h(a), h(b))$ completely defines h in H_0 and $\mathrm{ext}\, h$ in H_1.

Pick $m_a \neq m_b$ such that $\psi_0(m_a, m_b)$ holds in $\mathbf{M}$. This allows us to construct a homomorphism $h \in H_0$. To start, set $h(a) = m_a$ and $h(b) = m_b$. As $\mathbf{E}_0$ is generated by $\{a, b\}$ and $\overline{f}(a) = \overline{f}(b)$, for every $e \in E_0 \setminus \{a, b\}$ there is a non-identity term f_i with $i \in \{0, 1, \ldots, s\}$ such that $f_i(a) = f_i(b) = e$. Set $h(e) = f_i(h(a)) = f_i(m_a)$. If $f_i(a) = f_j(a)$ then $\psi_0(m_a, m_b)$ implies that $f_i(m_a) = f_j(m_a)$, so h is a well-defined homomorphism. By the preceding paragraph this homomorphism extends uniquely to H_1. □

The following lemma connects the formula $\psi_e(w_a, w_b, w_e)$ with the ability to lift homomorphisms.

Lemma 14. *Let $\mathbf{E}_2$ be a proper subalgebra of $\mathbf{E}$ that contains $\mathbf{E}_0 \cup \mathbf{E}_\omega$, and let $h : \mathbf{E}_2 \to \mathbf{M}$ be a homomorphism with $h|_{\mathbf{E}_\omega} \equiv 0$. Then, for $e \in E \setminus E_2$ there is a homomorphism $\hat{h} \colon \mathbf{E}_2 \cup \{e\} \to \mathbf{M}$ extending h if and only if there exists an $m \in \mathbf{M}$ with $\psi_e(h(a), h(b), m)$ holding in $\mathbf{M}$.*

Proof. Set $L = \{i \in \{1, \ldots, s\} \colon f_i(e) \notin E_w\}$. Note that, since $f_i(e) \in C \subseteq E_0 \cup E_\omega$, we have $f_i(e) \notin E_\omega$ implies $f_i(e) \in E_0$. Suppose that there exists an $m \in \mathbf{M}$ such that $\psi_e(h(a), h(b), m)$ holds in $\mathbf{M}$, that is,

$$\psi_0(h(a), h(b)) \;\;\&\; \mathop{\&}_{f_i(e) = f_j(a)} f_i(m) = f_j(h(a)).$$

Define $\zeta \in \{0, 1\}^s$ by

$$\zeta(i) = 1 \quad \text{if and only if} \quad f_i(m) = 1 \text{ and } i \in L.$$

Apply Lemma 7 (with $\mathbf{E} = \mathbf{M}$, $e = m$, $c = 1$) to conclude that $\zeta \in \mathbf{Rows}(\mathbf{M})$. Let m' witness ζ. Then m' satisfies

$$f_i(m') = \zeta(i) = \begin{cases} f_i(m) & \text{when } i \in L, \\ 0 & \text{when } i \notin L. \end{cases}$$

Since $C \subseteq E_0 \cup E_\omega$ and for any i we have $f_i(e) \in C \cup \{0^{\mathbf{E}}\}$, it follows that $E_2 \cup \{e\}$ is a subuniverse of $\mathbf{E}$. Extend h to $E_2 \cup \{e\}$ by $\hat{h}(e) = m'$. To see that $\hat{h}$ is a homomorphism consider $\hat{h}(f_i(e))$. For $i \in L$ we have $f_i(e) \notin E_\omega$ so $f_i(e) \in E_0$ and $f_i(e) = f_j(a)$ for some j. Thus $\hat{h}(f_i(e)) = h(f_i(e)) = h(f_j(a)) = f_j(h(a))$. But $f_j(h(a)) = f_i(m)$ because $f_j(a) = f_i(e)$ and $\psi_e(h(a), h(b), m)$ holds. Thus $\hat{h}(f_i(e)) = f_j(h(a)) = f_i(m) = f_i(m') = f_i(\hat{h}(e))$. For $i \notin L$, we have $f_i(e) \in E_\omega$ and $\zeta(i) = 0$; whence $\hat{h}(f_i(e)) = h(f_i(e)) = 0 = \zeta(i) = f_i(m') = f_i(\hat{h}(e))$. Thus $\hat{h}$ is a homomorphism.

On the other hand if $\hat{h}(e) = m$ is a homomorphism extending h, use the fact that $\psi_e(a, b, e)$ holds and apply $\hat{h}$ to the identities in $\psi_e(a, b, e)$ to get that $\psi_e(h(a), h(b), m)$ holds. □

We now have the machinery to prove that we can separate elements of a pair in $J_1 = \{(a, b) \in D^2 \colon a \neq b \text{ and } \overline{f}(a) = \overline{f}(b)\}$.

