Basic Concepts of Module Theory

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Definition. Essential submodule.

Definition. M/N

Definition. Linear independence (infinite sets). Basis.

Theorem. If K is a field then $K[X]$ is a principal ideal domain.

Proposition. If an ideal in a **commutative** ring is free then it is principal.

Euclidean rings and greatest common divisors.

Proposition. If M has a basis and N is any R-module, there exists a homomorphism from M to N taking the basis elements to any arbitrary set of elements in N.

Rephrased. If M has a basis $\{m_i\}_{i\in I}$ then for any R-module N there is an isomorphism of abelian groups $\theta\colon \operatorname{Hom}(M,N) \to \bigoplus_I N$. This is defined by $\theta(\varphi) = \{\varphi(m_i)\}_{i\in I}$. **Exercise?**

Proposition. $\{1\}$ is a basis for the R-module R.

Proposition. Existence of the free R-module on a set X.

Proposition. Every module is a homomorphic image of a free module.

Internal and external direct sums. Direct products. Characterization of the internal direct sum. Equivalent conditions for a family of submodules to be independent.

Proposition. Every subspace of a vector space is a direct summand.

Questions. For what rings R is it true that every submodule of an R-module M is a direct summand? What rings R have the property that every R-module is free?

Proposition. Let $N \subseteq M$ and let $\varphi\colon M \to M/N$ be the quotient map. (In Hungerford p. 172, this is called the canonical epimorphism.) Then N is a direct summand of M if and only if there exists a submodule $L \subseteq M$ such that φ maps L isomorphically onto M/N. In this case, $M = L \oplus N$. **(Exercise.)**

Proposition. Let N be a submodule of M. The following conditions are equivalent:

(1) N is a direct summand of M.
(2) There exists $\pi \in \operatorname{End} M$ such that $N = \pi(M)$ and π restricts to the identity map on N.
(3) There exists $\pi \in \operatorname{End} M$ such that $N = \pi(M)$ and $\pi^2 = \pi$.

(Exercise.)

Theorem. If φ is an **epimorphism** from M onto P and P is a free R-module, then $\operatorname{Ker}\varphi$ is a direct summand of M.

Restatement. If N is a submodule of M such that M/N is free, then N is a direct summand of M.

Proposition. If M is a **finitely generated** torsion free module over an integral domain R then M is isomorphic to a submodule of a free R-module.

Example. $\mathbb{Q}$ is not isomorphic to a submodule of a free $\mathbb{Z}$-module.
($\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$.)

Proposition. If M is a finitely generated torsion free module over an integral domain R then some homomorphic image of M is isomorphic to an ideal in R.

Example. No homomorphic image of $\mathbb{Z}(p)$ is isomorphic to an ideal of $\mathbb{Z}$.

Proposition. If R is a principal ideal domain then a submodule of a finitely generated free R-module is free.

Theorem. A **finitely generated** torsion free module over a principal ideal domain is free.

Corollary. If M is a **finitely generated** module over a principal ideal domain R and $t(M)$ is the torsion submodule of M, then $t(M)$ is a direct summand of M.

Example. If $R = \mathbb{Z}$ and $M = \prod_p \mathbb{Z}(p)$, then $t(M) = \bigoplus_p \mathbb{Z}(p)$ and $t(M)$ is not a direct summand of M.

Definition. An irreducible element in an integral domain and a prime ideal in a commutative ring. The associated primes for a module over a principal ideal domain and the p-primary components.

Proposition. An ideal (a) in a principal ideal domain R is a prime ideal if and only if a is an irreducible element.

Example. In the ring $\mathbb{Z}[\sqrt{-5}]$, 2 is irreducible but (2) is not a prime ideal.

Theorem. If M is a torsion module (not necessarily finitely generated) over a principal ideal domain then $M = \bigoplus_p M_p$.

Lemma. Suppose that M is a module over a principal ideal domain such that $p^n M = 0$ for some prime element p and $n \geq 1$. If n is the smallest integer such that $p^n M = 0$ then there exists an element $m \in M$ such that $p^{n-1} m \neq 0$. Furthermore, the cyclic submodule of M generated by m is a direct summand.