LOCAL RINGS

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FACT. Any proper left ideal in a ring is contained in a maximal left ideal.

Lemma. If r and s are elements in a non-trivial ring such that $rs = 1$ but $sr \neq 1$, then neither sr nor $1 - sr$ is either left or right invertible.

PROOF: If $rs = 1$ then $sr(1 - sr) = sr - srsr = sr - sr = 0 = (1 - sr)sr$. Thus if, say, sr is left invertible then $usr = 1$ for some u and so $1 - sr = usr(1 - sr) = u0 = 0$, contrary to the assumption that $sr \neq 1$. Likewise if $1 - sr$ is left invertible then $v(1 - sr) = 1$ for some v and so $sr = v(1 - sr)sr = v0 = 0$. But then $1 = 1^2 = rsrs = 0$, so $R = 0$, contrary to the hypothesis.

Thus neither sr nor $1 - sr$ can be left invertible and similarly we see that they cannot be right invertible. ☑

Proposition. If a non-trivial ring R has a unique maximal left ideal J, then J is two-sided and is also the unique maximal right ideal in R.

PROOF: Let $r \in R$. Then Jr is a left ideal. If $Jr \neq R$ then by the above FACT, Jr is contained in a maximal left ideal. Since J is the only maximal left ideal, $Jr \subseteq J$.

On the other hand, if $Jr = R$ then $jr = 1$ for some $j \in J$. Now $rj \in J$ so $rj \neq 1$. Thus by the above Lemma, $1 - rj$ is not left invertible, which is to say that $R(1 - rj) \neq R$. But then $R(1 - rj)$ is contained in a maximal left ideal, i. e. $1 - rj \in R(1 - rj) \subseteq J$, and so $1 = rj + (1 - rj) \in J$, a CONTRADITION. Thus $Jr = R$ is impossible, and we see that $r \in R \Rightarrow Jr \subseteq J$. Therefore J is a two-sided ideal. ☑

Proposition. A non-trivial ring R is has a unique maximal left ideal if and only if the sum of any two non-invertible elements of R is not invertible. In this case, the unique maximal left ideal consists of the set of non-invertible elements in the ring.

PROOF: ($\Rightarrow$): Suppose that R has a unique maximal left ideal J. Note first that in this case an element $r \in R$ is invertible if and only if it is left invertible. In fact, if r is left invertible then $Rr = R$, so $r \notin J$ and thus $rR \not\subseteq J$. But by the preceding proposition, J is also the unique maximal right ideal in R, so by the FACT above the right ideal rR cannot be a proper right ideal. Thus $1 \in rR$, so r is also right invertible.

Now let r, $s \in R$ be not invertible. Then they are not left invertible so the principal left ideals Rr and Rs are proper left ideals and hence by the FACT above contained in maximal left ideals. Since J is the unique maximal left ideal, r, $s \in J$, so $r + s \in J$, so $r + s$ is not invertible. (Otherwise $1 \in J$ so $J = R$.)

($\Leftarrow$): Suppose now that the set of elements which are not invertible is closed under addition and let I denote this set. We claim first that under this hypothesis, any left invertible element $s \in R$ is invertible. In fact, if $rs = 1$ but $sr \neq 1$ then by the Lemma above sr and $1 - sr$ are not invertible. But if the set of non-invertible elements is closed under addition, it would then follow that $1 = sr + (1 - sr)$ is not invertible, which is absurd. Thus every left invertible element is invertible.

Thus I is the set of elements of R which is not *left* invertible. We claim that I is a left ideal. In fact, if $j \in I$ and $r \in R$ then rj cannot be left invertible since from $srj = 1$ it would follow that j is left invertible, which by assumption it is not. Thus $rj \in J$. Since by hypothesis I is closed under addition, I is a left ideal.

Furthermore, we claim that any other proper left ideal must be contained in I. In fact, if L is a proper left ideal then no element of L can be left invertible, otherwise $1 \in RL = L$, so $RL = R$, a contradiction. Since I contains all elements of R which are not left invertible, $L \subseteq I$. Thus I is the unique maximal left ideal in R. $\boxed{\checkmark}$

Definition. A ring which has a unique maximal left ideal is called a **local ring**.

To summarize, we have seen that a ring is a local ring if and only if the set of non-invertible elements in the ring is closed under addition. In this case, the set of non-invertible elements is the unique maximal left ideal in the ring and is in fact a two-sided ideal.