Lemma 15. *If (a, b) in J_1, then there is a homomorphism $h_{a,b} \colon \mathbf{E} \to \mathbf{M}$ such that $h_{a,b}(a) \neq h_{a,b}(b)$.*

Proof. Fix (a, b) in J_1. With notation as above, define

$$\begin{aligned} Q = \;\; & \{\, (u, v) \in M^2 \setminus \Delta_M : \\ & \qquad \exists h \in H_1 \text{ with } (u, v) = (h(a), h(b)) \text{ and } h \text{ does } \textit{not} \text{ lift to } \mathbf{E}. \,\} \end{aligned}$$

If Q is empty then every $h \in H_1$ lifts to $\mathbf{E}$, and since Lemma 13 says H_1 is non-empty, we are done. Thus we assume Q is non-empty. Since $\psi_0(a, b)$ holds in $\mathbf{E}$, by Lemma 13, for each pair q in Q there is a unique $h_q \colon \mathbf{E}_0 \cup \mathbf{E}_\omega \to \mathbf{M}$ in H_1 such that $(h_q(a), h_q(b)) = q$. As $q \in Q$ there must be at least one element $e_q \in E$ such that some maximal extension of h_q does not contain e_q. Indeed, by Lemma 14, e_q is not in the domain of any extension of h_q. We proceed to demonstrate that there must be an $h_1 \in H_1$ that has an extension whose domain includes each e_q. This will imply that $(h_1(a), h_1(b))$ is not in Q, that is, h_1 extends to $\mathbf{E}$.

Consider now the formula $\psi(w_a, w_b, \{w_{e_q} \colon q \in Q\})$ given by

$$\psi(w_a, w_b, \{w_{e_q} \colon q \in Q\}) \colon \qquad \psi_0(w_a, w_b) \ \& \ \mathop{\&}_{q \in Q} \psi_{e_q}(w_a, w_b, w_{e_q}).$$

The formula ψ essentially describes the relationships in $\mathbf{E}$ between the elements a, b, and $\{e_q \mid q \in Q\}$. Notice that this formula has at most $|Q| + 2 \leq |M|\,(|M| - 1) + 2 = N$ variables. The quasi-equation

$$\psi(w_a, w_b, \{w_{e_q} \colon q \in Q\}) \to w_a \approx w_b$$

fails on $\mathbf{E}$ as $a \neq b$ and we can satisfy the left hand side with $w_a = a$, $w_b = b$, and $w_{e_q} = e_q$. Thus it must also fail on $\mathbf{M}$, as $\mathbf{E}$ satisfies all the N-variable quasi-equations satisfied by $\mathbf{M}$.

That means that there are elements m_q in M for $q \in Q$, and distinct m_a, m_b, such that $\psi(m_a, m_b, \{m_q : q \in Q\})$ holds in $\mathbf{M}$. In particular $\psi_0(m_a, m_b)$ holds in $\mathbf{M}$ and (by Lemma 13) there is a homomorphism $h_1 \in H_1$ such that $h_1(a) = m_a$ and $h_1(b) = m_b$. As each $\psi_{e_q}(m_a, m_b, m_q)$ holds in $\mathbf{M}$, each formula $\exists w\, \psi_{e_q}(m_a, m_b, w)$ holds in $\mathbf{M}$, and by Lemma 14 any maximal extension of h_1 contains $\{e_q \mid q \in Q\}$ in its domain. Thus $h_1 \neq h_q$ for all $q \in Q$, which implies that $(m_a, m_b) = \big(h_1(a), h_1(b)\big) \notin Q$, so h_1 lifts to $\mathbf{E}$. Set $h_{a,b}$ to be an extension of h_1 to $\mathbf{E}$. □

We now turn our attention to J_2. Recall that $J_2 = \{(c, d) \in C \times D \colon \overline{f}(c) = \overline{f}(d)\}$. In addition $I_0 = \{i \in \{0, 1, 2, \ldots, s\} \colon f_i(1) = 0\}$ and $I_1 = \{i \in \{1, 2, \ldots, s\} \colon f_i(1) = 1\}$. For $i_0 \in I_0$ and $i_1 \in I_1$ and any j, we have $f_{i_0} \circ f_j = f_0$ and $f_{i_1} \circ f_j = f_j$.

Lemma 16. *Assume $(c, d) \in J_2$. Then $I_1 \neq \emptyset$ and for $1 \leq i \leq s$,*

$$f_i(d) = \begin{cases} c & \text{if } i \in I_1, \\ 0^{\mathbf{E}} & \text{if } i \in I_0. \end{cases}$$

Proof. By Lemma 4,

$$f_i(c) = \begin{cases} c & \text{if } i \in I_1, \\ 0^{\mathbf{E}} & \text{if } i \in I_0. \end{cases}$$

Since $\overline{f}(d) = \overline{f}(c)$, the same holds for each $f_i(d)$. Again by Lemma 4, there is some i with $f_i(d) \in C$, whence I_1 is non-empty. □

Note that the proofs of the next few lemmas use at most two-variable quasi-equations, whereas the proof of Lemma 15 used an N-variable quasi-equation.

Lemma 17. *If J_2 is non-empty, then either* $\mathbf{row}(0)$ *has two distinct witnesses or* $\mathbf{row}(1)$ *has two distinct witnesses.*

Proof. Assume that $(c, d) \in J_2$ with $\overline{f}(c) = \overline{f}(d)$. By Lemma 16, $I_1 \neq \emptyset$. For each $i_1 \in I_1$ we have $f_{i_1}(d) = f_{i_1}(c) = c$, and for $i_0 \in I_0$ we have $f_{i_0}(d) = f_{i_0}(c) = 0^{\mathbf{E}}$.

