## Divisibility and Factorial Rings

**Definition.** Irreducible element in an integral domain. Prime ideal. Maximal ideal.

**Proposition.** If $R$ is an integral domain and the principal ideal $(r)$ is a prime ideal then $r$ is irreducible.

**Proposition.** If $R$ is a PID then an element $r$ is irreducible if and only if $(r)$ is a maximal ideal.

**Proposition.** Every element in a principal ideal domain has a unique factorization as a product of irreducible elements.

Some properties of principal ideal domains.

(1) All ideals are finitely generated.
(2) The principal ideal generated by an irreducible element is prime.
(3) All non-trivial prime ideals are maximal.

An integral domain satisfying (1) and (2) will be a **unique factorization domain**. However not all unique factorization domains satisfy (1).

An integral domain is a principal ideal domain if and only if it is a unique factorization domain and satisfies (1) and (3).

**Proposition.** A not-necessarily-commutative ring $R$ is a skew field [division ring] if and only if the only left ideals in $R$ are 0 and $R$.

**Proposition.** Maximal ideals are prime.

**Definition.** Associated prime, $p$-primary module.

Chinese Remainder Theorem. If $I$ and $J$ are ideals in a commutative ring $R$ such that $I + J = R$, then $R/IJ \approx R/I \oplus R/J$. (note: Ideals $I$ and $J$ with this property are called **comaximal**.)

**Corollary.** Every cyclic module over a principal ideal domain is a direct sum of $p$-primary modules.