Lemma. (1) If $\mathfrak{p}$ is an ideal which is maximal with respect to the property that $\mathfrak{p}$ is not finitely generated then $\mathfrak{p}$ is prime.

(2) If $\mathfrak{p}$ is an ideal maximal with respect to the property that $\mathfrak{p}$ is not principal, then $\mathfrak{p}$ is prime.

Theorem. (1) If every prime ideal in a commutative ring R is finitely generated, then R is noetherian.

(2) If every prime ideal in an integral domain R is principal then R is a principal ideal domain.

Localization
(Hungerford, Section 3.4)

Concept. Additive functor.

Definition. The localization functor with respect to a multiplicative set in the center of a ring and the natural transformation $\theta\colon M \to S^{-1}M$.

Concept. The total quotient ring for a commutative ring and the quotient field of an integral domain.

Proposition. The map $\theta\colon M \to S^{-1}M$ is an isomorphism if and only if for all $s \in S$, the map $m \to sm$ is an automorphism.

Universal Characterization.

$$\begin{array}{ccc} M & & \\ {\scriptstyle\theta}\downarrow & & \\ S^{-1}M & \longrightarrow & P \end{array}$$

Proposition. For given S, localization w.r.t. S is a functor and θ is natural, i.e.

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle\theta_M}\downarrow & & {\scriptstyle\theta_N}\downarrow \\ S^{-1}M & \xrightarrow{S^{-1}\varphi} & S^{-1}N. \end{array}$$

Proposition. The localization functor is exact and additive.

HW **Corollary.** If N is a submodule of an R-module M and S is a multiplicative set in the center of R, then $S^{-1}N$ can be identified as a submodule of $S^{-1}M$, and $S^{-1}M/S^{-1}N \approx S^{-1}(M/N)$.

If I is a (two-sided) ideal in R and $\bar{S} = \{s + I \mid s \in S\} \subseteq R/I$, then $S^{-1}I$ is an ideal in $S^{-1}R$ and the rings $S^{-1}R/S^{-1}I$ and $\bar{S}^{-1}(R/I)$ are isomorphic.

Proposition. If R is noetherian then $S^{-1}R$ is noetherian. (NOTE: We need not assume that R is commutative. As always, though, we require that $S \subseteq \operatorname{Center} R$.)

Proposition. If M and N are R-modules and S is a multiplicative set in the center of R, then

$$\operatorname{Hom}_{S^{-1}R}(S^{-1}M,\, S^{-1}N) \approx \operatorname{Hom}_R(S^{-1}M,\, S^{-1}N)\,.$$

Localization and Associated Primes for Commutative Rings

Note. An ideal $\mathfrak{a}$ in a commutative ring R is prime if and only if the set of elements of R not in $\mathfrak{a}$ forms a multiplicative set in R.

Lemma [Hungerford, Theorem 2.2, p. 378]**.** If S is a multiplicative set in a commutative ring R and $\mathfrak{a}$ is an ideal such that $\mathfrak{a} \cap S = \varnothing$ then there exists at least one ideal $\mathfrak{p}$ maximal with respect to the properties $\mathfrak{p} \supseteq \mathfrak{a}$ and $\mathfrak{p} \cap S = \varnothing$. Furthermore, any such ideal is prime.

Consequence [Hungerford, Theorem 2.6, p. 379]**.** Let $\mathfrak{a}$ be an ideal in R and let $r \in R$. The following are equivalent:

(1) r belongs to every prime ideal which contains $\mathfrak{a}$.
(2) r does not belong to any multiplicative set S such that $S \cap \mathfrak{a} = \varnothing$.
(3) For some positive integer k, $r^k \in \mathfrak{a}$.

Theorem. The set of nilpotent elements in a commutative ring forms an ideal. This is called the **nil radical** of the ring and it equals the intersection of all the prime ideals of the ring. (The nil radical itself is usually not prime.)

Notation. If $\mathfrak{p}$ is a prime ideal in a commutative ring R and $S = R \setminus \mathfrak{p}$, then we write $M_{\mathfrak{p}} = S^{-1}M$.

Proposition [Hungerford, Theorem 4.10, p. 146]**.** The prime ideals of $S^{-1}R$ are in one-to-one correspondence with the prime ideals of R which are disjoint from S.

Corollary. If $\mathfrak{p}$ is a prime ideal in R then $R_{\mathfrak{p}}$ is a local ring and $\mathfrak{p}R_{\mathfrak{p}} = J(R_{\mathfrak{p}})$.

