1. Let M be a module over a commutative ring R and assume that the family of ideals $\{\operatorname{ann} m \mid m \neq 0 \in M\}$ has a maximal member I. Prove that I is a prime ideal.

2. We will write $Z(p^\infty)$ for the *p-primary component* of $\mathbb{Q}/\mathbb{Z}$, i.e. $Z(p^\infty) = \{x \in \mathbb{Q}/\mathbb{Z} \mid (\exists n)\, p^n x = 0\}$.

a) Prove that every proper subgroup of $Z(p^\infty)$ is cyclic.
b) Prove that $Z(p^\infty)$ is artinian but not noetherian.
c) Prove that the ring $\operatorname{End}_{\mathbb{Z}} Z(p^\infty)$ is an integral domain with characteristic 0.

3. Let $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$. What this means is that R is the set of all matrices $\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$ such that $a_{11} \in \mathbb{Z}$, a_{21}, $a_{22} \in \mathbb{Q}$.

a) Prove that R is a ring.
b) Identify all the left ideals of R.
c) Identify all the right ideals of R. (NOTE: To answer this fully, you would have to describe all the $\mathbb{Z}$-submodules of $\mathbb{Q}$, which is beyond you. So your answer will have to be a little bit vague.)
d) Prove that all left ideas of R are finitely generated, i.e. that R is **left noetherian**.
e) Prove that not all right ideas of R are finitely generated, so R is not **right noetherian**.
f) Prove that a left R-module is uniquely determined by a $\mathbb{Z}$-module G, a $\mathbb{Q}$-vector space V, and a $\mathbb{Z}$-linear map $\varphi\colon G \to V$.
g) Prove that a right R-module is uniquely determined by a $\mathbb{Z}$-module G, a $\mathbb{Q}$-vector space V, and a $\mathbb{Z}$-linear map $\varphi'\colon V \to G$.

4. Let F be a skew-field and $R = \begin{pmatrix} F & F \\ F & F \end{pmatrix}$, i.e. R is the ring of 2×2 matrices with entries in F.

a) Find all left ideals in R.
b) Find all right ideals in R.
c) Prove that all non-trivial proper left ideals are mutually isomorphic.
d) Let M be a left R-module and N a left F-subspace of M. Prove that N is an R-submodule of M $\iff$ N is closed under multiplication by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
e) Prove that any R-module is isomorphic to a direct sum of left ideals.

3. **a)** The only thing that really needs to be checked is that R is closed under multiplication. This is why it would not have worked, for instance, to have chosen $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Q} \end{pmatrix}$.

b) First note for future reference that in any ring like this made up of matrices, the "columns" are always left ideals, because when two matrices are multiplied, the procedure is to multiply the left hand matrix times each column of the right hand matrix. I.e. we have left ideals $\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$. One also readily identifies the left ideals $\begin{pmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Q} \end{pmatrix}$. In fact, when multiplying any matrix by (on the left) $\begin{pmatrix} z_{11} & 0 \\ q_{21} & q_{22} \end{pmatrix}$, the bottom row simply gets multiplied by q_{22}. So if we take any subspace V of the vector space $\mathbb{Q} \oplus \mathbb{Q}$, we can obtain a left ideal by taking the set of matrices with top row trivial and bottom row ranging over V. More mundanely, we can describe such ideals as $\left\{ \begin{pmatrix} 0 & 0 \\ x & xr \end{pmatrix} \mid x \in \mathbb{Q} \right\}$ where r is a fixed element of $\mathbb{Q}$. This clearly yields all left ideals with trivial top row.
Now if the top row is non-trivial for a left ideal L, then we easily see that the set of all z_{11} that occur for $\begin{pmatrix} z_{11} & 0 \\ q_{21} & q_{22} \end{pmatrix} \in L$ if a subgroup of the cyclic group $\mathbb{Z}$, hence has the form $n\mathbb{Z}$ for some non-zero n. But now we see that L will contain all matrices $\begin{pmatrix} 0 & 0 \\ qn & 0 \end{pmatrix}$, i.e. L contains $\begin{pmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$. Now the set of possible second rows occuring in L is a $\mathbb{Q}$-vector subspace of $\mathbb{Q} \oplus \mathbb{Q}$ containing $\mathbb{Q} \oplus 0$. We now quickly see that either $L = \begin{pmatrix} n\mathbb{Z} & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$ or $L = \begin{pmatrix} n\mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$.

c) Obviously the reasoning is analogous. To start with, the "rows" $\begin{pmatrix} n\mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ are right ideals, and then we get more by letting the first column range over any sub**group** of $\begin{pmatrix} \mathbb{Z} \\ \mathbb{Q} \end{pmatrix}$, while keeping the second column trivial. (These latter ideals are finitely generated as R-modules $\iff$ they are finitely generated as abelian groups (WHY?), so there are lots of non-finitely generated ones, so R is not right noetherian.) Finally, if the second column is non-trivial then the right ideal must contain $\begin{pmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$ and it looks like $\begin{pmatrix} n\mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$, where n is an integer, possibly 0.

f) Give a $\mathbb{Z}$-module G, a $\mathbb{Q}$-vector space V and $\varphi\colon G \to V$, consider the set

$$\begin{pmatrix} G \\ V \end{pmatrix} = \left\{ \begin{pmatrix} g \\ v \end{pmatrix} \mid g \in G,\ v \in V \right\}.$$

We can make this into an R-module by defining

$$\begin{pmatrix} z_{11} & 0 \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} z_{11} g \\ q_{21}\varphi(g) + q_{22} v \end{pmatrix}.$$

Note that

$$\begin{aligned} \begin{pmatrix} z_{11} & 0 \\ q_{21} & q_{22} \end{pmatrix} \left(\begin{pmatrix} z'_{11} & 0 \\ q'_{21} & q'_{22} \end{pmatrix} \begin{pmatrix} g \\ v \end{pmatrix} \right) &= \begin{pmatrix} z_{11} z'_{11} g \\ q_{21}\varphi(z'_{11} g) + q_{22} q'_{21} \varphi(g) + q_{22} q'_{22} v \end{pmatrix} \\ &= \left(\begin{pmatrix} z_{11} & 0 \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} z'_{11} & 0 \\ q'_{21} & q'_{22} \end{pmatrix} \right) \begin{pmatrix} g \\ v \end{pmatrix}. \end{aligned}$$

(Although this looks like a mess, it's just the usual associativity for matrix multiplication.)

Conversely, if we start with an R-module M then we define $G = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) M$ and $V = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) M$. Furthermore, we define φ by $\varphi(g) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) g$ and note that

$$\varphi(G) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) M = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) M = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) M \subseteq \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) M = V.$$

If $m \in M$ let $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) m$ and $v = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) m$. Then $m = g + v$ and we see that

$$\left(\begin{smallmatrix} z_{11} & 0 \\ q_{12} & q_{22} \end{smallmatrix}\right) g = \left(\begin{smallmatrix} z_{11} & 0 \\ q_{12} & q_{22} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) m = \left(z_{11} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + q_{12} \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \right) m = z_{11} g + q_{12} \varphi(g),$$

and

$$\left(\begin{smallmatrix} z_{11} & 0 \\ q_{12} & q_{22} \end{smallmatrix}\right) v = \left(\begin{smallmatrix} z_{11} & 0 \\ q_{12} & q_{22} \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) m == q_{22} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) m = q_{22} v,$$

so that M is isomorphic to the module $\begin{pmatrix} G \\ V \end{pmatrix}$ constructed above.