Consider the two-variable quasi-equation $\Upsilon(x, y)$

$$\Big[\overline{f}(x) \approx \overline{f}(y) \;\&\; \mathop{\&}_{i_1 \in I_1} f_{i_1}(x) \approx x \;\&\; \mathop{\&}_{i_0 \in I_0} f_{i_0}(x) \approx 0\Big] \to x \approx y.$$

In $\mathbf{E}$ the pair (c, d) satisfies the hypothesis of $\Upsilon(x, y)$ but $c \neq d$. Thus Υ does not hold in $\mathbf{E}$, and consequently it must not hold in $\mathbf{M}$ either. In $\mathbf{M}$, if the hypothesis of $\Upsilon(x, y)$ holds then $x \in \{0, 1\}$, so the failure of $\Upsilon(x, y)$ in $\mathbf{M}$ implies that either $\mathbf{row}(0)$ or $\mathbf{row}(1)$ is not uniquely witnessed. □

Lemma 18. *Suppose that $(c, d) \in J_2$ and that 0 and m are distinct witnesses of $\mathbf{row}(0)$. Then $h\colon \mathbf{E} \to \mathbf{M}$ defined by*

$$h(x) = \begin{cases} m & \text{if } x = d \\ 0 & \text{otherwise} \end{cases}$$

is a homomorphism with $h(c) \neq h(d)$.

Proof. For $1 \leq j \leq s$, we have $f_j(h(d)) = f_j(m) = 0$ as m witnesses $\mathbf{row}(0)$; and $h(f_j(d)) = h(f_j(c)) = 0$ as $f_j(c) \neq d$. For $u \neq d$ we have $f_j(h(u)) = f_j(0) = 0$ and $h(f_j(u)) = 0$. Thus h is a homomorphism with the desired property. □

Lemma 19. *Suppose that $(c, d) \in J_2$ and that 1 and m are distinct witnesses of $\mathbf{row}(1)$. Then there is a homomorphism $h\colon \mathbf{E} \to \mathbf{M}$ defined so that $h(c) = 1$ and $h(d) = m$.*

Proof. For $u \in D$ define $\rho_u \in \{0, 1\}^s$ by

$$\rho_u(j) = 1 \quad \text{if and only if} \quad f_j(u) = c.$$

By Corollary 8, ρ_u is in $\mathbf{Rows}(\mathbf{M})$ so has a witness y_u in M. Define $h'\colon D \to M$ by

$$h'(u) = \begin{cases} m & \text{if } u = d \\ y_u & \text{otherwise.} \end{cases}$$

By Lemma 16, $f_j(d) = c$ if and only if $j \in I_1$. Thus, by Equation (3), $\rho_d = \mathbf{row}(1)$ which is also $\mathbf{row}(m)$. The map h' satisfies the hypothesis of Lemma 11 so extends to a homomorphism $h\colon \mathbf{E} \to \mathbf{M}$ with $h(c) = 1$. □

Lemma 20. *If $(c, d) \in J_2$, then there is a homomorphism $h_{c,d}\colon \mathbf{E} \to \mathbf{M}$ such that $h_{c,d}(c) \neq h_{c,d}(d)$.*

Proof. Assume that $(c, d) \in J_2$. By Lemma 17 either $\mathbf{row}(0)$ has distinct witnesses or $\mathbf{row}(1)$ has distinct witnesses. Use Lemma 18 or Lemma 19, respectively, for the existence of the desired homomorphism. □

Finally we show that $\mathbf{E}$ is actually in $\mathbb{ISP}(\mathbf{M})$ by verifying that for all $(e_1, e_2) \in E^2 \setminus \Delta_E$ there is a homomorphism h separating e_1 and e_2.

Lemma 21. *For all $(e_1, e_2) \in E^2 \setminus \Delta_E$ there is a homomorphism $h\colon \mathbf{E} \to \mathbf{M}$ such that $h(e_1) \neq h(e_2)$. The homomorphism may be chosen from*

$$\{h_c \mid c \in C\} \cup \{h_{a,b} \mid (a, b) \in J_1\} \cup \{h_{c,d} \mid (c, d) \in J_2\}$$

as defined in Corollary 12, Lemma 15, and Lemma 20 respectively.

Proof. Let e_1 and e_2 be distinct elements in E. If both e_1 and e_2 are in $E \setminus D = C \cup \{0^{\mathbf{E}}\}$, then without loss of generality $e_1 = c \in C$. By Corollary 12, $h_c(e_1) \neq h_c(e_2)$.

For the rest of the proof we assume that $e_2 \in D$. If $\overline{f}(e_1) \neq \overline{f}(e_2)$, then there is a basic term operation f_i with $f_i(e_1) \neq f_i(e_2)$. Pick $c \in \{f_i(e_1), f_i(e_2)\} \cap C$. Let c' denote the other element of $\{f_i(e_1), f_i(e_2)\}$. Using the homomorphism h_c of Corollary 12 we have $\{f_i(h_c(e_1)), f_i(h_c(e_2))\} = \{h_c(f_i(e_1)), h_c(f_i(e_2))\} = \{h_c(c), h_c(c')\}$, a two-element set. Thus $f_i(h_c(e_1)) \neq f_i(h_c(e_2))$ and consequently $h_c(e_1) \neq h_c(e_2)$.