Nakayama's Lemma [Hungerford, Lemma 4.5, p. 388]**.** If R is a not necessarily commutative ring and J its Jacobson radical and if M is an R-module and N a submodule such that $M = N + JM$, then $N = M$.

Lemma. Let $m \in M$ and let $\operatorname{ann} m = \{r \in R \mid rm = 0\}$. Then $m/1 \neq 0 \in S^{-1}M$ if and only if $S \cap \operatorname{ann} m = \varnothing$.

Localization-Globalization Theorem. Let M, N, P be modules over a commutative ring R.

(1) If $m_1, m_2 \in M$, then $m_1 = m_2$ if and only if $m_1/1 = m_2/1 \in M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$.
(2) $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$.
(3) Suppose that $N, P \subseteq M$. Then $N = P$ if and only if $N_{\mathfrak{m}} = P_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$.
(4) If $\varphi \in \operatorname{Hom}_R(M, N)$ then φ is monic [epic] if and only if $\varphi_{\mathfrak{m}} \colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is monic [epic] for all maximal ideals $\mathfrak{m}$.
(5) A sequence $M \to N \to P$ is exact if and only if the induced sequence $M_{\mathfrak{m}} \to N_{\mathfrak{m}} \to P_{\mathfrak{m}}$ is exact for all maximal ideals $\mathfrak{m}$.

Warning. $M_{\mathfrak{m}} \approx N_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ does not, in general, imply that $M \approx N$.

Corollary: Chinese Remainder Theorem. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals in a commutative ring R such that $\mathfrak{a}_i + \mathfrak{a}_j = R$ for all $i \neq j$. Let M be an R-module. Then

$$\frac{M}{\cap \mathfrak{a}_i M} \approx \bigoplus_1^n \frac{M}{\mathfrak{a}_i M}.$$

Theorem. Hilbert Basis Theorem [Hungerford, Theorem 4.9, p. 391.]

Definition. We say that a prime ideal $\mathfrak{p}$ is an **associated prime** for M if there exists $m \in M$ such that $\mathfrak{p} = \operatorname{ann} m$. We write $\operatorname{Ass} M$ (sometimes called the **assassinator** of M) for the set of associated primes for M.

A module M is called **$\mathfrak{p}$-primary** if $\operatorname{Ass} M = \{\mathfrak{p}\}$. (Somewhat inconsistently, a submodule N of M is called a $\mathfrak{p}$-primary submodule if M/N is $\mathfrak{p}$-primary. C.f. Hungerford, top of p. 384. Note that Hungerford assumes that the modules are finitely generated.)

Lemma. A prime ideal $\mathfrak{p}$ is an associated prime for M if and only if M contains a submodule isomorphic to $R/\mathfrak{p}$.

HW **Proposition.** If M is an R-module and $\mathfrak{p}$ is a prime ideal, the following conditions are equivalent:

(1) M is $\mathfrak{p}$-primary.

(2) The natural map $\theta \colon M \to M_{\mathfrak{p}}$ is monic and

$$(\forall r \in \mathfrak{p})\,(\forall m \in M)\,(\exists k \geq 1)\;\; r^k m = 0.$$

HW **Proposition.** If $\mathfrak{p}$ is a prime ideal then $R/\mathfrak{p}$ is $\mathfrak{p}$-primary.

Proposition. If $\mathfrak{p}$ is maximal in the family of ideals $\{\operatorname{ann} m \mid m \in M\}$, then $\mathfrak{p}$ is prime. Consequently **if R is noetherian** then

$$\operatorname{Ass} M = \varnothing \iff M = 0.$$

HW **Proposition.** If $N \subseteq M$ then $\operatorname{Ass} N \subseteq \operatorname{Ass} M \subseteq \operatorname{Ass} N \cup \operatorname{Ass} M/N$.

Definition. $\operatorname{Supp} M = \{\mathfrak{p} \mid M_{\mathfrak{p}} \neq 0\}$.

Note. By the Localization-Globalization Theorem, $M = 0 \iff \operatorname{Supp} M = \varnothing$.

Proposition. If M is finitely generated then

$$\operatorname{Supp} M = \{\mathfrak{p} \mid \mathfrak{p} \quad \text{is prime and} \quad \mathfrak{p} \supseteq \operatorname{ann} M\}.$$

Example. Let $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$. Then 0 is a prime ideal and $\operatorname{ann} M = 0$ but $0 \notin \operatorname{Supp} M$.

HW **Proposition.** $\operatorname{Ass} M \subseteq \operatorname{Supp} M$. Conversely, if $\mathfrak{p}$ is minimal among the prime ideals in $\operatorname{Supp} M$ then $\mathfrak{p} \in \operatorname{Ass} M$.

