What is a Finite Lattice?

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A **lattice** is an ordered set in which every pair of elements has a l.u.b. and g.l.b.

A *lattice* is an algebra $L = \langle L, \lor, \land \rangle$ satisfying

\[
\begin{align*}
    x \lor x & \approx x & x \land x & \approx x \\
    x \lor y & \approx y \lor x & x \land y & \approx y \land x \\
    x \lor (y \lor z) & \approx (x \lor y) \lor z & x \land (y \land z) & \approx (x \land y) \land z \\
    x \lor (x \land y) & \approx x & x \land (x \lor y) & \approx x
\end{align*}
\]
A finite **lattice** is a join semilattice with 0, or dually, a meet semilattice with 1.

A finite lattice can be represented as a closure system/Moore family on any set $S$ with $J(L) \subseteq S \subseteq L$.

A finite lattice can be represented as Sub $A$ for some finite algebra $A$. 
FBA or Formal Concept Analysis

A finite lattice can be represented by the closure lattice of the Galois connection $LE \subseteq S \times T$ whenever $J(L) \subseteq S \subseteq L$ and $M(L) \subseteq T \subseteq L$. 
Moore families

- Each finite lattice is isomorphic to a subsemilattice of $\langle 2^S, \cdot, 1 \rangle$ with $J(L) \subseteq S \subseteq L$.
- The lattice of closure systems on $S$ is isomorphic to $\text{Sub}(2^S, \cdot, 1)$ ordered by reverse inclusion.

A finite lattice can be obtained from $2^S$ by a sequence of operations removing a meet irreducible element.

More generally, remove $\uparrow a - \uparrow b$ for some pair $a < b$. 
Closure operators

- Each finite lattice is isomorphic to a homomorphic image of the free 0-semilattice on $S$.
- The lattice of closure operators on $S$ is isomorphic to $\text{Con} (F(S), \lor, 0)$.
- In particular, $\text{Clop}(S)$ is upper bounded and satisfies $\text{SD} \land$.

Join irreducibles

- $J(\text{Clop}(S))$ consists of all nontrivial implications of the form $X \rightarrow y$.
- $(X \rightarrow z) \leq (Y \rightarrow z)$ iff $X \supseteq Y$.
- $(X \rightarrow z) \leq \lor (Y_i \rightarrow t_i)$ iff this follows from the order relations and a sequence of applications of the cut rule $(X \cup Y \rightarrow b) \leq (X \rightarrow a) \lor (\{a\} \cup Y \rightarrow b)$.
A finite lattice is determined by a join representation $\Sigma = \bigvee T_i$ of the corresponding closure system (semilattice congruence) in Clop$(S)$:

- direct unit basis
- D-basis
- G-D canonical basis
- K-basis
Clop($S$) has canonical meets - the subdirect decomposition of $L$ as a join semilattice

Coatoms of Clop($S$) are $\psi_B$ where $S = B \cup T$ with $B \neq S$, given by

$$b \rightarrow b', \quad t \rightarrow t', \quad t \rightarrow b$$

for all $b, b' \in B$ and $t, t' \in T$.

$\psi_B \geq \bigwedge \psi_{C_j}$ minimally if $B = \bigcap C_j$. 

Nation | Finite Lattices
Given a finite lattice \( L \) and \( S = J(L) \), let

\[
\mathcal{B}(L) = \{ \psi_B : B = \downarrow p \cap J(L) \text{ for some } p \in M(L) \}
\]

\( L = F(S) / \theta \) where \( \theta = \bigwedge \{ \psi_B : B \in \mathcal{B}(L) \} \).

The number of meetands is \( |M(L)| \), out of a total of \( 2|J(L)| - 1 \) coatoms.

Removing one meet irreducible at a time is adding one relation \( b \rightarrow t \) that is below \( \theta \) but not below exactly one \( \psi_D \) with \( \psi_D \not\geq \theta \).
Likewise, every lattice is a subdirect product of subdirectly irreducible lattices.

Let $U$ be a filter on a lattice $L$. Then there is a unique largest congruence $\hat{\psi}_U$ on $L$ separating $U$.

Let $L$ be a finite subdirectly irreducible lattice, and let $u \in J(L)$ be a critical element. Let $X = J(L)$. Then there is an endomorphism $\sigma$ of $FL(X)$ such that (1) for all $x \in X$, $\sigma(x) \leq x$ and $\sigma(x) \in X^\wedge\wedge$ and (2) $L \cong FL(X)/\hat{\psi}_U$ for the filter generated by $\{\sigma^k(u) : k \in \omega\}$.

$U$ is lower bounded iff $U$ is principal.

Conversely, let $\sigma$ be an endomorphism of $FL(X)$, with $X$ finite, satisfying (1). If $u \in X$ and $U$ is defined as in (2), then $FL(X)/\hat{\psi}_U$ is finite.
Congruence lattices of finite lattices

Tischendorf extension

\[ \text{Con}(L) \cong^d \text{Id}(J(L), D) \text{ where } I \text{ is an ideal if } x \in I \text{ and } x D y \text{ implies } y \in I. \]

So the congruence lattice of \( L \) is determined by the non-binary part of its D-basis.

Removing the binary part of the D-basis produces an atomistic lattice \( K \) with \( \text{Con}(K) \cong \text{Con}(L) \).

OR you could add new binary relations \( x \rightarrow y \) to obtain a smaller lattice \( M \) with \( \text{Con}(M) \cong \text{Con}(L) \), so long as the D-relation remains intact.

Compare

- Belief revision
- Belief propagation
(Whitman, Pudlák, Tuma) Every finite lattice can be represented by equivalence relations on a finite set.

Every finite lattice can be embedded into the lattice of subgroups of a finite group.

Can we uniformly bound the exponent of the group?

Every finite lattice is isomorphic to a congruence lattice $\text{Con } A$ for some algebra $A$.

(Finite representation problem) Can we take $A$ to be finite?

Candidates for a counterexample: $2 \times 3$ with a wing, or nondesarguean projective planes.