The only cases that remain are when $\overline{f}(e_1) = \overline{f}(e_2)$ while still having $e_2 \in D$. Then $(e_1, e_2) \in J_1 \cup J_2$. (We cannot have $e_1 = 0^{\mathbf{E}}$ by Lemma 4.) Thus by either Lemma 15 or Lemma 20 we can construct a homomorphism h such that $h(e_1) \neq h(e_2)$. □

Corollary 22. *For* $\mathbf{E}$ *in* $\mathbb{HSP}(\mathbf{M})$ *with* $\mathbf{E}$ *finite, quasicritical, and satisfying the* N*-variable quasi-equations of* $\mathbf{M}$*, we have* $\mathbf{E}$ *in* $\mathbb{ISP}(\mathbf{M})$*.*

Proof. The map $\mathbf{E} \to \mathbf{M}^{\mathrm{Hom}(\mathbf{E},\mathbf{M})}$ given by $e \mapsto \langle h(e) : h \in \mathrm{Hom}(\mathbf{E}, \mathbf{M}) \rangle$ is an embedding. □

The proof of Theorem 6 is now complete, as every finite algebra that is quasicritical and satisfies the N-variable quasi-equations is actually already in the quasivariety. Lemma 1 shows that the N-variable quasi-equations form a basis of the quasi-equations of $\mathbf{M}$. Observe that for pairs in J_1, the proof of Lemma 15 potentially uses N-variables quasi-equations. However, for pairs not in J_1, the relevant proofs only require two-variable quasi-equations.

5. When the Rows Do Not Form an Order Ideal

The content of this section is the proof of the converse of Theorem 6: that a finite $\{0, 1\}$-valued unary algebra with 0 whose rows do not form an order ideal does not have a finite basis for its quasi-equations. We first illustrate this for algebras on a four-element universe and then generalize to an arbitrary finite algebra.

As before, throughout this section assume that $\mathbf{M}$ is a finite $\{0, 1\}$-valued unary algebra with 0 and that the clone is $f_0, f_1, f_2, \ldots, f_{s+1}$ where f_0 is the constant 0 function and f_{s+1} is the identity map. To prove the converse of Theorem 6 we need conditions for the non-existence of a finite basis for the quasi-equations of $\mathbf{M}$.

Results and definitions from [5] and [6] give these conditions. A **positive primitive formula** is an existentially quantified conjunction of atomic formulas. In the case of unary algebras the atomic formulas are of the form $f(x) \approx g(y)$ for not-necessarily-different variables x and y and terms (possibly the identity) f and g.

A unary algebra $\mathbf{M}$ is **pp-acyclic** if there is a positive primitive formula $\phi(x, y)$ defining an acyclic binary relation such that there exist 0 and 1 in $\mathbf{M}$ with $0 \neq 1$, $\phi(0, 0)$, $\phi(0, 1)$ and $\phi(1, 1)$. The simplest pp-acyclic relation is $\leq$ on the set $\{0, 1\}$.

Theorem 23 ([6])**.** *If* $\mathbf{M}$ *is a finite pp-acyclic unary algebra, then* $\mathbf{M}$ *does not have a finite basis for its quasi-equations.*

More generally, on an algebra $\mathbf{M}$, an n-ary relation R is **pp-defined** if there is a positive primitive formula $\phi(x_1, x_2, \ldots, x_n)$ such that $R = \{(a_1, a_2, \ldots, a_n) \in M^n : \phi(a_1, a_2, \ldots, a_n) \text{ holds in } \mathbf{M}\}$. We note in passing that $\mathbf{Rows}(\mathbf{M})$ is a pp-defined relation.

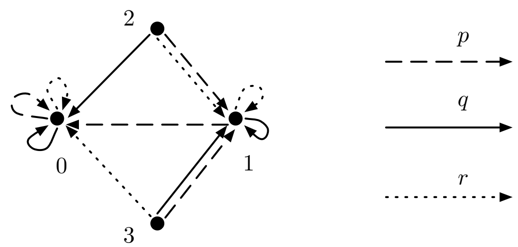
Theorem 24 ([5])**.** *If* $\mathbf{M}$ *is a finite unary algebra that has a pp-defined relation that is the graph of a non-trivial group, then* $\mathbf{M}$ *does not have a finite basis for its quasi-equations.*

The example given in Figure 3 has the graph of addition modulo 2 defined via

$$\exists w\, [\, x \approx p(w) \;\&\; y \approx q(w) \;\&\; z \approx r(w)] \,.$$

That is, the set of triples $\{(p(m), q(m), r(m)) \in {\mathbf{M}_2}^3 \mid m \in \mathbf{M}_2\}$ is $\{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$, which in turn is $\{(x,y,z) \in \{0,1\}^3 \colon x+y=z\}$. By Theorem 24, $\mathbf{M}_2$ does not have a finite basis for its quasi-equations.

However, for many examples, the positive primitive formulas are much more complex as will become evident in the upcoming proofs.



	p	q	r
0	0	0	0
1	0	1	1
2	1	0	1
3	1	1	0

FIGURE 3. The graph of the two-element group is pp-defined in $\mathbf{M}_2$.

