THE INJECTIVE ENVELOPE OF A MODULE

E. L. Lady

Theorem. Let E' be an injective module and $M \subseteq E \subseteq E'$. The following conditions are equivalent:

(1) E is a maximal essential extension of M.
(2) E is a minimal injective submodule of E' containing M.
(3) E is an essential extension of M and is injective.
(4) E is injective and every **monomorphism** from M into an injective module Q extends to a **monomorphism** from E into Q.

DEFINITION. If the above conditions are satisfied, we say that E is an *injective envelope* of M.

Lemma. If condition 1) is satisfied, then E is a summand of E'.

PROOF: By Zorn's Lemma there exists a submodule $F \subseteq E'$ maximal with respect to the property $F \cap E = 0$. We will show that $E' = E \oplus F$. Let $\pi \colon E \oplus F \to E$ be the projection. Since E' is injective, π extends to a map $\pi' \colon E' \to E'$.

We now claim that E is an essential submodule of $\pi'(E')$. In fact, let $\pi'(x) \in \pi'(E')$. If $x \in E \oplus F$ then $\pi'(x) = \pi(x) \in E$, so we only need consider the case $x \notin E \oplus F$. Then $x \notin F$ so by the maximality of F, $(F \oplus Rx) \cap E \neq 0$, so there exists $f + rx \neq 0 \in E$ with $f \in F$, $r \in R$. Then $r\pi'(x) = \pi'(f + rx) \in \pi'(E) = \pi(E) = E$ and $r\pi'(x) = \pi'(f + rx) \neq 0$ since $f + rx \notin F$, otherwise $F \cap E \neq 0$. This shows that $\pi'(E')$ is an essential extension of E, so by the maximality of E, $\pi'(E') = E$. Thus π' is an endomorphism of E' such that $\pi'(e) = \pi(e) = e$ for $e \in E$. This shows that E is a summand of E'. ☑

We claim that E'/F is isomorphic to a submodule of E' which is an essential extension of E. In fact the quotient map $E' \to E'/F$ restricts to a *monomorphism* $\zeta \colon E \to E'/F$ because $E \cap F = 0$. Now if $x + F \neq 0 \in E'/F$ then $x \notin F$ so $Rx + F$ properly contains F, so by the maximality of F we see that $(Rx + F) \cap E \neq 0$, which says that in E'/F the coset $x + F$ has a non-trivial multiple in $\zeta(E) = (E + F)/F$. This means that E'/F is an essential extension of $\zeta(E)$. Now since E' is injective, then inclusion map $E \hookrightarrow E'$ extends to a map $\theta\colon E'/F \to E'$, and since E'/F is an essential extension of $\zeta(E)$ it follows that θ is a monomorphism.

$$\begin{array}{ccccc} 0 & \longrightarrow & E & \xrightarrow{\ \zeta\ } & E'/F \\ & & \big\downarrow{\scriptstyle \subseteq} & & \\ & & E' & & \end{array}$$

Thus θ embeds E'/F into a submodule $\theta(E'/F)$ of E' which is an essential extension of E. By the transitivity of essential extensions $\theta(E'/F)$ is also an essential extension of M. Since E is a maximal essential extension of M, we conclude that $\theta(E'/F) = E$. From this we in turn conclude that $E'/F = \zeta(E) = (E+F)/F$ and so $E + F = E'$, so that $E' = E \oplus F$.

PROOF OF THEOREM: (1) $\Rightarrow$ (2): By the Lemma, E is a summand of E', hence is injective. Now if $M \subseteq E_1 \subseteq E$ and E_1 is injective then E_1 is a summand of E. Since E is an essential extension of E_1, this is only possible if $E_1 = E$. Thus E is a minimal injective submodule of E' containing M.

(2) $\Rightarrow$ (3): Assume that E is a minimal injective module containing M. Using Zorn's Lemma, we get a maximal essential extension E_1 of M with $E_1 \subseteq E$. Since E is injective, (1) $\Rightarrow$ (2) shows that E_1 is injective. Thus by the minimality of E we conclude $E_1 = E$, so that E is an essential extension of M.

(3) $\Rightarrow$ (4): Immediate from the properties of injective modules and essential extensions.

(4) $\Rightarrow$ (3): Again by Zorn's Lemma we get an essential extension E_1 of M satisfying (1) with $E_1 \subseteq E$. So E_1 is injective, and so by (4) the inclusion map $M \hookrightarrow E_1$ extends to a monomorphism from E into E_1. Then since E_1 is an essential extension of M, E must also be an essential extension of M. (In fact, of course, we must have $E_1 = E$.)

(3) $\Rightarrow$ (1): Since an injective module is a summand of every module that contains it, it can have no proper essential extensions. It then follows from the transitivity of injective extensions that no module properly containing E can be an essential extension of M.