Proposition.

$$\begin{aligned} \operatorname{Supp}_{S^{-1}R} S^{-1}M &= \{\, \mathfrak{p}S^{-1}R \mid \mathfrak{p} \in \operatorname{Supp} M \quad \& \quad \mathfrak{p} \cap S = \varnothing \,\} \\ \operatorname{Ass}_{S^{-1}R} S^{-1}M &= \{\, \mathfrak{p}S^{-1}R \mid \mathfrak{p} \in \operatorname{Supp} M \quad \& \quad \mathfrak{p} \cap S = \varnothing \,\} \\ \operatorname{Ass}_{R} S^{-1}M &= \{\, \mathfrak{p}S^{-1}R \mid \mathfrak{p} \in \operatorname{Supp} M \quad \& \quad \mathfrak{p} \cap S = \varnothing \,\}. \end{aligned}$$

Proposition. If M is a finitely generated module over a commutative noetherian ring then $\operatorname{Ass} M$ is a finite set.
(NOTE: In general, $\operatorname{Supp} M$ will not be finite.)

Definition. We say that $r \in R$ is a **zero divisor on a module M** if $rm = 0$ for some $m \neq 0 \in M$.

Important Lemma [Hungerford, Theorem 2.3, p. 378]**.** If $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are prime ideals in a commutative ring R and $\mathfrak{a}$ is an ideal such that $\mathfrak{a} \subseteq \bigcup_1^n \mathfrak{p}_i$, then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.

Proposition. If R is noetherian then $\bigcup\{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass} M\}$ is the set of elements in R which are zero divisors in M.

Corollary. If R is noetherian then $\bigcup\{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass} R\}$ equals the set of zero divisors in R.

Modules with Finite Length over a Commutative Noetherian Ring

Definition and Proposition [Hungerford, pp. 375–376]**.** A module M over a ring R is said to have **finite length** if and only if it has a composition series, and in this case we define $\operatorname{length} M$ to be the length of this composition series. The **Jordan-Holder Theorem** asserts that $\operatorname{length} M$ is independent of the particular composition series. A module has finite length if and only if it is both noetherian and artinian.

Proposition. If M is an artinian module then $\operatorname{Ass} M$ consists of maximal ideals.

Proposition. If M is a module such that $\operatorname{Ass} M$ consists of maximal ideals, then $\operatorname{Ass} M = \operatorname{Supp} M$ and for every $\mathfrak{p} \in \operatorname{Ass} M$, the canonical map $M \to M_{\mathfrak{p}}$ is a surjection and

$$M_{\mathfrak{p}} \approx \{m \in M \mid (\exists k)\ \mathfrak{p}^k m = 0\}.$$

Theorem. A module over a commutative noetherian ring has finite length if and only if it is finitely generated and all its associated primes are maximal.

Proposition. If M is a module with finite length over a commutative noetherian ring R then $\operatorname{Ass} M$ is finite and the canonical maps $M \to M_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Ass} M = \operatorname{Supp} M$ induce an isomorphism

$$M \approx \bigoplus_{\operatorname{Ass} M} M_{\mathfrak{p}}.$$

Corollary. A commutative noetherian ring is artinian if and only if every prime ideal is maximal (including the zero ideal, if applicable). If this is the case, then R is a finite product of local rings each of which has a unique prime ideal.

Theorem. Let R be a commutative noetherian local ring and let $\mathfrak{m}$ be its unique maximal prime ideal. The following conditions are equivalent:

(1) R is artinian.
(2) $\mathfrak{m}$ is the only prime ideal in R.
(3) $\operatorname{Ass} R = \{\mathfrak{m}\}$.
(4) $\mathfrak{m}^k = 0$ for some positive integer k.
(5) The injective envelope of $R/\mathfrak{m}$ is finite generated.
(6) There exists a finitely generated injective R-module.

Injective Modules over Commutative Noetherian Rings

Theorem. If $\mathfrak{p}$ is a prime ideal in a commutative noetherian ring R, then the injective envelope E of $R/\mathfrak{p}$ is indecomposable. Moreover, every indecomposable injective R-module is isomorphic to the injective envelope of $R/\mathfrak{p}$ for some $\mathfrak{p}$. Furthermore $E \approx E_{\mathfrak{p}}$ and $\operatorname{Ass} E = \{\mathfrak{p}\}$.

Corollary. Every injective module over a commutative noetherian ring R is a (possibly infinite) direct sum of indecomposable injective modules.

Note. This is never true over a non-noetherian ring.