Example 25. *If* $\mathbf{M}$ *is a four-element* $\{0,1\}$*-valued unary algebra with* 0*, then one of the following holds:*

(1) *the* $\leq$ *relation on* $\{0,1\}$ *can be positive primitively defined via a formula of the form* $\exists w\; x \approx f(w) \;\&\; y \approx g(w)$*;*
(2) *the graph of addition modulo* 2 *on* $\{0,1\}$ *can be positive primitively defined via a formula of the form* $\exists w\; x \approx p(w) \;\&\; y \approx q(w) \;\&\; z \approx r(w)$*;*
(3) *the rows of* $\mathbf{M}$ *form an order ideal.*

In the first two cases there is no finite basis for the quasi-equations, and in the last case there is a finite basis for the quasi-equations.

Proof. Let $\mathcal{F}$ be the non-constant, non-identity operations in the clone of $\mathbf{M}$. On a four-element set the possible non-identity functions for a $\{0,1\}$-valued unary algebra with 0 are given in Table 1. If $\mathcal{F}$ is empty or has one element then, by

	g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	0	0	1	1
3	0	1	0	1	0	1	0	1

TABLE 1. All possible operations of a four-element $\{0,1\}$-valued unary algebra with 0.

Lemma 5, there is a finite basis for the quasi-equations. Either $\mathbf{Rows}(\mathbf{M})$ is the empty ideal or there is one non-trivial function and the ideal formed by $\mathbf{Rows}(\mathbf{M})$ consists of $\langle 0\rangle$ and $\langle 1\rangle$. So we assume $|\mathcal{F}| \geq 2$.

The formula $\exists w\ x \approx g_3(w)\ \&\ y \approx g_5(w)\ \&\ z \approx g_6(w)$ defines the graph of addition modulo 2, so we may assume not all of g_3, g_5, and g_6 are in $\mathcal{F}$.

If $g_7 \in \mathcal{F}$ then for any $f \in \mathcal{F} \setminus \{g_7\}$ the formula $\exists w\ x \approx f(w)\ \&\ y \approx g_7(w)$ defines $\leq$ on $\{0,1\}$. Thus we may assume $g_7 \notin \mathcal{F}$. Similarly, any of the pairs $\{g_1, g_3\}$, $\{g_1, g_5\}$, $\{g_2, g_3\}$, $\{g_2, g_6\}$, $\{g_4, g_5\}$, or $\{g_4, g_6\}$ can be used to define $\leq$. Thus we may assume none of these pairs are in $\mathcal{F}$.

The remaining possible clones are $\{g_1, g_2\}$, $\{g_1, g_4\}$, $\{g_1, g_6\}$, $\{g_2, g_4\}$, $\{g_2, g_5\}$, $\{g_3, g_4\}$, $\{g_3, g_5\}$, $\{g_3, g_6\}$, $\{g_5, g_6\}$, and $\{g_1, g_2, g_4\}$. It is straightforward to check that all of these clones give order ideals. □

The remainder of this section is the generalization of Example 25 to an arbitrary finite $\{0,1\}$-valued unary algebra with 0 where we do not specify the form of the positive primitive formulas.

Lemma 26. *If* $\mathbf{Rows}(\mathbf{M})$ *is not relatively complemented, then* $\leq$ *on* $\{0,1\}$ *can be positive primitively defined in* $\mathbf{M}$*.*

Proof. As $\mathbf{Rows}(\mathbf{M})$ is not relatively complemented, there exists an interval of at least three elements, so we may assume $a < c < b$ are in $\mathbf{Rows}(\mathbf{M})$ and the complement of c between a and b is not in $\mathbf{Rows}(\mathbf{M})$. Let

$$\begin{aligned} S_{000} &= \{i\colon a(i) = c(i) = b(i) = 0\} \\ S_{001} &= \{i\colon a(i) = c(i) = 0 \text{ and } b(i) = 1\} \\ S_{011} &= \{i\colon a(i) = 0 \text{ and } c(i) = b(i) = 1\} \\ S_{111} &= \{i\colon a(i) = b(i) = c(i) = 1\}. \end{aligned}$$

As $a < c < b$ the disjoint union of these sets, $S_{000} \cup S_{001} \cup S_{011} \cup S_{111}$, is all of $\{1, 2, \ldots, s\}$. In addition, as a, b, and c are distinct, the sets S_{001} and S_{011} are non-empty. Define $\hat{C}$ to be the formula that identifies witnesses in $\mathbf{M}$ of the rows which have a constant value on each of these four sets and are 0 on S_{000}. That is, $\hat{C}(w)$ is

$$\hat{C}(w)\colon \quad \mathop{\&}_{i \in S_{000}} f_i(w) \approx 0\ \ \&\ \mathop{\&}_{i,j \in S_{001}} f_i(w) \approx f_j(w) \\ \&\ \mathop{\&}_{i,j \in S_{011}} f_i(w) \approx f_j(w)\ \ \&\ \mathop{\&}_{i,j \in S_{111}} f_i(w) \approx f_j(w).$$

When S_{000} or S_{111} is empty, then the conjunctions in $\hat{C}$ over S_{000} or S_{111} are omitted.

