THE KRULL-SCHMIDT-AZUMAYA THEOREM

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Proposition 1. Let K and N be submodules of M. The following conditions are equivalent:

(1) $M = K \oplus N$.
(2) There exists a map $\kappa\colon M \to K$ such that $\operatorname{Ker}\kappa = N$ and $\kappa(k) = k$ for all $k \in K$.
(3) There exists $\varphi \in \operatorname{End} M$ such that $\varphi^2 = \varphi$ and $\operatorname{Ker}\varphi = N$ and $K = \varphi(M)$.

Proposition 2. If $M = L \oplus P$ and H is a submodule of M such that $P \subseteq H \subseteq M$, then $H = P \oplus (H \cap L)$.

PROOF: Let $\pi\colon M \to P$ be the projection with kernel L. Since $\pi(M) = P \subseteq H$, π restricts to a map $\pi'\colon H \to P$ and for all $h \in H \subseteq P$, $\pi'(h) = \pi(h) = h$. Thus by Proposition 1, $H = P \oplus \operatorname{Ker}\pi'$. But $\operatorname{Ker}\pi' = H \cap \operatorname{Ker}\pi = H \cap L$. ☑

Proposition 3. Suppose that

$$M = K \oplus N = L_1 \oplus \cdots \oplus L_n$$

and that $\operatorname{End}_R K$ is a local ring. Then for some i there exist submodules L_i', $L_i'' \subseteq L_i$ such that $L_i' \approx K$ and

$$M = K \oplus L_1 \oplus \cdots \oplus L_i'' \oplus \cdots \oplus L_n.$$

PROOF: First, note that if $M = K \oplus L'' \oplus P$, then L'' is a summand of M and so if $L'' \subseteq L$ then by Proposition 2 L'' is a summand of L. Therefore there exists L' as claimed. Likewise if $M = K \oplus L \oplus P''$ then there exists P' as claimed.

Now let κ be the projection of M onto K with kernel N and, provisionally, let λ_i be the projection of M onto L_i with kernel $L_1 \oplus \cdots \oplus \widehat{L}_i \oplus \cdots \oplus L_n$. If $m \in M$ then $m = \sum \lambda_i(m)$ and so if $k \in K$ then

$$k = \kappa(k) = \sum \kappa(\lambda_i(k)).$$

At first glance, we can express this by saying that $\sum \kappa\lambda_i$ restricts to the identity map on K.

Unfortunately this doesn't quite make sense because technically the composition $\kappa\lambda_i$ is not defined, since

$$M \xrightarrow{\lambda_i} L_i \quad \text{and} \quad M \xrightarrow{\kappa} K.$$

However, this is purely a notional difficulty which which get around by **redefining** λ_i to be the idempotent in $\mathrm{End}_R M$ with $\lambda_i(\ell_i) = \ell_i$ if $\ell_i \in L_i$ and $\lambda_i(L_j) = 0$ for $j \neq i$ (c. f. Proposition 1). (We haven't really changed λ_i *as a function*, we've simply declared its codomain to be M instead of L_i.) With this change, $\sum \kappa\lambda_i$ restricts to the identity map in $\mathrm{End}_R K$. Thus not all $\kappa\lambda_i|_K$ can belong to the unique maximal left idea of $\mathrm{End}_R K$, so for some i, $\kappa\lambda_i|_K$ is an automorphism of K. Thus there exists $\theta \in \mathrm{End}_R K$ such that $\kappa\tilde{\lambda}_i\theta = 1_K$, where $\tilde{\lambda}_i$ is the restriction of λ_i to K.

[First Ending.] This says that $\tilde{\lambda}_i\theta$ is a splitting map for the exact sequence

$$0 \to N \to M \xrightarrow{\kappa} K \to 0,$$

and that therefore $M = N \oplus \tilde{\lambda}_i\theta(K)$. Write $L' = \tilde{\lambda}_i\theta(K) = \lambda_i(K) \subseteq L_i$. Then L' is a summand of M, hence is a summand of L_i (Proposition 2), say $L_i = L_i' \oplus L_i''$. Then

$$M = L_1 \oplus \cdots \oplus L_i' \oplus L_i'' \oplus \cdots \oplus L_n.$$

Let $\lambda_i' \in \mathrm{End}_R M$ be the idempotent with $L_i' = \lambda_i'(M)$ and $\lambda_i'(L_i'') = \lambda_i'(L_j) = 0$ for $j \neq i$. Note that since $\lambda_i(K) = L_i'$, then λ_i' and λ_i agree on K. In particular, λ_i' maps K isomorphically onto L_i'. If $\sigma\colon L_i' \to K$ is the inverse of the restriction of λ_i' to K, then σ corresponds to a map $L_i' \to M$ which splits the short exact sequence

$$0 \to L_1 \oplus \cdots \oplus L_i'' \oplus \cdots \oplus L_n \to M \xrightarrow{\lambda_i'} L_i' \to 0.$$

