Baer's Characterization of Injectives

E. L. Lady

The proof of Baer's characterization of injective modules usually seems fairly obscure. But in fact, the idea is quite simple. If one knows that any homomorphism from an ideal of R into M extends to all of R, then it would seem that any mapping from a submodule of a cyclic module would extend to the whole cyclic module. This is true, although it's not quite that obvious, since not all cyclic modules are isomorphic to the ring. Once one has proved the result for cyclic modules, though, then what one sees is that whenever one has a mapping φ from a submodule X of a module Y, one can always extend this mapping one step further. Namely, consider the cyclic submodule generated by an element y not in X. Then one can extend (the restriction of) φ to a mapping φ' on this entire cyclic module Ry. The fact that φ and φ' agree on the intersection of Ry and X means that they determine an extension of φ on the submodule of Y generated by X together with y. This justifies the claim that one can always extend any mapping from a submodule one step further, and in such a way that the domain of the new homomorphism will contain any given desired element.

From this, it seems fairly clear intuitively that the mapping extends to all of Y. In the case where Y is finite, we can formalize this by mathematical induction. In the general case, we need to continue the induction transfinitely or, more conveniently, invoke Zorn's Lemma.

Theorem [Baer]. Let M be a R-module with the following property:

> Every R-linear mapping from a left ideal of R into M extends to an R-linear mapping from R into M.

THEN whenever $X \subseteq Y$ are R-modules and φ is an R-linear map from X into M, then φ extends to a map $\varphi' \colon Y \to M$.

The proof will use two lemmas:

Lemma. The result is true if Y is cyclic.

PROOF: If Y is cyclic, then $Y \approx R/K$ for some left ideal K, and it suffices to consider the case $Y = R/K$. Since there is a one-to-one correspondance between the submodules of R/K and the left ideals in R containing K, $X = L/K$ for some left ideal L with $K \subseteq L$. Now the quotient map $L \to L/K$ induces an exact sequence

$$0 \to \mathrm{Hom}_R(L/K,\, M) \to \mathrm{Hom}_R(L,\, M) \to \mathrm{Hom}_R(K,\, M).$$

Thus $\varphi \in \mathrm{Hom}_R(X,\, M) = \mathrm{Hom}_R(L/K,\, M)$ is induced by a map $\hat{\varphi}\colon L \to M$ such that $\hat{\varphi}(K) = 0$. By hypothesis, $\hat{\varphi}$ extends to a map $\hat{\varphi}'\colon R \to M$. Since $\hat{\varphi}'(K) = \hat{\varphi}(K) = 0$, $\hat{\varphi}'$ induces a map φ' on R/K which agrees with φ on L/K.

Lemma. If X_1, $X_2 \subseteq Y$ and $\varphi_i\colon X_i \to M$ are R-linear maps whose restrictions to $X_1 \cap X_2$ agree, then there is an R-linear map $\varphi\colon X_1 + X_2 \to M$ which extends both φ_1 and φ_2.

PROOF: This is because the square

$$\begin{array}{ccc} X_1 \cap X_2 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_1 + X_2 \end{array}$$

is a push-out.

PROOF OF THEOREM: By Zornifying (details given in class) we may assume WLOG that X is a maximal submodule of Y such that φ extends to X. Now suppose BWOC that $X \neq Y$. Then we can choose $y \in Y$ with $y \notin X$. Then φ restricts to a mapping of $X \cap Ry$ into M. Applying the first lemma to the cyclic module Ry, we see that this extends to a mapping $\varphi_1\colon Ry \to M$. (NOTE: It is not necessary to exclude the case $X \cap Ry = 0$.) Then the second lemma shows that φ extends to a map $\varphi_2\colon X + Ry \to M$. This contradicts the maximality of X. This contradiction shows that $X = Y$. ☑