Pick $t_1 \in S_{001}$, $t_2 \in S_{011}$. First assume S_{111} is empty. This implies that a is $\mathbf{row}(0)$. The positive primitive formula $\hat{R}(x,y)$ defined by

$$\hat{R}(x,y)\colon \qquad \exists w\ \hat{C}(w)\ \&\ x \approx f_{t_1}(w)\ \&\ y \approx f_{t_2}(w)$$

defines $\leq$ on $\{0,1\}$. To see this, note that the witnesses of the rows a, c and b witness $\hat{R}(0,0)$, $\hat{R}(0,1)$, and $\hat{R}(1,1)$. If the witness of some row d were a witness to $\hat{R}(1,0)$ then d would be the relative complement of c in the interval between a and b. This contradicts the assumption that the relative complement of c between a and b is not in $\mathbf{Rows}(\mathbf{M})$.

Now assume S_{111} is non-empty and pick $t_3 \in S_{111}$. Note that $a \neq \mathbf{row}(0)$ because $a(t_3) = 1$. Consider the relation $E(x_1, x_2, x_3)$ defined by

$$E(x_1,x_2,x_3): \qquad \exists w\ \hat{C}(w)\ \&\ x_1 \approx f_{t_1}(w)\ \&\ x_2 \approx f_{t_2}(w)\ \&\ x_3 \approx f_{t_3}(w).$$

There are two cases. First assume $E(0,1,0)$ is witnessed in $\mathbf{M}$. Define $\bar{R}(x,y)$ via

$$\bar{R}(x,y): \qquad \exists w\ \hat{C}(w)\ \&\ x \approx f_{t_1}(w) \approx f_{t_3}(w)\ \&\ y \approx f_{t_2}(w).$$

The zero element is a witness to $\bar{R}(0,0)$, and the witness of b gives $\bar{R}(1,1)$ while the witness of $E(0,1,0)$ gives $\bar{R}(0,1)$. If $\bar{R}(1,0)$ holds, then the witness to $\bar{R}(1,0)$ is a witness to a complement of c between a and b, a contradiction. Thus $\leq$ is defined.

Now assume $E(0,1,0)$ is not witnessed in $\mathbf{M}$. Define $R^*(x,y)$ via

$$R^*(x,y): \qquad \exists w\ \hat{C}(w)\ \&\ 0 \approx f_{t_1}(w)\ \&\ x \approx f_{t_2}(w)\ \&\ y \approx f_{t_3}(w).$$

The witnesses of the zero row, a, and c witness $R^*(0,0)$, $R^*(0,1)$, and $R^*(1,1)$ respectively. However, $R^*(1,0)$ does not hold, as otherwise a witness of $R^*(1,0)$ is a witness of $E(0,1,0)$. Thus $\leq$ is again defined. □

Lemma 27. *Assume* $\mathbf{M}$ *is a* $\{0,1\}$*-valued unary algebra with* 0 *such that* $\mathbf{Rows}(\mathbf{M})$ *is relatively complemented but is not an order ideal. Then there is a pair of elements* $a, b \in \mathbf{Rows}(\mathbf{M})$ *with*

(1) $a < b$;
(2) *there exists* $c \in \{0,1\}^s \setminus \mathbf{Rows}(\mathbf{M})$ *with* $a < c < b$;
(3) *for* $d < b$ *with* $d \in \mathbf{Rows}(\mathbf{M})$*, the set of elements below* d *in* $\mathbf{Rows}(\mathbf{M})$ *form an order ideal; and*
(4) *the interval from* a *to* b *in* $\mathbf{Rows}(\mathbf{M})$ *is* $\{a,b\}$.

Proof. As $\mathbf{Rows}(\mathbf{M})$ is not an order ideal and $\mathbf{row}(0)$ is in $\mathbf{Rows}(\mathbf{M})$, there exists a minimal element b in $\mathbf{Rows}(\mathbf{M})$ and c in $\{0,1\}^s \setminus \mathbf{Rows}(\mathbf{M})$ with $\mathbf{row}(0) < c < b$. Fix c minimal with respect to this property. Thus for all $d < b$, if $d \in \mathbf{Rows}(\mathbf{M})$ then the interval below d in $\{0,1\}^s$ is in $\mathbf{Rows}(\mathbf{M})$. By the minimality of c, each element strictly below c in $\{0,1\}^s$ is also in $\mathbf{Rows}(\mathbf{M})$. Fix a a maximal element with $\mathbf{row}(0) \leq a < c$.

We now show that the interval from a to b in $\mathbf{Rows}(\mathbf{M})$ consists exactly of a and b. Suppose d in $\mathbf{Rows}(\mathbf{M})$ with $a \leq d \leq b$. If $d \wedge c = c$ then $d \geq c$ and in fact $d > c$ and by the minimality of b, we have $d = b$. Otherwise, using $a \leq d \wedge c < c$ and the minimality of c we get that $d \wedge c$ is in $\mathbf{Rows}(\mathbf{M})$. In fact $d \wedge c = a$ by maximality of a. Let $\hat{d}$ be the complement of d in the interval from a to b. As $\mathbf{Rows}(\mathbf{M})$ is relatively complemented, $\hat{d}$ is in $\mathbf{Rows}(\mathbf{M})$. Because $d \wedge c = a$, by distributivity of $\{0,1\}^s$ we have $\hat{d} \geq c$. In fact, $\hat{d} > c$ as $c \notin \mathbf{Rows}(\mathbf{M})$. By the minimality of b, the element $\hat{d}$ is actually b. By uniqueness of complements $d = a$, showing that the interval from a to b in $\mathbf{Rows}(\mathbf{M})$ consists exactly of a and b. □

Lemma 28. *Assume* $\mathbf{M}$ *is a* $\{0,1\}$*-valued unary algebra with* 0 *such that* $\mathbf{Rows}(\mathbf{M})$ *is relatively complemented but is not an order ideal. Then either there is a positive primitive formula that defines* $\leq$ *on* $\{0,1\}$*, or there is a positive primitive formula that defines the graph of addition modulo* 2.