Thus

$$M = \sigma(L_i') \oplus \mathrm{Ker}\,\lambda_i' = K \oplus \mathrm{Ker}\,\lambda_i' = K \oplus L_1 \oplus \cdots \oplus L_i'' \oplus \cdots \oplus L_n.$$

[Alternate Ending.] If $\kappa\lambda_i$ restricts to an automorphism of K then there exists $\theta \in \mathrm{End}_R K$ such that for any $k \in K$,

$$k = \theta\kappa\lambda_i(k)$$

and $\theta\kappa\lambda_i(M) = K$. Thus by Proposition 1,

$$M = K \oplus \mathrm{Ker}\,\theta\kappa\lambda_i.$$

Now since

$$L_1 \oplus \cdots \oplus \widehat{L}_i \oplus \cdots \oplus L_n \subseteq \mathrm{Ker}\,\theta\kappa\lambda_i \subseteq L_1 \oplus \cdots \oplus L_i \oplus \cdots \oplus L_n,$$

by Proposition 2,

$$\mathrm{Ker}\,\theta\kappa\lambda_i = L_1 \oplus \cdots \oplus \widehat{L}_i \oplus \cdots \oplus L_n \oplus L_i''$$

for some $L_i'' \subseteq L_i$ and so

$$M = K \oplus L_1 \oplus \cdots \oplus L_i'' \oplus \cdots \oplus L_n,$$

as claimed.

Note that since $L_i'' \subseteq L$, by Proposition 2, $L = L_i' \oplus L_i''$ for some L_i', and furthermore

$$L_i' \approx M/(L_1 \oplus \cdots \oplus L_i'' \oplus \cdots \oplus L_n) \approx K. \quad \boxed{\checkmark}$$

Krull-Schmidt-Azumaya Theorem. Let

$$M = M_1 \oplus \cdots \oplus M_s = L_1 \oplus \cdots \oplus L_t,$$

where the L_j are indecomposable and for each i, $\operatorname{End}_R M_i$ is a local ring. Then $t = s$ and the L_j can be renumbered so that for each $1 \le k \le s$,

$$M = M_1 \oplus \cdots \oplus M_{k-1} \oplus L_k \oplus \cdots \oplus L_s.$$

In particular, after renumbering, $M_i \approx L_i$ for all i.

PROOF: Apply Proposition 3 to the direct decomposition

$$M = M_1 \oplus (M_2 \oplus \cdots \oplus M_s) = L_1 \oplus L_2 \oplus \cdots \oplus L_t$$

to see that for some i, $L_i = L_i' \oplus L_i''$ with $L_i' \approx M_1$ and

$$M = M_1 \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_i'' \oplus \cdots \oplus L_n.$$

Renumbering, we may as well suppose $i = 1$. Furthermore, since L_1 is by assumption indecomposable and $L_1' \neq 0$, it follows that $L_1' = L_1$ and $L_1'' = 0$ and

$$M = M_1 \oplus L_2 \oplus \cdots \oplus L_n.$$

Now apply Proposition 3 to the direct decomposition

$$M = M_2 \oplus (M_1 \oplus M_3 \oplus \cdots \oplus M_s) = M_1 \oplus L_2 \oplus \cdots \oplus L_t,$$

to see that M_2 replaces one of the indecomposable summands on the RHS. This enables us to continue inductively, *provided that the summand replaced by M_2 is not M_1*. But checking the proof of Proposition 3 shows that in order for this summand to be M_1, the idempotent endomorphism μ_2 mapping M onto M_2 with kernel $M_1 \oplus M_3 \oplus \cdots \oplus M_s$ would have to map M_1 isomorphically onto M_2, which is impossible since $\mu_2(M_1) = 0$.

It is now clear how to complete the proof by induction. ☑

Corollary [Krull-Schmidt Theorem]. If M has finite length then any two decompositions of M into a direct sum of indecomposable modules satisfy the conclusion of the Krull-Schmidt-Azumaya Theorem.

PROOF: If M has finite length and $M = M_1 \oplus \cdots \oplus M_s$, then each M_i also has finite length. Thus if each M_i is indecomposable, then $\operatorname{End}_R M_i$ is a local ring. ☑