MATH 611 **Due Wed Sept 30, 1992**

Definitions. Let $\varphi\colon M \to N$.

(1) If φ is one-to-one we say that φ is a **monomorphism** or **monic**.
(2) If φ is **onto** (i.e. $N = \varphi(M)$) then we say that φ is a **surjection** or **surjective** or an **epimorphism** or **epic**.

1. Recall the fact from linear algebra that if φ is a linear operator o a finite dimensional vector space V then φ is a monomorphism if and only if φ is surjective.
Now let K be a field and let A be a finite dimensional K-algebra (Hungerford, p. 227). Let $a \in A$. Prove that the following conditions are equivalent:

(1) a is not a left zero divisor in A, i.e. $ab = 0 \Rightarrow b = 0$.
(2) a is right invertible in A, i.e. $(\exists b)\, ab = 1$.
(3) The principal right ideal generated by a is all of A.
(4) a is not a right zero divisor in A.
(5) a if left invertible in A.
(6) The principal left ideal generated by a is A.
(7) a has a (two-sided) inverse in A.

2. Let $N \subseteq M$ and let $\varphi\colon M \to M/N$ be the quotient map. (In Hungerford p. 172, this is called the canonical epimorphism.) Then N is a direct summand of M if and only if there exists a submodule $L \subseteq M$ such that φ maps L isomorphically onto M/N. In this case, $M = L \oplus N$.

3. Let $M = L \oplus N$. Define a map $\pi\colon M \to N$ as follows:
If $m \in M$ then there exist $\ell \in L$ and $n \in N$ such that $m = \ell + n$. Define $\pi(m) = n$.
Prove that π is a well-defined homomorphism of R-modules from M **onto** N.

4. Let N be a submodule of M. Prove that the following conditions are equivalent:

(1) N is a direct summand of M.
(2) There exists $\pi \in \operatorname{End} M$ such that $\pi(M) = N$ and π restricts to the identity map on N.
(3) There exists $\pi \in \operatorname{End} M$ such that $\pi(M) = N$ and $\pi^2 = \pi$.