MATH 611 **Due Wed Oct 7, 1992**

1. Let N be a submodule of M and let $\theta\colon M \to M/N$ be the quotient map (canonical epimorphism). Prove that there is a one-to-one correspondence between the submodules of M/N and the submodule of M containing N, where a submodule L of M/N corresponds to the submodule $\theta^{-1}(L)$ of M.

NOTE: The notation does not refer to a map θ^{-1}. Unless $N = 0$ there is no inverse to θ (WHY?). Instead, *by definition*

$$\theta^{-1}(L) = \{\, m \in M \mid \theta(m) \in L \,\}.$$

2. Let R be a ring and let R_1 and R_2 be additive subgroups of R closed under multiplication such that $R = R_1 + R_2$. (NOTE: According to our conventions, we do not refer to R_1 and R_2 as subrings of R since they probably (in fact, almost certainly!) will not contain 1.)

Prove that the following conditions are equivalent:

(1) $R_1 \cap R_2 = 0$ and for all $r_1, r_1' \in R_1$ and $r_2, r_2' \in R_2$,

$$(r_1 + r_2)(r_1' + r_2') = r_1 r_1' + r_2 r_2'.$$

(2) $R_1 R_2 = R_2 R_1 = 0$. I.e. for all $r_1 \in R_1$ and $r_2 \in R_2$, $r_1 r_2 = r_2 r_1 = 0$.

(3) R_1 and R_2 are (two-sided) ideals of R and $R_1 \cap R_2 = 0$.

(4) There exist $e_1, e_2 \in R$ such that $R_1 = Re_1$, $R_2 = Re_2$ and

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 + e_2 = 1, \quad e_1 e_2 = e_2 e_1 = 0$$

and for all $r \in R$, $re_1 = e_1 r$, $re_2 = e_2 r$.
(This last condition means that e_1 and e_2 are in the **center** of R.)

3. Let F be a skewfield and V a vector space over F with $\dim_F V = n < \infty$. Let $\alpha \in \operatorname{End}_F V$. We say that α is **nilpotent** if $\alpha^r = 0$ for some positive integer r.

a) Prove that α is nilpotent if and only if $\alpha^n = 0$, where $n = \dim_F V$.

b) Prove that α is nilpotent if and only if there exists a basis for V such that the matrix for α with respect to the basis is strictly upper triangular (i.e. has zeros on and below the main diagonal).

4. Let M be an A-module and $\alpha \in \operatorname{End}_A M$. Suppose there exists a positive integer n such that $\alpha^n(M) = \alpha^{n+1}(M)$ and $\operatorname{Ker} \alpha^n = \operatorname{Ker} \alpha^{n+1}$. Prove that $M = \alpha^n(M) \oplus \operatorname{Ker} \alpha^n$.