1. If M is a left R-module and $m \in M$, then we have defined $\operatorname{ann} m$ as the set of elements $r \in R$ with $rm = 0$. We have seen that $\operatorname{ann} m$ is a left ideal.

Now define $\operatorname{ann} M = \{r \in R \mid rM = 0\}$.

a) Prove that $\operatorname{ann} M = \bigcap_{m\in M} \operatorname{ann} m$.

b) Prove that $\operatorname{ann} M$ is a two-sided ideal.

2. Let R be a ring and e_1, e_2 be elements in the *center* of R such that $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$. Let $R_1 = e_1R$ and $R_2 = e_2R$. Then R_1 and R_2 are rings, although not subrings of R. (C.f. homework for October 7.)

a) Let M be an R-module and set $M_1 = e_1M$, $M_2 = e_2M$. Prove that M_1 and M_2 are R-submodules of M and that $M = M_1 \oplus M_2$.

b) Show that it is natural to think of M_i as an R_i-module. Show also that $e_1M_2 = 0 = e_2M_1$.

c) Conversely, show that if M_1 is an R_1-module and M_2 is an R_2 module then $M = M_1 \oplus M_2$ can be uniquely given the structure of an R-module such that $M_1 = e_1M$ (as R_1-modules) and $M_2 = e_2M$.

d) Prove that if M and N are R-modules and φ is a function from M to N then φ is R-linear if and only if φ restricts to an R_i-linear map from e_iM into e_iN ($i = 1, 2$).

3. **a)** Let M be an R-module and $\varphi \in \operatorname{End}_R M$. Prove that if φ is an epimorphism and $(\exists r) \operatorname{Ker}\varphi^r = \operatorname{Ker}\varphi^{r+1}$ then φ is an isomorphism from M onto itself. (In this case, φ is called an **automorphism** of M.)

b) Prove that if M satisfies the maximum condition (maximum principle) and $\varphi \in \operatorname{End}_R M$ is an epimorphism, then φ is an automorphism.

Definition. We say that an R-module M satisfies the **minimum condition** if every non-empty family of submodules of M has a minimal member.

4. Prove that if M satisfies the minimum condition and $\varphi \in \operatorname{End}_R M$ is monic, then φ is an automorphism. (HINT: Consider the family of submodules of the form $\varphi^r(M)$ for $r = 1, 2, \ldots$.)

Definition. An R-module M is called **indecomposable** if

$$M = L \oplus N \quad \Rightarrow \quad L = 0 \quad \text{or} \quad N = 0.$$

5. Prove that if an R-module M satisfies **either** the maximum condition or minimum condition, then M can be written as a finite direct sum of indecomposable modules.