Noetherian & Artinian Modules
(Hungerford, Section 6.1)

Theorem. The following conditions on an R-module M are equivalent:

(1) M satisfies the maximum condition.
(2) Every submodule of M is finitely generated.
(3) There do not exist infinite strictly ascending chains or submodules of M.

If these conditions hold, then M is called **noetherian**.
If the ring R is noetherian as a left R-module then we say that R is a left noetherian ring.

Proposition. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then M is noetherian [artinian] if and only if both L and N are.

Proposition. If a ring R is noetherian [artinian] then every finitely generated R-module is noetherian [artinian].

Corollary. A ring R is left noetherian if and only if all submodules of finitely generated left R-modules are finitely generated.

Definition. Jacobson radical.

Nakayama's Lemma [Hungerford, Section 6.4].

Jordan-Hölder Theorem.

Fitting's Lemma.

Corollary. If an **indecomposable** R-module M has finite length, then every endomorphism of M is either an automorphism or nilpotent. Furthermore, the set of nilpotent endomorphisms forms a two-sided ideal in $\operatorname{End} R$.

Replacement Lemma. Let $M = L \oplus N$ and let π be the associated projection map onto L. Let P be a submodule of M. If π maps P isomorphically onto L, then $M = P \oplus N$.

Krull-Schmidt-Azumaya Theorem.