Proof. By Lemma 27, there are elements a, b in $\mathbf{Rows}(\mathbf{M})$ such that $a < b$, the interval from a to b in $\mathbf{Rows}(\mathbf{M})$ is $\{a,b\}$, and there is an element $c \in \{0,1\}^s \setminus$

$\mathbf{Rows}(\mathbf{M})$ with $a < c < b$. Set

$$\begin{aligned} T_{00} &= \{i \colon b(i) = 0\} \\ T_{11} &= \{i \colon a(i) = 1\} \\ T_{01} &= \{1, 2, \dots, s\} \setminus (T_{00} \cup T_{11}). \end{aligned}$$

We show that when T_{11} is non-empty we can define $\leq$ with a positive primitive formula. When T_{11} is empty we may be able to define $\leq$ but, if we cannot, then addition modulo 2 is definable.

Notice that T_{00} and T_{11} are disjoint and that a is $\mathbf{row}(0)$ if and only if T_{11} is empty. Note that T_{01} is the set of co-ordinates on which a and b differ. As $a < c < b$ we have that $|T_{01}| \geq 2$.

Case 1: Assume T_{11} is non-empty. Let $C_1(w)$ be the formula that identifies rows which are constantly 0 on T_{00}, and are constant on T_{11} and on T_{01}. That is, $C_1(w)$ is

$$C_1(w)\colon \qquad \underset{i \in T_{00}}{\&} f_i(w) \approx 0 \ \& \underset{i,j \in T_{11}}{\&} f_i(w) \approx f_j(w) \ \& \underset{i,j \in T_{01}}{\&} f_i(w) \approx f_j(w).$$

Pick ℓ_1 in T_{01} and $j_1 \in T_{11}$. Define the formula $R_1(x, y)$ as

$$R_1(x, y)\colon \qquad \exists w \ C_1(w) \ \& \ x \approx f_{\ell_1}(w) \ \& \ y \approx f_{j_1}(w).$$

From the rows a, b, and $\mathbf{row}(0)$ we obtain $R_1(0, 1)$, $R_1(1, 1)$, and $R_1(0, 0)$ respectively. When R_1 defines $\leq$ on $\{0, 1\}$ we are done with this case. Otherwise $R_1(1, 0)$ holds for some witness. This means there is a row d in $\mathbf{Rows}(\mathbf{M})$ with

$$d(i) = \begin{cases} 0 & \text{if } i \in T_{00} \cup T_{11}, \\ 1 & \text{if } i \in T_{01}. \end{cases}$$

Thus $d < b$. By Lemma 27, we must have that $\mathbf{Rows}(\mathbf{M})$ is an order ideal below d. Thus the atom e which is 1 exactly on the co-ordinate ℓ_1 is in $\mathbf{Rows}(\mathbf{M})$.

Let $C_2(w)$ be the formula that identifies rows which are constantly 0 on T_{00}, and are constant on $T_{11} \cup T_{01} \setminus \{\ell_1\}$. That is, $C_2(w)$ is

$$C_2(w)\colon \qquad \underset{i \in T_{00}}{\&} f_i(w) \approx 0 \ \ \& \underset{i,j \in T_{11} \cup T_{01} \setminus \{\ell_1\}}{\&} f_i(w) \approx f_j(w).$$

With the same j_1 as above, define the formula $R_2(x, y)$ as

$$R_2(x, y)\colon \qquad \exists w \ C_2(w) \ \& \ x \approx f_{j_1}(w) \ \& \ y \approx f_{\ell_1}(w).$$

From the rows e, b, and $\mathbf{row}(0)$ we obtain $R_2(0, 1)$, $R_2(1, 1)$, and $R_2(0, 0)$ respectively. Suppose $R_2(1, 0)$ holds for some witness, then there is a row r with

$$r(i) = \begin{cases} 0 & \text{if } i \in T_{00} \cup \{\ell_1\}, \\ 1 & \text{if } i \in T_{11} \cup T_{01} \setminus \{\ell_1\}. \end{cases}$$

Thus $a < r < b$ because $|T_{01}| \geq 2$, which is a contradiction as the interval in $\mathbf{Rows}(\mathbf{M})$ from a to b contains just a and b. We have shown that when T_{11} is non-empty we can define $\leq$ on $\{0, 1\}$.

Case 2: Now assume T_{11} is empty. This means a is $\mathbf{row}(0)$. Pick ℓ_1 and ℓ_2 distinct in T_{01}. As f_{ℓ_1} and f_{ℓ_2} are distinct functions, there must be a row e in $\mathbf{Rows}(\mathbf{M})$ with $e(\ell_1) \neq e(\ell_2)$. Pick such an e minimal and, without loss of generality, assume $e(\ell_1) = 0$ and $e(\ell_2) = 1$. As the interval from a to b in $\mathbf{Rows}(\mathbf{M})$ contains exactly a and b we have $e \not\leq b$.

For $\beta, \gamma \in \{0, 1\}$ define the subsets of $\{1, 2, \ldots, s\}$

$$S_{\beta\gamma} = \{i \colon b(i) = \beta \text{ and } e(i) = \gamma\}.$$

The co-ordinate ℓ_1 is in S_{10} and ℓ_2 is in S_{11}. The set S_{01} is non-empty as otherwise $a = \mathbf{row}(0) < e < b$ which cannot hold. Define $C_3(w)$ be the formula that identifies rows which are constantly 0 on S_{00}, and are constant on each of S_{01}, S_{10}, and S_{11}. That is, $C_3(w)$ is

$$C_3(w)\colon \quad \mathop{\&}_{i \in S_{00}} f_i(w) \approx 0 \;\&\; \mathop{\&}_{i,j \in S_{01}} f_i(w) \approx f_j(w) \;\&\; \mathop{\&}_{i,j \in S_{10}} f_i(w) \approx f_j(w) \;\&\; \mathop{\&}_{i,j \in S_{11}} f_i(w) \approx f_j(w).$$

Let $R(x, y)$ be the formula

$$R(x, y)\colon \qquad \exists w \; C_3(w) \;\&\; x \approx f_{\ell_1}(w) \;\&\; y \approx f_{\ell_2}(w).$$

$R(0, 0)$, $R(0, 1)$, and $R(1, 1)$ hold via $\mathbf{row}(0)$, e, and b respectively. If $R(1, 0)$ does not occur then we have defined $\leq$. We shall see that when $R(1, 0)$ does occur we can define addition modulo 2.

Assume that there is an $m \in \{0, 1\}$ and a row d with

$$d(i) = \begin{cases} 0 & \text{if } i \in S_{00} \cup S_{11}, \\ m & \text{if } i \in S_{01}, \\ 1 & \text{if } i \in S_{10}. \end{cases}$$

That is, a witness to row d is a witness to $R(1, 0)$. If $m = 0$ then $a < d < b$, contradicting the non-existence in $\mathbf{Rows}(\mathbf{M})$ of rows between a and b, so $m = 1$.

Pick ℓ_3 in S_{01}. Let $P(x, y, z)$ be the formula

$$P(x, y, z)\colon \qquad \exists w \; C_3(w) \;\&\; x \approx f_{\ell_1}(w) \;\&\; y \approx f_{\ell_2}(w) \;\&\; z \approx f_{\ell_3}(w).$$

$P(0, 0, 0)$, $P(1, 1, 0)$, $P(1, 0, 1)$, and $P(0, 1, 1)$ hold via witnesses of $\mathbf{row}(0)$, b, d, and e respectively. Any witness of $P(0, 1, 0)$ or $P(1, 0, 0)$ would be a witness for a non-zero row below b which cannot happen. If $P(0, 0, 1)$ holds then a non-zero row is witnessed that is below e. Take the relative complement to this row between $\mathbf{row}(0)$ and e. The witness to this new row is a witness to $P(0, 1, 0)$, a contradiction. Finally if $P(1, 1, 1)$ has a witness then use the relative complement of e to obtain a witness to $P(1, 0, 0)$ and a corresponding row below b, another contradiction. Thus the formula P defines the graph of addition modulo 2. □

6. Classifying $\{0, 1\}$-Valued Unary Algebras with 0

We are now able to summarize the above material in the following result.

Theorem 29. *If* $\mathbf{M}$ *is a* $\{0, 1\}$*-valued unary algebra with* 0 *then one of the following holds:*

1. *the* $\leq$ *relation on* $\{0, 1\}$ *can be positive primitively defined;*
2. *the graph of addition modulo* 2 *on* $\{0, 1\}$ *can be positive primitively defined; or*

(3) *the rows of* $\mathbf{M}$ *form an order ideal.*

In the first two cases there is no finite basis for the quasi-equations, and in the last case there is a finite basis for the quasi-equations.

Proof. If $\mathbf{M}$ has one or fewer non-identity, non-zero operations then $\mathbf{Rows}(\mathbf{M})$ forms a (possibly empty) order ideal. By Lemma 5, there is a finite basis for the quasi-equations.

If $\mathbf{Rows}(\mathbf{M})$ has an interval that is not relatively complemented then by Lemma 26 we can positive primitively define $\leq$ on $\{0, 1\}$. By Theorem 23 there is no finite basis for the quasi-equations.

If $\mathbf{Rows}(\mathbf{M})$ is relatively complemented but is not an order ideal then, by Lemma 28, there is either a positive primitive formula that defines $\leq$ on $\{0, 1\}$ or there is a positive primitive formula that defines the graph of addition modulo 2. By Theorem 23 or Theorem 24 there is no finite basis for the quasi-equations.

If none of the above situations holds, then $\mathbf{Rows}(\mathbf{M})$ forms a non-empty order ideal. By Theorem 6, $\mathbf{M}$ has a finite basis for its quasi-equations. In fact, the $(2 + |M|\,(|M| - 1))$-variable quasi-equations form a basis. □

As a consequence of Theorem 29 we obtain the main result of this paper.

Theorem 30. *Let* $\mathbf{M}$ *be a finite* $\{0, 1\}$*-valued unary algebra with* 0*. Then* $\mathbf{M}$ *has a finite basis for its quasi-equations if and only if the rows of* $\mathbf{M}$ *form an order ideal.*

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